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Lax Formula for Obstacle Problems

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The first order obstacle problem $\min\{u_t + H(Du),\ g(x) - u\} = 0,\ u(T, x) = g(x)$ has a Hopf formula in the case when g *is convex.* It was first derived by A. Subbotin [11]. The case when g is continuous but the *Hamiltonian* H *is convex* is considered here. The corresponding Lax formula is derived to be

$$\begin{aligned} u(t,x) &= \sup_{y\in\mathbb{R}^n} \inf_{t\le\tau\le T} \Big\{ g(y) - (\tau - t) H^*\Big(\frac{y-x}{\tau - t}\Big)\Big\} \\ &= \sup_{y\in\mathbb{R}^n} \inf_{t\le\tau\le T} \Big\{ g(x + y(\tau - t)) - (\tau - t)H^*(y)\Big\}. \end{aligned}$$

This formula is shown to provide a viscosity solution of the obstacle problem. The argument to derive and prove this is based on optimal control in L^∞.

Keywords: Lax formula, Hopf formula, optimal control, obstacle problem.

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1. Introduction

The classic important Lax formula for the solution of

$$u_t + H(Du) = 0,\ (t,x) \in [0,T) \times \mathbb{R}^n,\ u(T,x) = g(x),\ x \in \mathbb{R}^n \tag{1}$$

is

$$u(t,x) = \sup_{y\in\mathbb{R}^n} \Big\{ g(y) - (T - t) H^*\Big(\frac{y-x}{T-t}\Big)\Big\}, \tag{2}$$

where g is assumed at least continuous and $H\colon \mathbb{R}^n \to \mathbb{R}$ is assumed convex, continuous and coercive, i.e., $H(p)/|p| \to \infty$ as $|p| \to \infty$. Here

$$H^*(y) = \sup_{p\in\mathbb{R}^n} \{p \cdot y - H(p)\}$$

is the Fenchel conjugate of H. If g is assumed Lipschitz, then so is u. If $g\colon \mathbb{R}^n \to [-\infty, \infty)$ is assumed merely upper semicontinuous and $g(x) \le C(1 + |x|)$ for some $C > 0$, then u is also upper semicontinuous, bounded from above by a

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function of linear decay, and the unique upper semicontinuous viscosity solution of the Hamilton-Jacobi equation. Refer to [2, 3, 9, 12, 13, 11], and the references there for the large literature devoted to this problem. Notice however that the problem usually considered in the literature is forward in time and is the forward Cauchy problem $u_t + H(Du) = 0, u(0, x) = g(x)$ and the Lax formula is $u(t, x) = \inf_{y\in\mathbb{R}^n}\{g(y) + tH^*((x - y)/t)\}$. Many authors have considered and used the Lax formula in various contexts. For example, the Lax formula is essential in the duality theory of optimal transport [14].

If one instead assumes that g is convex and H is Lipschitz, the Hopf formula for the explicit solution of (1) is given by

$$u(t, x) = \sup_{y\in\mathbb{R}^n}\{y \cdot x - g^*(y) + (T - t)H(y)\}, \tag{3}$$

Bardi and Evans [3], Evans [9], were the first to prove that this function provides a continuous viscosity solution of (1). This formula too has been extended to quasiconvex or quasiconcave and semicontinuous data. Refer to [1] for details and the most general results.

In both the Lax and Hopf formulas, spatial independence of the Hamiltonian is essential. The formulas have been extended to include t dependence and u dependence, but this typically involves more assumptions. See T.D. Van [12] for a short survey of Hopf and Lax formulas and some further results. Refer also to [1] for results with semicontinuous as well as quasiconvex data.

Many important problems in calculus of variations and optimal control (see, for instance [2]), such as optimal stopping problems, or problems in L^∞ control lead to Hamilton-Jacobi equations which are obstacle problems of the following type

$$\min\{u_t + H(t, x, u, Du),\ g(t, x) - u\} = 0,\ u(T, x) = g(T, x). \tag{4}$$

Specifically, this problem arises as the Bellman equation for certain types of calculus of variations or optimal control problems in L^∞, see [6]. These are generalizations of deterministic optimal stopping problems in that the obstacle is allowed to depend on the control. In [7], we studied the obstacle problem from the point of view of a Hopf formula assuming the obstacle was convex (or concave). The first derivation of the Hopf formula for the first order obstacle problem was given by Subbotin in [11]. In this paper the Lax formula is derived assuming the Hamiltonian H is convex. In the following subsection we discuss this obstacle problem and how the Lax formula is obtained.

1.1. Lax From Hopf

In 1995, A. Subbotin [11] considered the obstacle problem

$$\begin{cases} \min\{u_t + H(Du),\ g(x) - u\} = 0, \\ (t, x) \in [0, T) \times \mathbb{R}^n,\ u(T, x) = g(x),\ x \in \mathbb{R}^n \end{cases} \tag{5}$$

with g a convex function. Subbotin proved, by using his theory of minimax solutions, that the function u is given by a Hopf formula

$$u(t,x) = \sup_{p\in\mathbb{R}^n} \{p\cdot x - g^*(p) + (T-t)H(p)\wedge 0\}\,. \tag{6}$$

We use the standard notation that $a\wedge b = \min\{a,b\}$ and $a\vee b = \max\{a,b\}$. Also, $B_r(x)$ denotes the open ball centered at x of radius $r>0$.

On the other hand, if g is concave, the Hopf solution of (5) is given by

$$u(t,x) = \inf_{p\in\mathbb{R}^n} \{p\cdot x - g_*(p) + (T-t)H(p)\wedge 0\}\,. \tag{7}$$

We will provide a proof of this claim below and refer also to [7] for Hopf formulas for convex single obstacles and double obstacles.

Assume $H\colon \mathbb{R}^n \to \mathbb{R}$ is continuous and $g\colon \mathbb{R}^n\to\mathbb{R}$ is continuous on *dom*$(g) = \{x \mid -\infty < g(x) < +\infty\}$.

For the convenience of the reader we provide the precises definition of viscosity solution.

Definition 1.1. A locally bounded function $u\colon [0,T]\times\mathbb{R}^n\to\mathbb{R}$ is a *viscosity subsolution* of (5) if $(t_0,x_0)\in \arg\max\{u-\varphi\}, u(t_0,x_0) = \varphi(t_0,x_0)$, for some $\varphi\in C^\infty$, implies that

$$\min\{\varphi_t(t_0,x_0) + H(D_x\varphi(t_0,x_0),\ g(x_0) - u(t_0,x_0)\} \geq 0.$$

The function u is a *supersolution* of (5) if $(t_0,x_0)\in\arg\min\{u-\varphi\}$ and $u(t_0,x_0) = \varphi(t_0,x_0)$, for some $\varphi\in C^\infty$, imply that

$$\min\{\varphi_t(t_0,x_0) + H(D_x\varphi(t_0,x_0),\ g(x_0) - u(t_0,x_0)\} \leq 0.$$

Theorem 1.2. *If* $g\colon\mathbb{R}^n\to\mathbb{R}$ *is concave, the unique viscosity solution of* (5) *is given by the Hopf formula*

$$\begin{aligned} u(t,x) &= \inf_{p\in\mathbb{R}^n}\{p\cdot x - g_*(p) + (T-t)H(p)\wedge 0\} \\ &= (g_*(p) - (T-t)H(p)\wedge 0)_*(x), \end{aligned} \tag{8}$$

where $g_*(p) = \inf_{x\in\mathbb{R}^n}(p\cdot x - g(x))$ *is the Fenchel concave conjugate of* g.

Proof. We begin by showing that u is a supersolution. Set

$$v(t,x;p) = p\cdot x - g_*(p) + (T-t)H(p)\wedge 0 \in C^1((0,T)\times\mathbb{R}^n),$$

for each $p\in\mathbb{R}^n$, and $u(t,x) = \inf_{p\in\mathbb{R}^n} v(t,x;p)$. Fix the point (t,x). Without loss of generality we may assume $u(t,x)\leq g(x) - \delta/2$ for some $\delta>0$ since if $u(t,x)\geq g(x)$ it is immediate that u is a supersolution and there is nothing to prove.

Choose $\hat{p} \in \mathbb{R}^n$ such that $u(t,x) \geq v(t,x;\hat{p}) - \delta/4$. Then, assuming $H(\hat{p}) \geq 0$,

$$\begin{aligned} g(x) - \delta/2 \geq u(t,x) \geq v(t,x;\hat{p}) - \delta/4 &= \hat{p} \cdot x - g_*(\hat{p}) + H(\hat{p}) \wedge 0(T-t) - \delta/4 \\ &= \hat{p} \cdot x - g_*(\hat{p}) - \delta/4 \geq g(x) - \delta/4, \end{aligned}$$

a contradiction. Therefore, $u(t,x) = \inf\{v(t,x;p) \mid p \in \mathbb{R}^n, H(p) < 0\}$ whenever $u(t,x) < g(x)$. Therefore, since $v_t = -H(p), D_x v = p$, we have $v_t + H(D_x v) = 0$ on $\{(t,x) \mid u(t,x) < g(x)\}$. This implies that v is a supersolution, for each p of $v_t + H(D_x v) = 0$ on $\{(t,x) \mid u(t,x) < g(x)\}$. Since the infimum of supersolutions is a supersolution, we conclude that $u(t,x) = \inf_p v(t,x,p)$ is a supersolution of problem (5).

Next, to show that u is a subsolution of (5), we use the fact that u is concave. Let $(t_0,x_0) \in (0,T) \times \mathbb{R}^n$ and $u(t_0,x_0) > -\infty$. Since u is concave, for any $(p_t,p_x) \in D^+u(t_0,x_0) \subset -\partial(-u(t_0,x_0))$, where D^+u denotes the subdifferential of u (see [2], for instance), and $\partial(-u)$ denotes the convex subdifferential of the convex function $-u$,

$$u(t,x) \leq u(t_0,x_0) + p_t(t-t_0) + p_x(x-x_0), \;\; \forall (t,x) \in (0,T] \times \mathbb{R}^n. \tag{9}$$

Set $t = T$ to get

$$\begin{aligned} &g(x) + H(p_x) \wedge 0\,(T-t_0) \leq u(t_0,x_0) + (T-t_0)(p_t + H(p_x) \wedge 0) + p_x \cdot x - p_x \cdot x_0, \\ &g(x) - p_x \cdot x + p_x \cdot x_0 + (T-t_0)H(p_x) \wedge 0 - u(t_0,x_0) \leq (T-t_0)(p_t + H(p_x) \wedge 0). \end{aligned}$$

Now take the supremum over $x \in \mathbb{R}^n$ to get

$$0 \leq p_x \cdot x_0 - g_*(p_x) + (T-t_0)H(p_x) \wedge 0 - u(t_0,x_0) \leq (T-t_0)(p_t + H(p_x) \wedge 0),$$

since $u(t_0,x_0) = \inf_{p \in \mathbb{R}^n}\{p \cdot x_0 - g_*(p) + (T-t_0)H(p) \wedge 0\}$. We conclude that $p_t + H(p_x) \wedge 0 \geq 0$ and then $p_t + H(p_x) \geq 0$. Next,

$$u(t_0,x_0) = \inf_{p \in \mathbb{R}^n}\{p \cdot x_0 - g_*(p) + (T-t_0)H(p) \wedge 0\} \leq \inf_{p \in \mathbb{R}^n}\{p \cdot x_0 - g_*(p)\} = g(x_0),$$

so that $g(x_0) - u(t_0,x_0) \geq 0$. We have proved

$$\min\{p_t + H(p_x),\, g(x_0) - u(t_0,x_0)\} \geq 0, \;\; (p_t,p_x) \in D^+u(t_0,x_0),$$

which means that u is a subsolution of (5). □

Remark 1.3. To summarize the result, we see that if g is convex, the Hopf solution of (5) is

$$\begin{aligned} u(t,x) &= \sup_{p \in \mathbb{R}^n}\{p \cdot x - g^*(p) + (T-t)H(p) \wedge 0\} \\ &= (g^*(p) - (T-t)H(p) \wedge 0)^*(x), \end{aligned} \tag{10}$$

while if g is concave, the unique solution of (5) is

$$\begin{aligned} u(t,x) &= \inf_{p \in \mathbb{R}^n}\{p \cdot x - g_*(p) + (T-t)H(p) \wedge 0\} \\ &= (g_*(p) - (T-t)H(p) \wedge 0)_*(x). \end{aligned} \tag{11}$$

We will now derive the Lax formula for the obstacle problem (5) by using a minimax formal interchange. A similar interchange gives the duality between the classical Hopf and Lax formulas. We begin with the Hopf formula for concave g.

$$\begin{aligned}
u(t,x) &= \inf_{p\in\mathbb{R}^n}\{p\cdot x - g_*(p) + (T-t)H(p)\wedge 0\}\\
&= \inf_{p\in\mathbb{R}^n}\inf_{t\le\tau\le T}\sup_{y\in\mathbb{R}^n}\{p\cdot x-(p\cdot y-g(y))+(\tau-t)H(p)\}\\
&= \inf_{p\in\mathbb{R}^n}\inf_{t\le\tau\le T}\sup_{y\in\mathbb{R}^n}\{p\cdot(x-y)+g(y)+(\tau-t)H(p)\}\\
&= \sup_{y\in\mathbb{R}^n}\inf_{t\le\tau\le T}\inf_{p\in\mathbb{R}^n}\{p\cdot(x-y)+(\tau-t)H(p)+g(y)\}\\
&= \sup_{y\in\mathbb{R}^n}\inf_{t\le\tau\le T}\Big\{g(y)-\sup_{p\in\mathbb{R}^n}\{p\cdot(y-x)-(\tau-t)H(p)\}\Big\}\\
&= \sup_{y\in\mathbb{R}^n}\inf_{t\le\tau\le T}\Big\{g(y)-(\tau-t)H^*\Big(\frac{y-x}{\tau-t}\Big)\Big\}.
\end{aligned}$$

The interchange of the $\inf_p$ and $\sup_y$ in the fourth line is unjustified. The interchange in the second line is justified because of separated variables. Interestingly, beginning with the Hopf formula for concave g, we may drop the concavity of g and arrive at a Lax formula for convex H. A similar formula will result if we begin with the Hopf formula for convex g and interchange the infimum and supremum. Specifically, we have

$$\begin{aligned}
u(t,x) &= \sup_{p\in\mathbb{R}^n}\{p\cdot x - g_*(p) + (T-t)H(p)\wedge 0\}\\
&= \inf_{y\in\mathbb{R}^n}\inf_{t\le\tau\le T}\Big\{g(y)-(\tau-t)H_*\Big(\frac{y-x}{\tau-t}\Big)\Big\},
\end{aligned}$$

after going through a similar unjustified interchange. Now starting with the assumption of a convex g, we obtain a formula involving a concave H.

In this short paper we will prove that

$$\begin{aligned}
u(t,x) &= \sup_{y\in\mathbb{R}^n}\inf_{t\le\tau\le T}\Big\{g(y)-(\tau-t)H^*\Big(\frac{y-x}{\tau-t}\Big)\Big\}\\
&= \sup_{y\in\mathbb{R}^n}\inf_{t\le\tau\le T}\{g(x+y(\tau-t))-(\tau-t)H^*(y)\},
\end{aligned}$$

is indeed the viscosity solution of (5). Also, the supremum may be replaced by maximum and the infimum may be replaced by minimum.

2. A Lax formula for upper obstacle problems

We consider the first order Hamilton-Jacobi obstacle problem

$$\min\{u_t+H(Du),\ g(x)-u\}=0,\ (t,x)\in[0,T)\times\mathbb{R}^n,\ u(T,x)=g(x),\ x\in\mathbb{R}^n. \quad (12)$$

Given a convex Hamiltonian $H\colon \mathbb{R}^n \to \mathbb{R}$, we define the *Fenchel conjugate* by

$$H^*(z) = \sup_{p\in\mathbb{R}^n} \{p \cdot z - H(p)\}.$$

The goal is to explicitly solve the problem (12) by deriving a Lax formula for u. This will be done by constructing a control problem for which u is the value function with data g and running cost H^*.

Consider the control problem

$$u(t,x) = \sup_{\zeta\in\mathcal{Z}[t,T]} \inf_{t\le\tau\le T} \left\{ g(\xi(\tau)) - \int_t^\tau H^*(\zeta(s))\, ds \right\}, \tag{13}$$

where $\mathcal{Z}[t,T]$ is the class of all measurable functions $\zeta\colon [t,T] \to \mathbb{R}^n$ which are almost everywhere bounded. For a given control $\zeta \in \mathcal{Z}[t,T]$, the trajectory $\xi\colon [t,T] \to \mathbb{R}^n$ is given by $\dot{\xi}(s) = \zeta(s)$, $t < s \le T$, $\xi(t) = x \in \mathbb{R}^n$.

For $R > 0$ we set $\mathcal{Z}_R[t,T]$ as the class of control functions $\zeta \in \mathcal{Z}[t,T]$ such that $|\zeta(s)| \le R$, for almost every $t \le s \le T$.

We will assume throughout this paper that

$$H\colon \mathbb{R}^n \to \mathbb{R} \text{ is convex, } \lim_{|p|\to\infty} \frac{H(p)}{|p|} = +\infty, \tag{14}$$

and $g\colon \mathbb{R}^n \to \mathbb{R}$ is uniformly Lipschitz continuous with Lipschitz constant L_g.

Now we show that in view of (14) we may in fact restrict the control functions to $\mathcal{Z}_R[t,T]$ for all large enough R.

Lemma 2.1. *There is $R_0 > 0$ such that for all $R > R_0$,*

$$u(t,x) = \sup_{\zeta\in\mathcal{Z}_R[t,T]} \inf_{t\le\tau\le T} \left\{ g(\xi(\tau)) - \int_t^\tau H^*(\zeta(s))\, ds \right\}. \tag{15}$$

Proof. First, choose $\zeta(\tau) \equiv 0$ in (13) to get $\xi(\tau) \equiv x$ and

$$u(t,x) \ge \inf_{t\le\tau\le T} \{g(x) - (\tau - t)H^*(0)\} = g(x) - (T-t)H^*(0) \vee 0.$$

Consequently $g(x) \le u(t,x) + (T-t)H^*(0) \vee 0$.

Next, since $H(p)/|p| \to \infty$ as $|p| \to \infty$, this is also true for $H^*(p)$ and therefore, there is $M > 0$ such that

$$|z| > M \implies H^*(z) \ge 2(L_g + 1)|z|.$$

Now we estimate for each ζ such that $|\zeta(\tau)| > M,\ t \le \tau \le T$,

$$\begin{aligned}
&\inf_{t\le\tau\le T}\left\{g(\xi(\tau)) - \int_t^\tau H^*(\zeta(s))\,ds\right\}\\
&\qquad\le \inf_{t\le\tau\le T}\left\{g(x) + L_g|\xi(\tau)-x| - \int_t^\tau 2(L_g{+}1)|\zeta(s)|\,ds\right\}\\
&\qquad\le g(x) + \inf_{t\le\tau\le T}\left\{L_g\int_t^\tau |\zeta(s)|\,ds - \int_t^\tau 2(L_g+1)|\zeta(s)|\,ds\right\}\\
&\qquad= g(x) - \int_t^T (L_g+2)|\zeta(s)|\,ds\\
&\qquad\le u(t,x) + \int_t^T H^*(0)\vee 0 - (L_g+2)|\zeta(s)|\,ds\\
&\qquad\le u(t,x)\ \text{ for }\ |\zeta(s)| > \frac{H^*(0)\vee 0}{L_g+2},\quad t\le s\le T.
\end{aligned}$$

Therefore, for $R_0 = \max\{\frac{H^*(0)\vee 0}{L_g+2},\ M\}$ we have (15) for all $R > R_0$. □

Now for fixed $R > 0$ define the value function

$$u^R(t,x) = \sup_{\zeta\in\mathcal{Z}_R[t,T]}\ \inf_{t\le\tau\le T}\left\{g(\xi(\tau)) - \int_t^\tau H^*(\zeta(s))\,ds\right\}. \tag{16}$$

Theorem 2.2. *The function u^R is the unique continuous viscosity solution of*

$$\begin{cases} \min\{u^R_t + \max\limits_{|z|\le R}(D_x u^R\cdot z - H^*(z)),\ g(x) - u^R\} = 0,\\ (t,x)\in[0,T)\times\mathbb{R}^n \ \textit{and}\ u^R(T,x) = g(x). \end{cases} \tag{17}$$

Proof. The proof is similar to the proof in [6, Theorems 3.2 & 4.2]. To make this paper self-contained and for the convenience of the reader we will provide the proof of this theorem. The proof is based on the following dynamic programming principle.

Lemma 2.3. *The function u^R satisfies*

$$\begin{aligned}
u^R(t,x) = \sup_{\zeta\in\mathcal{Z}_R[t,t+\delta]}\ \inf_{t\le\tau\le t+\delta}&\left(g(\xi(\tau)) - \int_t^\tau H^*(\zeta(s))\,ds\right)\\
&\wedge\left(u(t+\delta,\xi(t+\delta)) - \int_t^{t+\delta} H^*(\zeta(s))\,ds\right).
\end{aligned}\tag{18}$$

Proof. Let $\zeta\in\mathcal{Z}_R[t,T]$, $\zeta_1 = \zeta\Big|_{[t,t+\delta]}\in\mathcal{Z}_R[t,t+\delta]$, $\zeta_2 = \zeta\Big|_{[t+\delta,T]}\in\mathcal{Z}_R[t+\delta,T]$.

Let $\xi(\cdot)$ be the trajectory associated with ζ. We have

$$\begin{aligned}
&\inf_{t\le\tau\le T} g(\xi(\tau)) - \int_t^\tau H^*(\zeta(s))\,ds\\
&= \min\left(\inf_{t\le\tau\le t+\delta} g(\xi(\tau)) - \int_t^\tau H^*(\zeta(s))\,ds, \inf_{t+\delta\le\tau\le T} g(\xi(\tau)) - \int_t^\tau H^*(\zeta(s))\,ds\right)\\
&= \min\left(\inf_{t\le\tau\le t+\delta} g(\xi(\tau)) - \int_t^\tau H^*(\zeta_1(s))\,ds,\right.\\
&\qquad\qquad \left.\inf_{t+\delta\le\tau\le T} g(\xi(\tau)) - \int_{t+\delta}^\tau H^*(\zeta_2(s))\,ds - \int_t^{t+\delta} H^*(\zeta_1(s))\,ds\right)\\
&\le \min\left(\inf_{t\le\tau\le t+\delta} g(\xi(\tau)) - \int_t^\tau H^*(\zeta_1(s))\,ds,\right.\\
&\qquad\qquad \left.u^R(t+\delta, \xi(t+\delta) - \int_t^{t+\delta} H^*(\zeta_1(s))\,ds\right)\\
&\le \sup_{\zeta\in\mathcal{Z}_R[t,t+\delta]} \min\left(\inf_{t\le\tau\le t+\delta} g(\xi(\tau)) - \int_t^\tau H^*(\zeta(s))\,ds,\right.\\
&\qquad\qquad \left.u^R(t+\delta, \xi(t+\delta) - \int_t^{t+\delta} H^*(\zeta(s))\,ds\right).
\end{aligned}$$

Consequently,
$$\begin{aligned}
u^R(t,x) \le \sup_{\zeta\in\mathcal{Z}_R[t,t+\delta]} \min&\left(\inf_{t\le\tau\le t+\delta} g(\xi(\tau)) - \int_t^\tau H^*(\zeta(s))\,ds,\right.\\
&\left.u^R(t+\delta, \xi(t+\delta) - \int_t^{t+\delta} H^*(\zeta(s))\,ds\right).
\end{aligned}$$

For the reverse inequality take arbitrary controls $\zeta_1 \in \mathcal{Z}_R[t,t+\delta], \zeta_2 \in \mathcal{Z}_R[t+\delta,T]$ and define

$$\zeta = \begin{cases} \zeta_1, & t\le\tau<t+\delta \\ \zeta_2, & t+\delta\le\tau\le T \end{cases}, \qquad \zeta\in\mathcal{Z}_R[t,T].$$

Let $\dot\xi = \zeta, \xi(t) = x$. Then,

$$\begin{aligned}
&\min\left(\inf_{t\le\tau\le t+\delta} g(\xi(\tau)) - \int_t^\tau H^*(\zeta_1(s))\,ds,\right.\\
&\qquad\qquad \left.\inf_{t+\delta\le\tau\le T} g(\xi(\tau)) - \int_{t+\delta}^\tau H^*(\zeta_2(s))\,ds - \int_t^{t+\delta} H^*(\zeta_1(s))\,ds\right)\\
&= \inf_{t\le\tau\le T} g(\xi(\tau)) - \int_t^\tau H^*(\zeta(s))\,ds \le u^R(t,x).
\end{aligned}$$

Since $\zeta_2 \in \mathcal{Z}_R[t+\delta,T]$ is arbitrary, this implies

$$\begin{aligned}
&\min\left(\inf_{t\le\tau\le t+\delta} g(\xi(\tau)) - \int_t^\tau H^*(\zeta_1(s))\,ds,\right.\\
&\qquad\qquad \left.u^R(t+\delta, \xi(t+\delta)) - \int_t^{t+\delta} H^*(\zeta_1(s))\,ds\right) \le u^R(t,x).
\end{aligned}$$

Finally, taking the supremum over $\zeta_1 \in \mathscr{Z}_R[t, t+\delta]$ we obtain (18). $\square$

Once we have (18) the argument to prove that u^R is a viscosity solution of (17) is as follows.

First, the fact that $(t, x) \to u^R(t, x)$ is Lipschitz continuous is proved exactly as in [6] and is omitted.

If u^R is not a supersolution, then let $u^R - \varphi$ achieve a strict zero minimum at (t_0, x_0) with $\varphi \in C^\infty$ and

$$\min\{\varphi_t + \max_{|z|\le R}(D_x\varphi \cdot z - H^*(z)),\ g(x_0) - u^R\} \ge \gamma > 0$$

at (t_0, x_0) for some $\gamma > 0$. This implies that for some $\overline{z}, |\overline{z}| \le R$, we have

$$\varphi_t(t_0, x_0) + D_x\varphi(t_0, x_0) \cdot \overline{z} - H^*(\overline{z}) \ge \gamma/2 > 0 \tag{19}$$

and

$$g(x_0) \ge u^R(t_0, x_0) + \gamma. \tag{20}$$

Using continuity we may assume that

$$\varphi_t(t, x) + D_x\varphi(t, x) \cdot \overline{z} - H^*(\overline{z}) \ge \gamma/4, \tag{21}$$

and, since $H^*(\overline{z}) < +\infty$,

$$g(x) - (t - t_0)H^*(\overline{z}) \ge u^R(t_0, x_0) + \gamma/2, \tag{22}$$

for any $(t, x) \in B_\delta(t_0, x_0)$ for some $\delta > 0$ sufficiently small.

Set $\zeta(\tau) \equiv \overline{z}$ and $\dot{\xi}(\tau) = \zeta(\tau), t < \tau, \xi(t) = x_0$. Choose δ small enough so that $(\tau, \xi(\tau)) \in B_\delta(t_0, x_0), t_0 \le \tau \le t_0 + \delta$. From (21) we have

$$\varphi_t(\tau, \xi(\tau)) + D_x\varphi(\tau, \xi(\tau)) \cdot \zeta(\tau) - H^*(\zeta(\tau)) \ge \gamma/4, \quad t_0 \le \tau \le t_0 + \delta. \tag{23}$$

Integrate from t_0 to $t_0 + \delta$ to get

$$\varphi(t_0 + \delta, \xi(t_0 + \delta)) - \varphi(t_0, x_0) - \int_{t_0}^{t_0+\delta} H^*(\zeta(s))\, ds \ge \delta\gamma/4.$$

Since $(t_0, x_0) \in \arg\min(u^R - \varphi)$ and $u^R(t_0, x_0) = \varphi(t_0, x_0)$ we have

$$u^R(t_0 + \delta, \xi(t_0 + \delta)) - u^R(t_0, x_0) - \int_{t_0}^{t_0+\delta} H^*(\zeta(s))\, ds \ge \delta\gamma/4.$$

Consequently,

$$u^R(t_0 + \delta, \xi(t_0 + \delta)) - \int_{t_0}^{t_0+\delta} H^*(\zeta(s))\, ds \ge u^R(t_0, x_0) + \delta\gamma/4. \tag{24}$$

Using (22) we have

$$g(\xi(\tau)) - \int_{t_0}^{\tau} H^*(\zeta(s))\, ds \ge u^R(t_0, x_0) + \gamma/4, \quad t_0 \le \tau \le t_0 + \delta.$$

This implies

$$\inf_{t_0\le\tau\le t_0+\delta}\left\{g(\xi(\tau))-\int_{t_0}^{\tau}H^*(\zeta(s))\,ds\right\}\ge u^R(t_0,x_0)+\delta\gamma/4. \tag{25}$$

Putting together (24) and (25) and taking the supremum over $\zeta\in\mathcal{Z}_R[t_0,t_0+\delta]$ we obtain a contradiction of (18) and conclude that u^R is a supersolution.

We have left to prove that u^R is a subsolution of (17).

Suppose u^R is not a subsolution. There is (t_0,x_0) and $\varphi\in C^\infty$ such that $u^R-\varphi$ achieves a strict zero maximum at (t_0,x_0) and

$$\min\left\{\varphi_t+\max_{|z|\le R}(D_x\varphi\cdot z-H^*(z)),\ g(x_0)-u^R\right\}\le-\gamma$$

at (t_0,x_0) for some $\gamma>0$. Since it is always true that $u^R(t_0,x_0)\le g(x_0)$ we have

$$\varphi_t(t_0,x_0)+\max_{|z|\le R}(D_x\varphi(t_0,x_0)\cdot z-H^*(z))\le-\gamma.$$

By continuity, there is $\sigma>0$ such that

$$\varphi_t(s,y)+\max_{|z|\le R}(D_x\varphi(s,y)\cdot z-H^*(z))\le-\gamma/2,\ (s,y)\in B_\sigma(t_0,x_0). \tag{26}$$

Choose arbitrary $\zeta\in\mathcal{Z}_R[t_0,T]$. and trajectory $\dot\xi=\zeta,\xi(t_0)=x_0$. There is a $\delta>0$ so that $(\tau,\xi(\tau))\in B_\sigma(t_0,x_0),t_0\le\tau\le t_0+\delta$. Therefore, by (26),

$$\varphi_t(\tau,\xi(\tau))+D_x\varphi(\tau,\xi(\tau))\cdot\zeta(\tau)-H^*(\zeta(\tau))\le-\gamma/2,\ \ t_0\le\tau\le t_0+\delta\,. \tag{27}$$

Integrate to get

$$\begin{aligned}&u^R(t_0+\delta,\xi(t_0+\delta))-u^R(t_0,x_0)-\int_{t_0}^{t_0+\delta}H^*(\zeta(s))\,ds\\ &\qquad\le\varphi(t_0+\delta,\xi(t_0+\delta))-\varphi(t_0,x_0)-\int_{t_0}^{t_0+\delta}H^*(\zeta(s))\,ds\le-\delta\,\gamma/2\,,\end{aligned}$$

and so

$$u^R(t_0+\delta,\xi(t_0+\delta))-\int_{t_0}^{t_0+\delta}H^*(\zeta(s))\,ds\le u^R(t_0,x_0)-\delta\,\gamma/2\,. \tag{28}$$

Next, there are two possibilities

$$\inf_{t_0\le\tau\le t_0+\delta}\left\{g(\xi(\tau))-\int_{t_0}^{\tau}H^*(\zeta(s))\,ds\right\}>u^R(t_0,x_0)-\delta\,\gamma/2\,, \tag{29}$$

or

$$\inf_{t_0\le\tau\le t_0+\delta}\left\{g(\xi(\tau))-\int_{t_0}^{\tau}H^*(\zeta(s))\,ds\right\}\le u^R(t_0,x_0)-\delta\,\gamma/2\,. \tag{30}$$

In case (30) holds, combining (30) with (28), we get

$$\left(\inf_{t_0\le\tau\le t_0+\delta}\left\{g(\xi(\tau))-\int_{t_0}^{\tau}H^*(\zeta(s))\,ds\right\}\right)$$
$$\wedge\left(u^R(t_0+\delta,\xi(t_0+\delta))-\int_{t_0}^{t_0+\delta}H^*(\zeta(s))\,ds\right)\le u^R(t_0,x_0)-\delta\,\gamma/2.$$

Since ζ is arbitrary, taking the $\sup_{\zeta\in\mathcal{Z}_R[t_0,t_0+\delta]}$ we immediately obtain a contradiction to (18).

In case (29) holds, using (28),

$$\inf_{t_0\le\tau\le t_0+\delta}\left\{g(\xi(\tau))-\int_{t_0}^{\tau}H^*(\zeta(s))\,ds\right\}>u^R(t_0,x_0)-\delta\,\gamma/2$$
$$\ge u^R(t_0+\delta,\xi(t_0+\delta))-\int_{t_0}^{t_0+\delta}H^*(\zeta(s))\,ds$$

so that

$$\left(\inf_{t_0\le\tau\le t_0+\delta}g(\xi(\tau))-\int_{t_0}^{\tau}H^*(\zeta(s))\,ds\right)\wedge\left(u^R(t_0+\delta,\xi(t_0+\delta))-\int_{t_0}^{t_0+\delta}H^*(\zeta(s))\,ds\right)$$
$$=u^R(t_0+\delta,\xi(t_0+\delta))-\int_{t_0}^{t_0+\delta}H^*(\zeta(s))\,ds\le u^R(t_0,x_0)-\delta\,\gamma/2\,.$$

Taking the supremum over $\zeta\in\mathcal{Z}_R[t_0,t_0+\delta]$ we conclude

$$u^R(t_0,x_0)=\sup_{\zeta\in\mathcal{Z}_R[t_0,t_0+\delta]}\left(\inf_{t_0\le\tau\le t_0+\delta}g(\xi(\tau))-\int_{t_0}^{\tau}H^*(\zeta(s))\,ds\right)$$
$$\wedge\left(u^R(t_0+\delta,\xi(t_0+\delta))-\int_{t_0}^{t_0+\delta}H^*(\zeta(s))\,ds\right)<u^R(t_0,x_0),$$

a contradiction. We conclude that u^R is a viscosity subsolution of (17) as well.

Finally, the uniqueness of the viscosity solution of (17) may be proved directly by slightly modifying the proof of [6, Theorem 4.2], or, since the Hamiltonian of this problem is continuous, applying standard uniqueness theorems for Hamilton-Jacobi equations as in [2, Theorems 3.12, 4.11] or [10]. □

Lemma 2.4. *We have* $u(t,x)=\lim_{R\to\infty}u^R(t,x)$ *and* u *is the unique continuous viscosity solution of* (12).

Proof. It is clear that since $\mathcal{Z}_R[t,T]\subset\mathcal{Z}[t,T]$ for all $R>0$, we have $u^R(t,x)\le u(t,x)$. Also, $R_1<R_2$ implies $u^{R_1}\le u^{R_2}$. According to Lemma 2.1, there is R_0 so that $R>R_0$ implies $u^R=u$ and we conclude that $u^R\nearrow u$.

According to Theorem 2.2 we know that u^R is the solution of

$$\min\{u^R_t + H_R(Du^R), g - u\} = 0,\ (t,x) \in [0,T) \times \mathbb{R}^n,\ u^R(T,x) = g(x),\ x \in \mathbb{R}^n,$$

where the Hamiltonian for the problem with control set $\mathcal{Z}_R[t,T]$ is

$$H_R(p) = \max_{|z| \le R} (p \cdot z - H^*(p)).$$

We have that $H_R(p) \nearrow H(p)$ as $R \to \infty$. Consequently, by basic stability results in viscosity solutions [2, Prop 2.2], since $u^R \nearrow u$ as $R \to \infty$, u is the unique continuous viscosity solution of (12). □

Now we may prove the Lax Formula for (12).

Theorem 2.5. *Assume $H \colon \mathbb{R}^n \to \mathbb{R}$ satisfies* (14) *and $g \colon \mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous with Lipschitz constant L_g. The unique uniformly continuous viscosity solution of* (12) *is given by*

$$\begin{aligned} u(t,x) &= \sup_{z \in \mathbb{R}^n} \inf_{t \le \tau \le T} \left(g(z) - (\tau - t) H^* \left(\frac{z - x}{\tau - t} \right) \right) \\ &= \sup_{z \in \mathbb{R}^n} \inf_{t \le \tau \le T} \big\{ g(x + z(\tau - t)) - (\tau - t) H^*(z) \big\}. \end{aligned} \tag{31}$$

Proof. Choose $z \in \mathbb{R}^n$ and $\zeta(s) = z$ so that $\xi(s) = x + z(s - t)$, $t \le s \le T$. Then

$$\begin{aligned} u(t,x) &= \sup_{\zeta \in \mathcal{Z}[t,T]} \inf_{t \le \tau \le T} \left(g(\xi(\tau)) - \int_t^\tau H^*(\zeta(s))\, ds \right) \\ &\ge \inf_{t \le \tau \le T} \big\{ g(x + z(\tau - t)) - H^*(z)(\tau - t) \big\}. \end{aligned}$$

Since this is true for any $z \in \mathbb{R}^n$,

$$u(t,x) \ge \sup_{z \in \mathbb{R}^n} \inf_{t \le \tau \le T} \big\{ g(x + z(\tau - t)) - H^*(z)(\tau - t) \big\}.$$

On the other hand, by Jensen's inequality

$$\begin{aligned} u(t,x) &= \sup_{\zeta \in \mathcal{Z}[t,T]} \inf_{t \le \tau \le T} \left\{ g(\xi(\tau)) - (\tau - t) \frac{1}{\tau - t} \int_t^\tau H^*(\zeta(s))\, ds \right\} \\ &\le \sup_{\zeta \in \mathcal{Z}[t,T]} \inf_{t \le \tau \le T} \left\{ g(\xi(\tau)) - (\tau - t) H^* \left(\frac{1}{\tau - t} \int_t^\tau \zeta(s)\, ds \right) \right\} \\ &= \sup_{\zeta \in \mathcal{Z}[t,T]} \inf_{t \le \tau \le T} \left\{ g(\xi(\tau)) - (\tau - t) H^* \left(\frac{\xi(\tau) - x}{\tau - t} \right) \right\} \\ &= \sup_{z \in \mathbb{R}^n} \inf_{t \le \tau \le T} \big\{ g(x + z(\tau - t)) - (\tau - t) H^*(z) \big\}. \end{aligned}$$

The proof is complete. □

Remark 2.6. Observe that if $H(p) \geq 0$, then $u(t,x) \equiv g(x)$ is the unique solution of (5), so this result is of interest only if H also takes on negative values.

Corollary 2.7. *Under the assumptions of Theorem* 2.5 *we have*

$$\begin{aligned} u(t,x) &= \max_{z\in\mathbb{R}^n} \min_{t\leq\tau\leq T} \left\{ g(z) - (\tau - t) H^*\left(\frac{z-x}{\tau - t}\right)\right\} \\ &= \max_{z\in\mathbb{R}^n} \min_{t\leq\tau\leq T} \big\{ g(x + z(\tau - t)) - (\tau - t) H^*(z)\big\}. \end{aligned} \tag{32}$$

Remark 2.8. If the obstacle problem involves a lower obstacle as in

$$\begin{cases} \max\{u_t + H(Du),\ g(x) - u\} = 0, \\ (t,x) \in [0,T) \times \mathbb{R}^n,\ (T,x) = g(x),\ x \in \mathbb{R}^n, \end{cases} \tag{33}$$

then, if H is assumed concave, $H_*(p) = \inf_{z\in\mathbb{R}^n} \{p\cdot z - H(z)\}$, we have

$$u(t,x) = \inf_{\zeta\in\mathcal{Z}[t,T]} \sup_{t\leq\tau\leq T} \left\{ g(\xi(\tau)) - \int_t^\tau H_*(\zeta(s))\, ds\right\},$$

where $\dot{\xi}(s) = \zeta(s), t < s \leq T, \xi(t) = x \in \mathbb{R}^n$. Arguing as before we get the Lax formula

$$\begin{aligned} u(t,x) &= \inf_{z\in\mathbb{R}^n} \sup_{t\leq\tau\leq T} \{g(x + z(\tau - t)) - (\tau - t) H^*(z)\} \\ &= \inf_{z\in\mathbb{R}^n} \sup_{t\leq\tau\leq T} \left\{ g(z) - (\tau - t) H_*\left(\frac{z-x}{\tau - t}\right)\right\}. \end{aligned}$$

From this formula we may again formally derive the associated Hopf formula as follows:

$$\begin{aligned} u(t,x) &= \inf_{z\in\mathbb{R}^n} \sup_{t\leq\tau\leq T} \left\{ g(z) - (\tau - t) H_*\left(\frac{z-x}{\tau - t}\right)\right\} \\ &= \inf_{z\in\mathbb{R}^n} \sup_{t\leq\tau\leq T} g(z) - \inf_{q\in\mathbb{R}^n} \{q\cdot(z-x) - (\tau - t)H(q)\} \\ &= \inf_{z\in\mathbb{R}^n} \sup_{q\in\mathbb{R}^n} g(z) - q\cdot z + q\cdot x + (T-t)H(q)) \vee 0 \\ &= \sup_{q\in\mathbb{R}^n} \inf_{z\in\mathbb{R}^n} q\cdot x + g(z) - q\cdot z + (T-t)H(q)) \vee 0 \\ &= \sup_{q\in\mathbb{R}^n} q\cdot x - \sup_{z\in\mathbb{R}^n} (q\cdot z - g(z)) + (T-t)H(q)) \vee 0 \\ &= \sup_{q\in\mathbb{R}^n} q\cdot x - g^*(q) + (T-t)H(q)) \vee 0. \end{aligned}$$

In the fourth line the inf and sup have been unjustifiably interchanged. The formula

$$u(t,x) = \sup_{q\in\mathbb{R}^n} \{q\cdot x - g^*(q) + (T-t)H(q)) \vee 0\}$$

is the Hopf formula for (33) derived in [7] using differential games, assuming that g is a convex lower obstacle. This is very similar to Subbotin's formula (6) for the upper obstacle problem (5) but $H \wedge 0$ is replaced by $H \vee 0$.

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