Minimax Theory and its Applications
Volume 4 (2019), No. 2, 355–386

Multifacility Minimax Location Problems via Multi-Composed Optimization*

Gert Wanka
Faculty of Mathematics, Chemnitz University of Technology, 09107 Chemnitz, Germany
gert.wanka@mathematik.tu-chemnitz.de

Oleg Wilfer
Faculty of Mathematics, Chemnitz University of Technology, 09107 Chemnitz, Germany
oleg.wilfer@mathematik.tu-chemnitz.de

Received: September 25, 2018
Accepted: March 7, 2019

We present a conjugate duality approach for multifacility minimax location problems with geometric constraints, where the underlying space is Banach and the distances are measured by gauges of closed convex sets. Besides assigning corresponding conjugate dual problems, we derive necessary and sufficient optimality conditions. Moreover, we introduce a further dual problem with less dual variables than the first formulated dual and deliver corresponding statements of strong duality and optimality conditions. To illustrate the results of the latter duality approach and to give a more detailed characterization of the relation between the location problem and its dual, we consider the situation in the Euclidean space.

Keywords: Conjugate duality, composed functions, minimax location problems, gauges, optimality conditions.

2010 Mathematics Subject Classification: 49K35, 49N15, 90B85, 90C46, 90C47.

1. Introduction

Facility location problems are known for their numerous applications in areas like computer science, telecommunication, transportation and emergency facilities programming. In the framework of continuous optimization where the distances are measured by gauges, two kinds of location problems are particularly significant. The first one consists of the so-called minisum location problems and has the objective to determine a new point such that the sum of distances between the new and given points is minimal (see [3, 4, 5, 14, 18, 21]). The second class contains the so-called minimax location problems, where a new point is sought such that the maximum of distances between the new and given points will be minimized (see [7, 8, 11, 16, 17, 20, 22, 24]). The latter type of location problems was extensively studied in [27] in the context of conjugate duality.

The central concern of this article is the consideration of a more general and complex problem, namely the so-called multifacility minimax location problem

*This work was supported by DFG (German Research Foundation) under Project WA 922/9-1.

ISSN 2199-1413 (printed), ISSN 2199-1421 (electronic) / $ 2.50

(see [9, 23]), which has attracted less attention in the literature compared to the multifacility minisum location problems (see [10, 15, 19, 25]). This may be due to the fact that the objective function of the minimax location problem is a composition of a nonlinear outer function defined by a pointwise maximum and an inner vector function given by distances, which are composed with linear operators. In particular, the outer function increases the level of complexity not only from the theoretical but also from the numerical point of view. In other words, the objective of the multifacility minimax location problems is to determine several new points such that either the maximum of distances between pairs of new points or the maximum of distances between new and existing points is minimal. In our analysis we will use the results recently presented in [26] for multi-composed optimization problems to deliver a detailed duality approach to this type of location problems. This duality concept allows more suitable and effective considerations than the classical methods due to the mentioned complexity of the objective function of the multifacility minimax location problem. In concrete terms, this means that we formulate an associated conjugate dual problem as well as derive necessary and sufficient optimality conditions. Especially, we show that in the settings where the underlying space is Banach and the distances are measured by gauges of closed convex sets strong duality can always be guaranteed.

Further, we introduce another dual problem reducing the number of dual variables compared to the first formulated dual problem. Continuing in this vein, we will also employ a duality approach including statements of strong duality and optimality conditions.

As the most location problems are considered in Euclidean spaces, we particularize the latter case in this context and show that we have a full symmetry between the location problem, its dual problem and the Lagrange dual problem of the dual problem, which means that the Lagrange dual is identical to the location problem. Finally, we close this paper with an example showing on the one hand how an optimal solution of the location problem can be recovered from an optimal solution of the associated conjugate dual problem and on the other hand how we can geometrically interpret an optimal dual solution.

To this end, we start with recalling some preliminary notions and results from the convex analysis needed for our approach.

2. Preliminaries

2.1. Elements of convex analysis

Let X be a Hausdorff locally convex space and X^* its topological dual space endowed with the weak* topology $w(X^*, X)$. For $x \in X$ and $x^* \in X^*$, let $\langle x^*, x\rangle := x^*(x)$ be the value of the linear continuous functional x^* at x. For a subset $A \subseteq X$, its *indicator function* $\delta_A : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is

$$\delta_A(x) := \begin{cases} 0, & \text{if } x \in A, \\ +\infty, & \text{otherwise} \end{cases}$$

and its *support function* $\sigma_A \colon X^* \to \overline{\mathbb{R}}$ is $\sigma_A(x^*) = \sup_{x \in A}\langle x^*, x\rangle$. For a given function $f \colon X \to \overline{\mathbb{R}}$ we consider its *effective domain*

$$\operatorname{dom} f := \{x \in X : \ f(x) < +\infty\}$$

and call $f \colon X \to \overline{\mathbb{R}}$ *proper* if $\operatorname{dom} f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in X$. The *conjugate function* of f with respect to the non-empty subset $S \subseteq X$ is defined by

$$f_S^* \colon X^* \to \overline{\mathbb{R}}, \ f_S^*(x^*) = (f + \delta_S)^*(x^*) = \sup_{x \in S}\{\langle x^*, x\rangle - f(x)\}.$$

In the case $S = X$, it is clear that f_S^* turns into the classical Fenchel-Moreau conjugate function of f denoted by f^*. Let us mention that it holds $f^*(x^*) = \sup_{x \in \operatorname{dom} f}\{\langle x^*, x\rangle - f(x)\}$ as well as $f(x) + f(x^*) \geq \langle x^*, x\rangle$ for all $x \in X, \ x^* \in X^*$, which is the so-called Young-Fenchel inequality. For the subset $S \subseteq X$ and $x \in S$, the *normal cone* to S at x is defined by

$$N_S(x) := \{x^* \in X^* : \langle x^*, y - x\rangle \leq 0 \ \forall \ y \in S\},$$

which is a convex cone. Additionally, we consider a non-empty convex cone $K \subseteq X$, which induces on X a partial ordering relation "$\leqq_K$", defined by

$$\leqq_K := \{(x, y) \in X \times X : y - x \in K\},$$

i.e. for $x, y \in X$ it holds $x \leqq_K y \Leftrightarrow y - x \in K$. Note that we assume that all cones we consider contain the origin. Further, we attach to X a greatest element with respect to "$\leqq_K$", denoted by $+\infty_K$, which does not belong to X and denote $X^\bullet = X \cup \{+\infty_K\}$. Then it holds $x \leqq_K +\infty_K$ for all $x \in X^\bullet$. We also define $x \leq_K y$ if and only if $x \leqq_K y$ and $x \neq y$. Further, we define $\leqq_{\mathbb{R}_+} =: \leq$ and $\leq_{\mathbb{R}_+} =: <$.

On $X^\bullet$ we consider the following operations and conventions: $x + (+\infty_K) = (+\infty_K) + x := +\infty_K \ \forall x \in X \cup \{+\infty_K\}$ and $\lambda \cdot (+\infty_K) := +\infty_K \ \forall \lambda \in [0, +\infty]$. Further, if $K^* := \{x^* \in X^* : \ \langle x^*, x\rangle \geq 0, \ \forall x \in K\}$ is the dual cone of K, then we define $\langle x^*, +\infty_K\rangle := +\infty$ for all $x^* \in K^*$. On the extended real space $\overline{\mathbb{R}}$ we add the following operations and conventions: $\lambda + (+\infty) = (+\infty) + \lambda := +\infty \ \forall \lambda \in (-\infty, +\infty]$, $\lambda + (-\infty) = (-\infty) + \lambda := -\infty \ \forall \lambda \in [-\infty, +\infty)$, $\lambda \cdot (+\infty) := +\infty \ \forall \lambda \in [0, +\infty]$, $\lambda \cdot (+\infty) := -\infty \ \forall \lambda \in [-\infty, 0)$, $\lambda \cdot (-\infty) := -\infty \ \forall \lambda \in (0, +\infty]$, $\lambda \cdot (-\infty) := +\infty \ \forall \lambda \in [-\infty, 0)$, $(+\infty) + (-\infty) = (-\infty) + (+\infty) := +\infty$ and $0(-\infty) := 0$.

Let Z be another Hausdorff locally convex space ordered by the convex cone $Q \subseteq Z$, then for a vector function $F \colon X \to Z^\bullet = Z \cup \{+\infty_Q\}$ the domain is the set $\operatorname{dom} F := \{x \in X : F(x) \neq +\infty_Q\}$. A function $f \colon X \to \overline{\mathbb{R}}$ is called convex if $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$ for all $x, y \in X$ and all $\lambda \in [0, 1]$. When $F(\lambda x + (1 - \lambda)y) \leqq_Q \lambda F(x) + (1 - \lambda)F(y)$ holds for all $x, y \in X$ and all $\lambda \in [0, 1]$ the function F is said to be *Q-convex*.

Further, we consider the *epigraph* of a function f defined by

$$\operatorname{epi} f := \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}.$$

The Q-*epigraph* of a vector function F is $\operatorname{epi}_Q F = \{(x, z) \in X \times Z : F(x) \leqq_Q z\}$ and when Q is closed we say that F is Q-epi closed if $\operatorname{epi}_Q F$ is a closed set.

If $Q^* := \{x^* \in X^* : \ \langle x^*, x\rangle \geq 0, \ \forall x \in Q\}$ is the dual cone of Q, then we define for $z^* \in Q^*$ the function $(z^*F) \colon X \to \overline{\mathbb{R}}$ by $(z^*F)(x) := \langle z^*, F(x)\rangle$, where it is not hard to see that $\operatorname{dom}(z^*F) = \operatorname{dom} F$. Moreover, it is easy to see that if F is Q-convex, then (z^*F) is convex for all $z^* \in Q^*$.

A function $f \colon X \to \overline{\mathbb{R}}$ is called *lower semicontinuous* at $\overline{x} \in X$ if $\liminf_{x \to \overline{x}} f(x) \geq f(\overline{x})$ and when this function is lower semicontinuous at all $x \in X$, then we call it lower semicontinuous (l.s.c. for short). The vector function F is called positively Q-lower semicontinuous at $x \in X$ if (z^*F) is lower semicontinuous at x for all $z^* \in Q^* \setminus \{0_{X^*}\}$. The function F is called *positively Q-lower semicontinuous* if it is positively Q-lower semicontinuous at every $x \in X$. Note that if F is positively Q-lower semicontinuous, then it is also Q-epi closed, while the inverse statement is not true in general (see: Proposition 2.2.19 in [2]). Let us mention that in the case $Z = \mathbb{R}$ and $Q = \mathbb{R}_+$, the notions of Q-epi closedness and positively Q-lower semicontinuity fall into the classical notion of lower semicontinuity.

A function $f \colon X \to \overline{\mathbb{R}}$ is called K-*increasing*, if from $x \leqq_K y$ follows $f(x) \leq f(y)$ for all $x, y \in X$.

Definition 2.1. The vector function $F \colon X \to Z^\bullet$ is called K-Q-increasing, if from $x \leqq_K y$ follows $F(x) \leqq_Q F(y)$ for all $x, y \in X$.

For a set $S \subseteq X$ the *conic hull* is defined by $\operatorname{cone}(S) := \{\lambda x : x \in S, \ \lambda \geq 0\}$ and sqri is used to denote the *strong quasi relative interior*, where in the case of having a convex set $S \subseteq X$ it holds

$$\operatorname{sqri}(S) \quad = \quad \{x \in S : \operatorname{cone}(S - x) \text{ is a closed linear subspace}\}.$$

In this paper we do not use the classical differentiability, but we use the notion of subdifferentiability to formulate optimality conditions. If we take an arbitrary $x \in X$ such that $f(x) \in \mathbb{R}$, then we call the set

$$\partial f(x) := \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x\rangle \ \forall y \in X\}$$

the (convex) *subdifferential* of f at x, where the elements are called the *subgradients* of f at x. Moreover, if $\partial f(x) \neq \emptyset$, then we say that f is subdifferentiable at x and if $f(x) \notin \mathbb{R}$, then we make the convention that $\partial f(x) := \emptyset$. Note that the subgradients can be characterized by the conjugate function, especially this means

$$x^* \in \partial f(x) \Leftrightarrow f(x) + f^*(x^*) = \langle x^*, x\rangle \ \forall x \in X, \ x^* \in X^*, \tag{1}$$

i.e. the Young-Fenchel inequality is fulfilled with equality.

Let $C \subseteq X$. In conclusion of this section we collect some properties of the *gauge function* of the subset C (known in the literature also as the Minkowski functional), $\gamma_C \colon X \to \overline{\mathbb{R}}$ defined by

$$\gamma_C(x) := \inf\{\lambda > 0 : x \in \lambda C\}.$$

At this point we want to emphasize that the gauge function is equal to $+\infty$ if $x \notin \lambda C$ for all $\lambda > 0$, as by definition one has that $\inf \emptyset = +\infty$. The following statements were proved in [27].

Theorem 2.2. *Let $C \subseteq X$ be a convex and closed set with $0_X \in C$, then the gauge function γ_C is proper, convex and lower semicontinuous.*

Lemma 2.3. *Let $C \subseteq X$ be a convex and closed set with $0_X \in C$, then the conjugate of the gauge function γ is given by*

$$\gamma_C^*(x^*) := \begin{cases} 0, & \text{if } \sigma_C(x^*) \leq 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

Remark 2.4. Note that the gauge function γ_C is not only convex but also sublinear. Moreover, if $0_X \in \operatorname{int} C$, then γ_C is well-defined, which means that $\operatorname{dom} \gamma_C = X$. □

Definition 2.5. Let $C \subseteq X$. The *polar set* of C is defined by

$$C^0 := \left\{ x^* \in X^* : \sup_{x \in C} \langle x^*, x \rangle \leq 1 \right\} = \{x^* \in X^* : \sigma_C(x^*) \leq 1\}$$

and by means of the polar set the *dual gauge* is defined by

$$\gamma_{C^0}(x^*) := \sup_{x \in C} \langle x^*, x \rangle = \sigma_C(x^*).$$

Remark 2.6. Note that C^0 is a convex and closed set containing the origin and by the definition of the dual gauge follows that the conjugate function of γ_C can equivalently be expressed by

$$\gamma_C^*(x^*) := \begin{cases} 0, & \text{if } \gamma_{C^0}(x^*) \leq 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

Furthermore, we have the generalized Cauchy-Schwarz inequality (see [13, Lemma 2.1])

$$\langle x^*, x \rangle \leq \gamma_{C^0}(x^*)\gamma_C(x) \ \ \forall x^* \in X^*, \ x \in X. \qquad \square \tag{2}$$

Finally, let $\mathcal{H}$ be a real Hilbert space equipped with the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, where the associated norm $\| \cdot \|_{\mathcal{H}}$ is defined by $\|y\|_{\mathcal{H}} := \sqrt{\langle y, y \rangle_{\mathcal{H}}}$ for all $y \in \mathcal{H}$. If $\mathcal{H} = \mathbb{R}^m$, then $\| \cdot \|_{\mathbb{R}^m}$ is the Euclidean norm associated to the Euclidean inner product on $\mathbb{R}^m$ and we will write for simplicity just $\| \cdot \|$.

2.2. Lagrange duality approach for multi-composed optimization problems

The purpose of this section is to recall some important results from [26] by studying multi-composed optimization problems. Let us consider an optimization problem with geometric and cone constraints having as objective function the composition of $n+1$ functions:

$$(P^C) \qquad \inf_{x\in\mathcal{A}} (f \circ F^1 \circ ... \circ F^n)(x) : \mathcal{A} = \{x \in S : g(x) \in -Q\},$$

where X_i is a Hausdorff locally convex space partially ordered by the non-empty convex cone $K_i \subseteq X_i$ for $i = 0, ..., n-1$. Moreover,

(1) $S \subseteq X_n$ is a non-empty convex set,

(2) $f\colon X_0 \to \overline{\mathbb{R}}$ is proper, convex and K_0-increasing on
$$F^1(\operatorname{dom} F^1) + K_0 \subseteq \operatorname{dom} f,$$

(3) $F^i\colon X_i \to X^\bullet_{i-1} = X_{i-1} \cup \{+\infty_{K_{i-1}}\}$ is proper, K_{i-1}-convex and K_i-K_{i-1}-increasing on $F^{i+1}(\operatorname{dom} F^{i+1}) + K_i \subseteq \operatorname{dom} F^i$ for $i = 1, ..., n-2$,

(4) $F^{n-1}\colon X_{n-1} \to X^\bullet_{n-2} = X_{n-2} \cup \{+\infty_{K_{n-2}}\}$ is proper, K_{n-2}-convex and K_{n-1}-K_{n-2}-increasing on $F^n(\operatorname{dom} F^n \cap \mathcal{A}) + K_{n-1} \subseteq \operatorname{dom} F^{n-1}$,

(5) $F^n\colon X_n \to X^\bullet_{n-1} = X_{n-1} \cup \{+\infty_{K_{n-1}}\}$ is a proper and K_{n-1}-convex function,

(6) $g\colon X_n \to Z^\bullet$ is a proper function fulfilling $S \cap g^{-1}(-Q) \cap ((F^n)^{-1} \circ ... \circ (F^1)^{-1})(\operatorname{dom} f) \neq \emptyset$.

Additionally, we make the convention that $f(+\infty_{K_0}) = +\infty$ and $F^i(+\infty_{K_i}) = +\infty_{K_{i-1}}$, i.e. $f\colon X_0^\bullet \to \overline{\mathbb{R}}$ and $F^i\colon X_i^\bullet \to X^\bullet_{i-1}$, $i = 1, ..., n-1$.

Remark 2.7. Let us point out that for the convexity of $(f \circ F^1 \circ ... \circ F^n)$ we ask that the function f be convex and K_0-increasing on $F^1(\operatorname{dom} F^1) + K_0$ and the functions F^i be K_{i-1}-convex and fulfill also the property of monotonicity for $i = 1, ..., n-1$, while the function F^n needs just be K_{n-1}-convex. It turns out, especially in the context of location problems, that in the case when F^n is an affine function, then the condition of monotonicity of F^{n-1} is too restrictive. But fortunately, this circumstance can be bypassed by the fact that the composition of an affine function and a function, which fulfills the property of convexity, fulfills also the property of convexity. That is to say, one can omit the monotonicity of F^{n-1} by setting $K_{n-1} = \{0_{X_{n-1}}\}$ to preserve the property of convexity of $(f \circ F^1 \circ ... \circ F^n)$ (for more details see Remark 3.1 and 4.1 in [26]). □

The corresponding conjugate dual problem to problem (P^C) is (see [26])

$$(D^C) \quad \sup_{\substack{z^{n*}\in Q^* \\ z^{i*}\in K_i^* \\ i=0,...,n-1}} \left\{ \inf_{x\in S} \left\{ \langle z^{(n-1)*}, F^n(x)\rangle + \langle z^{n*}, g(x)\rangle \right\} - f^*(z^{0*}) - \sum_{i=1}^{n-1} (z^{(i-1)*}F^i)^*(z^{i*}) \right\},$$

where $\widetilde{z}^* := (z^{0*}, ..., z^{(n-1)*}, z^{n*}) \in \widetilde{K}^* := K_0^* \times ... \times K_{n-1}^* \times Q^*$ is the dual variable.

We denote by $v(P^C)$ and $v(D^C)$ the optimal objective values of the optimization problems (P^C) and (D^C), respectively. To guarantee strong duality, i.e. the situation where $v(P^C) = v(D^C)$ and the conjugate dual problem has an optimal solution, we consider the following generalized interior point regularity condition introduced in [26]:

$$(RC) \quad \left| \begin{array}{l} f \text{ is l.s.c., } S \text{ is closed, } g \text{ is } Q\text{-epi closed, } K_{i-1} \text{ is closed,} \\ \operatorname{int} K_{i-1} \neq \emptyset,\ F^i \text{ is } K_{i-1}\text{-epi closed, } i = 1, ..., n, \\ 0_{X_0} \in \operatorname{sqri}(F^1(\operatorname{dom} F^1) - \operatorname{dom} f + K_0), \\ 0_{X_{i-1}} \in \operatorname{sqri}(F^i(\operatorname{dom} F^i) - \operatorname{dom} F^{i-1} + K_{i-1}), i = 2, ..., n-1, \\ 0_{X_{n-1}} \in \operatorname{sqri}(F^n(\operatorname{dom} F^n \cap \operatorname{dom} g \cap S) - \operatorname{dom} F^{n-1} + K_{n-1}) \text{ and} \\ 0_Z \in \operatorname{sqri}(g(\operatorname{dom} F^n \cap \operatorname{dom} g \cap S) + Q), \end{array} \right.$$

where the strong quasi relative interior can be replaced by the algebraic interior and the interior leading to stronger versions of (RC), which are in fact equivalent. Besides these regularity conditions we also want to mention in the framework of the multi-composed optimization problems the well-known Slater constraint qualification [26] as well as the closedness type regularity condition [12], which we will not further consider in this paper.

In [26] the following theorems have been stated.

Theorem 2.8. (Strong duality) *If the condition (RC) is fulfilled, then between (P^C) and (D^C) strong duality holds, i.e. $v(P^C) = v(D^C)$ and the conjugate dual problem has an optimal solution.*

Theorem 2.9. (Optimality conditions)

(a) *Suppose that the regularity condition (RC) is fulfilled and let $\overline{x} \in \mathcal{A}$ be an optimal solution of the problem (P^C). Then there exists*

$$(\overline{z}^{0*}, ..., \overline{z}^{(n-1)*}, \overline{z}^{n*}) \in K_0^* \times ... \times K_{n-1}^* \times Q^*,$$

an optimal solution to (D^C), such that

(i) $f((F^1 \circ ... \circ F^n)(\overline{x})) + f^*(\overline{z}^{0*}) = \langle \overline{z}^{0*}, (F^1 \circ ... \circ F^n)(\overline{x}) \rangle$,

(ii) $(\overline{z}^{(i-1)*} F^i)((F^{i+1} \circ ... \circ F^n)(\overline{x})) + (\overline{z}^{(i-1)*} F^i)^*(\overline{z}^{i*})$
$$= \langle \overline{z}^{i*}, (F^{i+1} \circ ... \circ F^n)(\overline{x}) \rangle, \ i = 1, ..., n-1,$$

(iii) $(\overline{z}^{(n-1)*} F^n)(\overline{x}) + (\overline{z}^{n*} g)(\overline{x}) + ((\overline{z}^{(n-1)*} F^n) + (\overline{z}^{n*} g))_S^*(0_{X_n^*}) = 0$,

(iv) $\langle \overline{z}^{n*}, g(\overline{x}) \rangle = 0$.

(b) *If there exists $\overline{x} \in \mathcal{A}$ such that the conditions* (i)–(iv) *are fulfilled for some $(\overline{z}^{0*}, ..., \overline{z}^{(n-1)*}, \overline{z}^{n*}) \in K_0^* \times ... \times K_{n-1}^* \times Q^*$ then $\overline{x}$ is an optimal solution of (P^C), $(\overline{z}^{0*}, ..., \overline{z}^{n*})$ is an optimal solution for (D^C) and $v(P^C) = v(D^C)$.*

Remark 2.10. If for some $i \in \{1, ..., n\}$ the function F^i is positively K_{i-1}-lower semicontinuous, then we can omit asking that K_{i-1} is closed, $\operatorname{int}(K_{i-1}) \neq \emptyset$ and F^i is K_{i-1}-epi closed in the regularity condition (RC) (for more details see Remark 4.2 in [26]). □

Theorem 2.11. *Let $a_i \in \mathbb{R}_+$ be a given point and $h_i\colon \mathbb{R} \to \overline{\mathbb{R}}$ with $h_i(x) \in \mathbb{R}_+$, if $x \in \mathbb{R}_+$, and $h_i(x) = +\infty$, otherwise, be a proper, lower semicontinuous and convex function, $i = 1, ..., n$. Then the conjugate of the function $g\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by*

$$g(x_1, ..., x_n) := \begin{cases} \max\{h_1(x_1) + a_1, ..., h_n(x_n) + a_n\}, & \text{if } x_i \in \mathbb{R}_+,\ i = 1, ..., n, \\ +\infty, & \text{otherwise,} \end{cases}$$

is given by $g^*\colon \mathbb{R}^n \to \mathbb{R}$,

$$g^*(x_1^*, ..., x_n^*) = \min_{\substack{\sum_{i=1}^n z_i^{0*} \leq 1 \\ z_i^{0*} \geq 0,\ i=1,...,n}} \left\{ \sum_{i=1}^n [(z_i^{0*} h_i)^*(x_i^*) - z_i^{0*} a_i] \right\}.$$

Lemma 2.12. *Let $a_i \in \mathbb{R}_+$ be a given point and $h_i\colon \mathbb{R} \to \overline{\mathbb{R}}$ with $h_i(x) \in \mathbb{R}_+$, if $x \in \mathbb{R}_+$, and $h_i(x) = +\infty$, otherwise, be a proper, lower semicontinuous and convex function, $i = 1, ..., n$. Then the function $g\colon \mathbb{R}^m \to \overline{\mathbb{R}}$,*

$$g(x_1, ..., x_n) = \begin{cases} \max\{h_1(x_1) + a_1, ..., h_n(x_n) + a_n\}, & \text{if } x_i \in \mathbb{R}_+,\ i = 1, ..., n, \\ +\infty, & \text{otherwise,} \end{cases}$$

can equivalently be expressed as

$$g(x_1, ..., x_n) = \sup_{\substack{\sum_{i=1}^n z_i^{0*} \leq 1 \\ z_i^{0*} \geq 0, i=1,...,n}} \left\{ \sum_{i=1}^n z_i^{0*}[h_i(x_i) + a_i] \right\} \ \forall x_i \geq 0,\ i = 1, ..., n.$$

For the proofs of the last two statements see [27].

3. Multifacility minimax location problems with mixed gauges

In this section we use the results of our previous approach to develop a conjugate dual problem of the multifacility minimax location problem with mixed gauges and geometric constraints. Furthermore, we will show the validity of strong duality and derive optimality conditions for the corresponding primal-dual pair.

Let X be a Banach space (note that most of the following investigations can be extended to a Fréchet space, too), $J := \{jk : 1 \leq j \leq m,\ 1 \leq k \leq m,\ j \neq k\}$ and $\widetilde{J} := \{ji : 1 \leq j \leq m, 1 \leq i \leq t\}$. Further, let $V \subseteq J$ and $\widetilde{V} \subseteq \widetilde{J}$ be given index sets, the sets $C_{jk} \subseteq X$ with $0_X \in \operatorname{int} C_{jk}$ for $jk \in V$ and $\widetilde{C}_{ji} \subseteq X$ with $0_X \in \operatorname{int} \widetilde{C}_{ji}$ for $ji \in \widetilde{V}$ be closed and convex as well as $S \subseteq X^m$ non-empty, closed and convex. Obviously, the gauges given by the sets C_{jk}, $jk \in V$, and $\widetilde{C}_{ji}$, $ji \in \widetilde{V}$, are convex, lower semicontinuous and well-defined.

For given distinct points $p_i \in X$, $1 \leq i \leq t$, the multifacility minimax location problem minimizes the maximum of gauges between pairs of m new facilities $x_1, ..., x_m$ and between pairs of m new and t existing facilities, concretely this means that

$$(P^M) \qquad \inf_{(x_1,...,x_m)\in S} \max\left\{ \left(\gamma_{C_{jk}}(x_j - x_k)\right)_{jk\in V}, \ \left(\gamma_{\widetilde{C}_{ji}}(x_j - p_i)\right)_{ji\in\widetilde{V}} \right\}.$$

The motivation of introducing the sets V and $\widetilde{V}$ is founded in the circumstance that some distances between pairs of new and pairs of new and existing facilities may not be relevant for the location problem (P^M) and thus, do not need to be considered in the objective function. In addition, take note that $|V| \leq |J| = m(m-1)$ and $|\widetilde{V}| \leq |\widetilde{J}| = mt$.

Now, we set $X_0 = \mathbb{R}^{|V|} \times \mathbb{R}^{|\widetilde{V}|}$ ordered by $K_0 = \mathbb{R}_+^{|V|} \times \mathbb{R}_+^{|\widetilde{V}|}$, $X_1 = X^{|V|} \times X^{|\widetilde{V}|}$ ordered by the trivial cone $K_1 = \{0_{X_1}\}$ and $X_2 = X^m$, where the corresponding dual spaces and dual variables are $(z^{0*}, \widetilde{z}^{0*}) = \left((z^{0*}_{jk})_{jk\in V}, (\widetilde{z}^{0*}_{ji})_{ji\in\widetilde{V}}\right) \in \mathbb{R}^{|V|} \times \mathbb{R}^{|\widetilde{V}|}$ and $(z^{1*}, \widetilde{z}^{1*}) = \left((z^{1*}_{jk})_{jk\in V}, (\widetilde{z}^{1*}_{ji})_{ji\in\widetilde{V}}\right) \in (X^*)^{|V|} \times (X^*)^{|\widetilde{V}|}$.

We continue with the decomposition of the objective function of the problem (P^M) into the following functions:

(1) $f\colon \mathbb{R}^{|V|} \times \mathbb{R}^{|\widetilde{V}|} \to \mathbb{R}$ defined by $f(y^0, \widetilde{y}^0) = \max\left\{ \left(y^0_{jk}\right)_{jk\in V}, \ \left(\widetilde{y}^0_{ji}\right)_{ji\in\widetilde{V}} \right\}$ if $y^0 = (y^0_{jk})_{jk\in V} \in \mathbb{R}_+^{|V|}$ and $\widetilde{y}^0 = (\widetilde{y}^0_{ji})_{ji\in\widetilde{V}} \in \mathbb{R}_+^{|\widetilde{V}|}$, otherwise $f(y^0, \widetilde{y}^0) = +\infty$,

(2) $F^1\colon X^{|V|} \times X^{|\widetilde{V}|} \to \mathbb{R}^{|V|} \times \mathbb{R}^{|\widetilde{V}|}$ defined by
$$F^1(y^1, \widetilde{y}^1) = \left((\gamma_{C_{jk}}(y^1_{jk}))_{jk\in V}, (\gamma_{\widetilde{C}_{ji}}(\widetilde{y}^1_{ji}))_{ji\in\widetilde{V}}\right),$$
where $y^1 = (y^1_{jk})_{jk\in V} \in X^{|V|}$ and $\widetilde{y}^1 = (\widetilde{y}^1_{ji})_{ji\in\widetilde{V}} \in X^{|\widetilde{V}|}$,

(3) $F^2\colon X^m \to X^{|V|} \times X^{|\widetilde{V}|}$ defined by $F^2(x) = \left((A_{jk}x)_{jk\in V}, (B_{ji}x - p_i)_{ji\in\widetilde{V}}\right)$, where for $jk \in V$ and $ji \in \widetilde{V}$,
$$A_{jk} = (\overset{1}{\mathbf{0}}, ..., \mathbf{0}, \overset{j}{I}, \mathbf{0}, ..., \overset{k}{-I}, \mathbf{0}, ..., \overset{m}{\mathbf{0}}), \quad B_{ji} = (\overset{1}{\mathbf{0}}, ..., \mathbf{0}, \overset{j}{I}, \mathbf{0}, ..., \overset{m}{\mathbf{0}}),$$
$\mathbf{0}$ is the zero mapping and I is the identity mapping, i.e. $\mathbf{0}x_i = 0_X$ and $Ix_i = x_i \ \forall x_i \in X$, $i = 1, ..., m$. In particular, $A_{jk}\colon X^m \to X$ is defined as the mapping $x = (x_1, ..., x_m) \mapsto$
$$\mathbf{0}x_1 + ... + \mathbf{0}x_{j-1} + Ix_j + \mathbf{0}x_{j+1} + ... + \mathbf{0}x_{k-1} - Ix_k + \mathbf{0}x_{k+1} + ... + \mathbf{0}x_m,$$
i.e. $(x_1, ..., x_m) \mapsto x_j - x_k$, $jk \in V$, and $B_{ji}\colon X^m \to X$ is defined as the map
$$(x_1, ..., x_m) \mapsto \mathbf{0}x_1 + ... + \mathbf{0}x_{j-1} + Ix_j + \mathbf{0}x_{j+1} + ... + \mathbf{0}x_m = x_j, \ ji \in \widetilde{V}.$$

Thus, the problem (P^M) can be represented in the form

$$\begin{aligned}
(P^M) \qquad & \inf_{x\in S} \max\left\{ \left(\gamma_{C_{jk}}(x_j - x_k)\right)_{jk\in V}, \ \left(\gamma_{\widetilde{C}_{ji}}(x_j - p_i)\right)_{ji\in\widetilde{V}} \right\} \\
&= \inf_{x\in S} f\left(\left(\gamma_{C_{jk}}(A_{jk}x)\right)_{jk\in V}, \ \left(\gamma_{\widetilde{C}_{ji}}(B_{ji}x - p_i)\right)_{ji\in\widetilde{V}} \right) \\
&= \inf_{x\in S} (f \circ F^1 \circ F^2)(x),
\end{aligned}$$

where the expression $F^1 \circ F^2$ in the last equality is defined as the component-wise composition of the vector-valued functions F^1 and F^2, i.e.

$$(F^1 \circ F^2)(x) = \Big(\big((\gamma_{C_{jk}} \circ A_{jk})(x)\big)_{jk\in V}, \big((\gamma_{\widetilde{C}_{ji}} \circ (B_{jk} \cdot - p_i))(x)\big)_{ji\in \widetilde{V}} \Big).$$

Remark 3.1. As already mentioned in the introduction, the intricacy of (P^M) is formed by the pointwise maximum of gauge functions composed with linear operators. For this reason, there is the desire to reduce the level of complexity by reformulating (P^M) into an unconstrained optimization problem where the objective function is a sum of proper, convex and lower semicontinuous functions. This can be achieved by using the notion of epigraph and by introducing an additional variable $t \in \mathbb{R}$ as follows

$$(P^M) \qquad \inf_{\substack{t\in\mathbb{R},\ x\in S \\ \gamma_{C_{jk}}(A_{jk}x)\leq t,\ jk\in V \\ \gamma_{\widetilde{C}_{ji}}(B_{ji}x-p_i)\leq t,\ ji\in\widetilde{V}}} t = \inf_{\substack{t\in\mathbb{R},\ x\in S \\ (x,t)\in \operatorname{epi}(\gamma_{C_{jk}}(A_{jk}\cdot)),\ jk\in V \\ (x,t)\in \operatorname{epi}(\gamma_{\widetilde{C}_{ji}}(B_{ji}\cdot-p_i)),\ ji\in\widetilde{V}}} t$$

$$= \inf_{(x,t)\in X^m\times\mathbb{R}} \Big\{ t + \delta_S(x) + \sum_{jk\in V} \delta_{\operatorname{epi}(\gamma_{C_{jk}}(A_{jk}\cdot))}(x,t) + \sum_{ji\in\widetilde{V}} \delta_{\operatorname{epi}(\gamma_{\widetilde{C}_{ji}}(B_{ji}\cdot-p_i))}(x,t) \Big\}. \quad (3)$$

Now we set $X = \mathcal{H}$, then the advantage of the formulation of (P^M) according to (3) is that we can use the parallel splitting algorithm from [1, Proposition 27.8] to solve the multifacility minimax location problem numerically. Notice that this method requires in each iteration step the computation of the proximal points of the functions involved in the objective function in (3).

As the proximal operator of an indicator function collapses into a projection operator, one can apply here the formulae given in [28, Theorem 2.2] and [28, Remark 2.5] for the projection onto the epigraph of the gauge function composed with a linear operator. □

As mentioned in Remark 2.7 we do not need the monotonicity assumption for the function F^1, because F^2 is an affine function. Furthermore, it is clear that (P^M) is a convex optimization problem. Besides, it can easily be verified that f is proper, convex, $\mathbb{R}^{|V|}_+ \times \mathbb{R}^{|V|}_+$-increasing on

$$F^1(\operatorname{dom} F^1) + K_0 = \operatorname{dom} f = \mathbb{R}^{|V|}_+ \times \mathbb{R}^{|\widetilde{V}|}_+$$

and lower semicontinuous, and that F^1 is proper and $\mathbb{R}^{|V|}_+ \times \mathbb{R}^{|\widetilde{V}|}_+$-convex as well as $\mathbb{R}^{|V|}_+ \times \mathbb{R}^{|\widetilde{V}|}_+$-epi closed.

To use the formula from the previous section for the dual problem of (P^M), we set $Z = X^m$ ordered by the trivial cone $Q = X^m$ and define the function $g\colon X^m \to X^m$ by $g(x_1, ..., x_m) := (x_1, ..., x_m)$. As $Q^* = \{0_{(X^*)^m}\}$, which means that $z^{2*} = 0_{(X^*)^m}$, we derive for the dual problem

$$(D^M) \quad \sup_{\substack{(z^{0*},\widetilde{z}^{0*})\in\mathbb{R}_+^{|V|}\times\mathbb{R}_+^{|\widetilde{V}|},\\ (z^{1*},\widetilde{z}^{1*})\in(X^*)^{|V|}\times(X^*)^{|\widetilde{V}|}}} \Bigg\{ \inf_{x\in S} \Bigg\{ \sum_{jk\in V} \langle z_{jk}^{1*}, A_{jk}x\rangle + \sum_{ji\in\widetilde{V}} \langle \widetilde{z}_{ji}^{1*}, B_{ji}x - p_i\rangle \Bigg\} - f^*(z^{0*},\widetilde{z}^{0*}) - ((z^{0*},\widetilde{z}^{0*})F^1)^*(z^{1*},\widetilde{z}^{1*}) \Bigg\},$$

and hence, we need to calculate the conjugate functions f^* and $((z^{0*},\widetilde{z}^{0*})F^1)^*$. Let $h_i\colon \mathbb{R}\to\overline{\mathbb{R}}$ be defined by

$$h_i(x_i) := \begin{cases} x_i, & \text{if } x_i\in\mathbb{R}_+,\\ +\infty, & \text{otherwise,}\end{cases}$$

then the conjugate function of $\lambda_i h_i$, $\lambda_i \geq 0$, is

$$(\lambda_i h_i)^*(x_i^*) = \begin{cases} 0, & \text{if } x_i^* \leq \lambda_i,\\ +\infty, & \text{otherwise,}\end{cases}, \quad i=1,...,n.$$

and by Theorem 2.11 we get for f^*,

$$f^*(z^{0*},\widetilde{z}^{0*}) = \begin{cases} 0, \text{ if } z_{jk}^{0*}\leq\lambda_{jk},\ \widetilde{z}_{ji}^{0*}\leq\widetilde{\lambda}_{ji},\ \sum\limits_{jk\in V}\lambda_{jk}+\sum\limits_{ji\in\widetilde{V}}\widetilde{\lambda}_{ji}\leq 1\\ \qquad (\lambda_{jk})_{jk\in V}\in\mathbb{R}_+^{|V|} \text{ and } (\lambda_{ji})_{ji\in\widetilde{V}}\in\mathbb{R}_+^{|\widetilde{V}|},\\ +\infty, \text{ otherwise,}\end{cases}$$

$$= \begin{cases} 0, \text{ if } \sum\limits_{jk\in V} z_{jk}^{0*} + \sum\limits_{ji\in\widetilde{V}} \widetilde{z}_{ji}^{0*} \leq 1,\ z^{0*}\in\mathbb{R}_+^{|V|},\ \widetilde{z}^{0*}\in\mathbb{R}_+^{|\widetilde{V}|},\\ +\infty, \text{ otherwise,}\end{cases}$$

while for $((z^{0*},\widetilde{z}^{0*})F^1)^*$ we obtain by using the definition of the conjugate function

$$\begin{aligned}((z^{0*},\widetilde{z}^{0*})F^1)^*(z^{1*},\widetilde{z}^{1*}) &= \sup_{y^1\in X^{|V|},\ \widetilde{y}^1\in X^{|\widetilde{V}|}} \Bigg\{ \sum_{jk\in V}\langle z_{jk}^{1*}, y_{jk}^1\rangle + \\ &\sum_{ji\in\widetilde{V}}\langle \widetilde{z}_{ji}^{1*},\widetilde{y}_{ji}^1\rangle - \sum_{jk\in V} z_{jk}^{0*}\gamma_{C_{jk}}(y_{jk}^1) - \sum_{ji\in\widetilde{V}}\widetilde{z}_{ji}^{0*}\gamma_{\widetilde{C}_{ji}}(\widetilde{y}_{ji}^1)\Bigg\}\\ &= \sum_{jk\in V}\sup_{y_{jk}^1\in X}\left\{\langle z_{jk}^{1*},y_{jk}^1\rangle - z_{jk}^{0*}\gamma_{C_{jk}}(y_{jk}^1)\right\} + \sum_{ji\in\widetilde{V}}\sup_{\widetilde{y}_{ji}^1\in X}\left\{\langle\widetilde{z}_{ji}^{1*},\widetilde{y}_{ji}^1\rangle - \widetilde{z}_{ji}^{0*}\gamma_{\widetilde{C}_{ji}}(\widetilde{y}_{ji}^1)\right\}\\ &= \sum_{jk\in V}(z_{jk}^{0*}\gamma_{C_{jk}})^*(z_{jk}^{1*}) + \sum_{ji\in\widetilde{V}}(\widetilde{z}_{ji}^{0*}\gamma_{\widetilde{C}_{ji}})^*(\widetilde{z}_{ji}^{1*})\end{aligned}$$

for all $(z^{0*},\widetilde{z}^{0*})\in\mathbb{R}_+^{|V|}\times\mathbb{R}_+^{|\widetilde{V}|}$, $z^{1*}=(z_{jk}^{1*})_{jk\in V}\in X^{|V|}$ and $\widetilde{z}^{1*}=(z_{ji}^{1*})_{ji\in\widetilde{V}}\in X^{|\widetilde{V}|}$.

Hence, the dual problem may be written as

$$(D^M) \qquad \sup_{\substack{(z^{0*},\widetilde{z}^{0*},z^{1*},\widetilde{z}^{1*})\in\mathbb{R}_+^{|V|}\times\mathbb{R}_+^{|\widetilde{V}|}\times X^{|V|}\times X^{|\widetilde{V}|} \\ \sum\limits_{jk\in V} z_{jk}^{0*}+\sum\limits_{ji\in\widetilde{V}} \widetilde{z}_{ji}^{0*}\leq 1}} \inf_{x\in S} \Phi(z^{0*},\widetilde{z}^{0*},z^{1*},\widetilde{z}^{1*}),$$

where

$$\Phi(z^{0*},\widetilde{z}^{0*},z^{1*},\widetilde{z}^{1*}) = \inf_{x\in S}\left\{\sum_{jk\in V}\langle z_{jk}^{1*}, A_{jk}x\rangle + \sum_{ji\in\widetilde{V}}\langle \widetilde{z}_{ji}^{1*}, B_{ji}x-p_i\rangle\right\} - \sum_{jk\in V}(z_{jk}^{0*}\gamma_{C_{jk}})^*(z_{jk}^{1*}) - \sum_{ji\in\widetilde{V}}(\widetilde{z}_{ji}^{0*}\gamma_{\widetilde{C}_{ji}})^*(\widetilde{z}_{ji}^{1*}).$$

Let $I := \{jk \in V : z_{jk}^{0*} > 0\}$ and $\widetilde{I} := \{ji \in \widetilde{V}_{ji} : \widetilde{z}_{ji}^{0*} > 0\}$, then we separate in the objective function Φ the sum into the terms with $z_{jk}^{0*},\ \widetilde{z}_{ji}^{0*} > 0$ and the terms with $z_{jk}^{0*},\ \widetilde{z}_{ji}^{0*} = 0$:

$$\begin{aligned}\Phi(z^{0*},\widetilde{z}^{0*},z^{1*},\widetilde{z}^{1*}) \;=\; & \inf_{x\in S}\left\{\sum_{jk\in V}\langle z_{jk}^{1*}, A_{jk}x\rangle + \sum_{ji\in\widetilde{V}}\langle \widetilde{z}_{ji}^{1*}, B_{ji}x-p_i\rangle\right\} \\ & -\sum_{jk\in I}(z_{jk}^{0*}\gamma_{C_{jk}})^*(z_{jk}^{1*}) - \sum_{ji\in\widetilde{I}}(\widetilde{z}_{ji}^{0*}\gamma_{\widetilde{C}_{ji}})^*(\widetilde{z}_{ji}^{1*}) \\ & - \sum_{jk\in V\setminus I}(0\cdot\gamma_{C_{jk}})^*(z_{jk}^{1*}) - \sum_{ji\in\widetilde{V}\setminus\widetilde{I}}(0\cdot\gamma_{\widetilde{C}_{ji}})^*(\widetilde{z}_{ji}^{1*}).\end{aligned}$$

Now, for $jk \in I$ we have (see [2])

$$(z_{jk}^{0*}\gamma_{C_{jk}})^*(z_{jk}^{1*}) = \begin{cases} 0, \text{ if } \gamma_{C_{jk}^0}(z_{jk}^{1*}) \leq z_{jk}^{0*}, \\ +\infty, \text{ otherwise}, \end{cases} \tag{4}$$

and analogously, it follows for $ji \in \widetilde{I}$ that

$$(\widetilde{z}_{ji}^{0*}\gamma_{\widetilde{C}_{ji}})^*(\widetilde{z}_{ji}^{1*}) = \begin{cases} 0, \text{ if } \gamma_{\widetilde{C}_{ji}^0}(\widetilde{z}_{ji}^{1*}) \leq \widetilde{z}_{ji}^{0*}, \\ +\infty, \text{ otherwise}. \end{cases} \tag{5}$$

For $jk \in V \setminus I$ we have

$$(0\cdot\gamma_{C_{jk}})^*(z_{jk}^{1*}) = \sup_{y_{jk}^1\in X}\{\langle z_{jk}^{1*}, y_{jk}^1\rangle\} = \begin{cases} 0, \text{ if } z_{jk}^{1*} = 0_{X^*}, \\ +\infty, \text{ otherwise}, \end{cases}$$

and analogously, we get for $ji \in \widetilde{V}\setminus\widetilde{I}$,

$$(0\cdot\gamma_{\widetilde{C}_{ji}})^*(\widetilde{z}_{ji}^{1*}) = \begin{cases} 0, \text{ if } \widetilde{z}_{ji}^{1*} = 0_{X^*}, \\ +\infty, \text{ otherwise}, \end{cases}$$

which implies that if $jk \in V \setminus I$, then $z_{jk}^{1*} = 0_{X^*}$ and if $ji \in \widetilde{V}\setminus\widetilde{I}$, then $\widetilde{z}_{ji}^{1*} = 0_{X^*}$.

Therefore, we obtain for the dual problem of the location problem (P^M):

$$(D^M) \quad \sup_{(z^{0*},\widetilde{z}^{0*},z^{1*},\widetilde{z}^{1*})\in\mathcal{B}} \inf_{x\in S}\left\{\sum_{jk\in I}\langle z_{jk}^{1*}, A_{jk}x\rangle + \sum_{ji\in\widetilde{I}}\langle \widetilde{z}_{ji}^{1*}, B_{ji}x - p_i\rangle\right\},$$

where

$$\mathcal{B} = \Big\{(z^{0*},\widetilde{z}^{0*},z^{1*},\widetilde{z}^{1*}) \in \mathbb{R}_+^{|V|}\times\mathbb{R}_+^{|\widetilde{V}|}\times(X^*)^{|V|}\times(X^*)^{|\widetilde{V}|} : I\subseteq V,\ \widetilde{I}\subseteq\widetilde{V},\ z_{jk}^{0*}>0,$$
$$z_{jk}^{1*}\in X^*, \gamma_{C_{jk}^0}(z_{jk}^{1*}) \le z_{jk}^{0*}, jk\in I, \widetilde{z}_{ji}^{0*}>0, \widetilde{z}_{ji}^{1*}\in X^*, \gamma_{\widetilde{C}_{ji}^0}(\widetilde{z}_{ji}^{1*}) \le \widetilde{z}_{ji}^{0*}, ji\in\widetilde{I}, z_{ef}^{0*}=0,$$
$$z_{ef}^{1*} = 0_{X^*},\ ef\in V\setminus I,\ \widetilde{z}_{ed}^{0*}=0,\ \widetilde{z}_{ed}^{1*}\in 0_{X^*},\ ed\in\widetilde{V}\setminus\widetilde{I},\ \sum_{jk\in I} z_{jk}^{0*} + \sum_{ji\in\widetilde{I}}\widetilde{z}_{ji}^{0*} \le 1\Big\}.$$

Since, the objective function of the conjugate dual problem (D^M) can also be written as

$$\inf_{x\in S}\left\{\sum_{jk\in I}\langle z_{jk}^{1*}, A_{jk}x\rangle + \sum_{ji\in\widetilde{I}}\langle \widetilde{z}_{ji}^{1*}, B_{ji}x - p_i\rangle\right\}$$
$$= \inf_{x\in S}\left\{\left\langle \sum_{jk\in I} A_{jk}^* z_{jk}^{1*} + \sum_{ji\in\widetilde{I}} B_{ji}^*\widetilde{z}_{ji}^{1*}, x\right\rangle\right\} - \sum_{ji\in I}\langle\widetilde{z}_{ji}^{1*}, p_i\rangle,$$

where $\langle A_{jk}^* z_{jk}^{1*}, x\rangle = \langle(\overset{1}{0_{X^*}}, ..., 0_{X^*}, \overset{j}{z_{jk}^{1*}}, 0_{X^*}, ..., 0_{X^*}, \overset{k}{-z_{jk}^{1*}}, 0_{X^*}, ..., \overset{m}{0_{X^*}}),$
$(x_1, ..., x_m)\rangle = \langle z_{jk}^{1*}, x_j - x_k\rangle$

and $\langle B_{ji}^*\widetilde{z}_{ji}^{1*}, x\rangle = \langle(\overset{1}{0_{X^*}}, ..., 0_{X^*}, \overset{j}{\widetilde{z}_{ji}^{1*}}, 0_{X^*}, ..., \overset{m}{0_{X^*}}), (x_1, ..., x_m)\rangle = \langle\widetilde{z}_{ji}^{1*}, x_j\rangle,$

we can express (D^M) as

$$(D^M) \quad \sup_{(z^{0*},\widetilde{z}^{0*},z^{1*},\widetilde{z}^{1*})\in\mathcal{B}} \Big\{-\sigma_S\Big(-\sum_{jk\in I} A_{jk}^* z_{jk}^{1*} - \sum_{ji\in\widetilde{I}} B_{ji}^*\widetilde{z}_{ji}^{1*}\Big) - \sum_{ji\in\widetilde{I}}\langle\widetilde{z}_{ji}^{1*}, p_i\rangle\Big\}.$$

Remark 3.2. Note that the problem (D^M) is equivalent to the following:

$$(\widehat{D}^M) \quad \sup_{(z^{0*},\widetilde{z}^{0*},z^{1*},\widetilde{z}^{1*})\in\mathcal{B}} \Big\{-\sigma_S\Big(-\sum_{jk\in V} A_{jk}^* z_{jk}^{1*} - \sum_{ji\in\widetilde{V}} B_{ji}^*\widetilde{z}_{ji}^{1*}\Big) - \sum_{ji\in\widetilde{V}}\langle\widetilde{z}_{ji}^{1*}, p_i\rangle\Big\},$$

where $\widehat{\mathcal{B}} = \Big\{(z^{0*},\widetilde{z}^{0*},z^{1*},\widetilde{z}^{1*})\in\mathbb{R}_+^{|V|}\times\mathbb{R}_+^{|\widetilde{V}|}\times(X^*)^{|V|}\times(X^*)^{|\widetilde{V}|} : \gamma_{C_{jk}^0}(z_{jk}^{1*}) \le z_{jk}^{0*},$

$$jk\in V,\ \gamma_{\widetilde{C}_{ji}^0}(\widetilde{z}_{ji}^{1*}) \le \widetilde{z}_{ji}^{0*},\ ji\in\widetilde{V},\ \sum_{jk\in V} z_{jk}^{0*} + \sum_{ji\in\widetilde{V}}\widetilde{z}_{ji}^{0*} \le 1\Big\},$$

which can be proved as follows.

Let $(z^{0*}, \widetilde{z}^{0*}, z^{1*}, \widetilde{z}^{1*}) \in \widehat{\mathcal{B}}$ be a feasible solution of problem $(\widehat{D}^M)$, then it follows for $jk \in V \setminus I$ and $ji \in \widetilde{V} \setminus \widetilde{I}$,

$$0 \leq \gamma_{C^0_{jk}}(z^{1*}_{jk}) = \sup_{x \in C_{jk}} \langle z^{1*}_{jk}, x \rangle \leq 0 \Leftrightarrow \langle z^{1*}_{jk}, x \rangle = 0 \ \forall x \in C_{jk} \Leftrightarrow z^{1*}_{jk} = 0_{X^*}$$

as well as

$$0 \leq \gamma_{\widetilde{C}^0_{ji}}(\widetilde{z}^{1*}_{ji}) = \sup_{x \in \widetilde{C}_{ji}} \langle \widetilde{z}^{1*}_{ji}, x \rangle \leq 0 \Leftrightarrow \langle \widetilde{z}^{1*}_{ji}, x \rangle = 0 \ \forall x \in \widetilde{C}_{ji} \Leftrightarrow \widetilde{z}^{1*}_{ji} = 0_{X^*}.$$

The latter implies that from $jk \in V \setminus I$, i.e. $z^{0*}_{jk} = 0$, follows $z^{1*}_{jk} = 0_{X^*}$ and from $ji \in \widetilde{V} \setminus \widetilde{I}$, i.e. $\widetilde{z}^{0*}_{ji} = 0$, $\widetilde{z}^{1*}_{ji} = 0_{X^*}$. This relation means that $\widehat{\mathcal{B}} = \mathcal{B}$, i.e. that $(z^{0*}, \widetilde{z}^{0*}, z^{1*}, \widetilde{z}^{1*})$ is also a feasible solution of (D^M) and as

$$\begin{aligned} &\sigma_S\left(-\sum_{jk \in V} A^*_{jk} z^{1*}_{jk} - \sum_{ji \in \widetilde{V}} B^*_{ji} \widetilde{z}^{1*}_{ji}\right) + \sum_{ji \in \widetilde{V}} \langle \widetilde{z}^{1*}_{ji}, p_i \rangle \\ = \ &\sigma_S\left(-\sum_{jk \in I} A^*_{jk} z^{1*}_{jk} - \sum_{ji \in \widetilde{I}} B^*_{ji} \widetilde{z}^{1*}_{ji}\right) + \sum_{ji \in \widetilde{I}} \langle \widetilde{z}^{1*}_{ji}, p_i \rangle, \end{aligned}$$

one has immediately that $v(D^M) = v(\widehat{D}^M)$.

Vice versa, if we take a feasible solution $(z^{0*}, \widetilde{z}^{0*}, z^{1*}, \widetilde{z}^{1*})$ of the problem (D^M), then it is obvious that we have then also a feasible solution of $(\widehat{D}^M)$, which again implies that $v(D^M) = v(\widehat{D}^M)$.

From the theoretical aspect a dual problem of the form (D^M) is very useful, as one has a more detailed characterization of the set of feasible solutions. But from the numerical viewpoint it is complicate to solve, as the index sets I and $\widetilde{I}$ bring an undesirable discretization in the dual problem. For this reason it is preferable to use the dual problem $(\widehat{D}^M)$ for numerical and (D^M) for theoretical studies. It is also important to emphasize that the index sets I and $\widetilde{I}$ are determinable for a fixed and feasible solution $(z^{0*}, \widetilde{z}^{0*}, z^{1*}, \widetilde{z}^{1*})$ to the dual problem (D^M). □

We know that the weak duality between the problem (P^M) and its corresponding dual problem (D^M) always holds. Now, we are interested to know whether we also can guarantee strong duality. For this purpose we use the results from Section 2.2. As $Z = X^m$ ordered by the trivial cone $Q = X^m$ and $g \colon X^m \to X^m$ is defined by $g(x_1, ..., x_m) = (x_1, ..., x_m)$, it is obvious that g is Q-epi closed and

$$0_{X^m} \in \operatorname{sqri}(g(x) + Q) = \operatorname{sqri}(X^m + Q) = X^m.$$

More than that, recall that f is lower semicontinous, $K_0 = \mathbb{R}^{|V|}_+ \times \mathbb{R}^{|\widetilde{V}|}_+$ is closed, S is closed and F^1 is $\mathbb{R}^{|V|}_+ \times \mathbb{R}^{|\widetilde{V}|}_+$-epi closed. As

$$\begin{aligned} 0_{\mathbb{R}_+^{|V|}\times\mathbb{R}_+^{|\widetilde{V}|}} &\in \operatorname{sqri}(F^1(\operatorname{dom} F^1) - \operatorname{dom} f + K_0) \\ &= \operatorname{sqri}(F^1(\operatorname{dom} F^1) - \mathbb{R}_+^{|V|} \times \mathbb{R}_+^{|\widetilde{V}|} + \mathbb{R}_+^{|V|} \times \mathbb{R}_+^{|\widetilde{V}|}) \\ &= \mathbb{R}^{|V|} \times \mathbb{R}^{|\widetilde{V}|}, \end{aligned}$$

$$\begin{aligned} 0_{X^{|V|}\times X^{|\widetilde{V}|}} &\in \operatorname{sqri}(F^2(\operatorname{dom} F^2) - \operatorname{dom} F^1 + K_1) \\ &= \operatorname{sqri}(X^{|V|} \times X^{|\widetilde{V}|} - \operatorname{dom} F^1 + K_1) = X^{|V|} \times X^{|\widetilde{V}|} \end{aligned}$$

and F^2 is $\{0_{X^{|V|}\times X^{|\widetilde{V}|}}\}$-epi closed, the generalized interior point regularity condition (RC) is fulfilled, it follows by Theorem 2.8 the following statement (note that we denote by $v(P^M)$ and $v(D^M)$ the optimal objective values of the problems (P^M) and (D^M), respectively).

Theorem 3.3. (Strong duality) *Between (P^M) and (D^M) holds strong duality, i.e. $v(P^M) = v(D^M)$ and the conjugate dual problem has an optimal solution.*

The previous theorem implies the following necessary and sufficient optimality conditions for the primal-dual pair (P^M)-(D^M).

Theorem 3.4. (Optimality conditions)

(a) *Let $\overline{x} \in S$ be an optimal solution to the problem (P^M). Then there exists an optimal solution $(\overline{z}^{0*}, \overline{\overline{z}}^{0*}, \overline{z}^{1*}, \overline{\overline{z}}^{1*}) \in \mathbb{R}_+^{|V|} \times \mathbb{R}_+^{|\widetilde{V}|} \times (X^*)^{|V|} \times (X^*)^{|\widetilde{V}|}$ to (D^M) with corresponding index sets $\overline{I} \subseteq V$ and $\overline{\overline{I}} \subseteq \widetilde{V}$ such that*

(i) $$\max\left\{ \left(\gamma_{C_{ef}}(\overline{x}_e - \overline{x}_f)\right)_{ef\in V}, \left(\gamma_{\widetilde{C}_{ed}}(\overline{x}_e - p_d)\right)_{ed\in\widetilde{V}} \right\} = \sum_{jk\in\overline{I}} \overline{z}_{jk}^{0*}\gamma_{C_{jk}}(\overline{x}_j - \overline{x}_k) + \sum_{ji\in\overline{\overline{I}}} \overline{\overline{z}}_{ji}^{0*}\gamma_{\widetilde{C}_{ji}}(\overline{x}_j - p_i),$$

(ii) $$\left\langle \sum_{jk\in\overline{I}} A_{jk}^*\overline{z}_{jk}^{1*} + \sum_{ji\in\overline{\overline{I}}} B_{ji}^*\overline{\overline{z}}_{ji}^{1*}, \overline{x} \right\rangle = \inf_{x\in S}\left\{ \left\langle \sum_{jk\in\overline{I}} A_{jk}^*\overline{z}_{jk}^{1*} + \sum_{ji\in\overline{\overline{I}}} B_{ji}^*\overline{\overline{z}}_{ji}^{1*}, x \right\rangle \right\},$$

(iii) $\sum_{jk\in\overline{I}} \overline{z}_{jk}^{0*} + \sum_{ji\in\overline{\overline{I}}} \overline{\overline{z}}_{ji}^{0*} = 1$, $\overline{z}_{jk}^{0*} > 0$, $jk \in \overline{I}$, $\overline{\overline{z}}_{ji}^{0*} > 0$, $ji \in \overline{\overline{I}}$ *and* $\overline{z}_{ef}^{0*} = 0$, $ef \in V \setminus \overline{I}$, $\overline{\overline{z}}_{ed}^{0*} = 0$, $ed \in \widetilde{V} \setminus \overline{\overline{I}}$,

(iv) $\overline{z}_{jk}^{0*}\gamma_{C_{jk}}(\overline{x}_j - \overline{x}_k) = \langle \overline{z}_{jk}^{1*}, \overline{x}_j - \overline{x}_k \rangle$, $jk \in \overline{I}$,

(v) $\overline{\overline{z}}_{ji}^{0*}\gamma_{\widetilde{C}_{ji}}(\overline{x}_j - p_i) = \langle \overline{\overline{z}}_{ji}^{1*}, \overline{x}_j - p_i \rangle$, $ji \in \overline{\overline{I}}$,

(vi) $\max\left\{\left(\gamma_{C_{ef}}(\overline{x}_e - \overline{x}_f)\right)_{ef\in V}, \left(\gamma_{\widetilde{C}_{ed}}(\overline{x}_e - p_d)\right)_{ed\in\widetilde{V}}\right\} = \gamma_{C_{jk}}(\overline{x}_j - \overline{x}_k)$, $jk \in \overline{I}$,

(vii) $\max\left\{\left(\gamma_{C_{ef}}(\overline{x}_e - \overline{x}_f)\right)_{ef\in V}, \left(\gamma_{\widetilde{C}_{ed}}(\overline{x}_e - p_d)\right)_{ed\in\widetilde{V}}\right\} = \gamma_{\widetilde{C}_{ji}}(\overline{x}_j - p_i)$, $ji \in \overline{\overline{I}}$,

(viii) $\gamma_{C_{jk}^0}(\overline{z}_{jk}^{1*}) = \overline{z}_{jk}^{0*}$, $\overline{z}_{jk}^{1*} \in X^*$, $jk \in \overline{I}$ *and* $\overline{z}_{ef}^{1*} = 0_{X^*}$, $ef \in V \setminus \overline{I}$,

(ix) $\gamma_{\widetilde{C}_{ji}^0}(\overline{\overline{z}}_{ji}^{1*}) = \overline{\overline{z}}_{ji}^{0*}$, $\overline{\overline{z}}_{ji}^{1*} \in X^*$, $ji \in \overline{\overline{I}}$ *and* $\overline{\overline{z}}_{ed}^{1*} = 0_{X^*}$, $ed \in \widetilde{V} \setminus \overline{\overline{I}}$.

(b) *If there exists* $\overline{x} \in S$ *such that for some* $(\overline{z}^{0*}, \overline{\overline{z}}^{0*}, \overline{z}^{1*}, \overline{\overline{z}}^{1*})$ *and the corresponding index sets* $\overline{I}$ *and* $\overline{\overline{I}}$ *the conditions* (i)–(ix) *are fulfilled, then* $\overline{x}$ *is an optimal solution to* (P^C), $(\overline{z}^{0*}, \overline{\overline{z}}^{0*}, \overline{z}^{1*}, \overline{\overline{z}}^{1*})$ *is an optimal solution to* (D^M) *and* $v(P^M) = v(D^M)$.

Proof. (a) From Theorem 2.9 one gets

(i) $\max\Big\{\big(\gamma_{C_{ef}}(\overline{x}_e - \overline{x}_f)\big)_{ef\in V},\ \Big(\gamma_{\widetilde{C}_{ed}}(\overline{x}_e - p_d)\Big)_{ed\in\widetilde{V}}\Big\}$
$$= \sum_{jk\in\overline{I}} \overline{z}^{0*}_{jk}\gamma_{C_{jk}}(\overline{x}_j - \overline{x}_k) + \sum_{ji\in\overline{\overline{I}}} \overline{\overline{z}}^{0*}_{ji}\gamma_{\widetilde{C}_{ji}}(\overline{x}_j - p_i),$$

(ii) $\sum_{jk\in\overline{I}} \overline{z}^{0*}_{jk}\gamma_{C_{jk}}(\overline{x}_j-\overline{x}_k)+\sum_{ji\in\overline{\overline{I}}} \overline{\overline{z}}^{0*}_{ji}\gamma_{\widetilde{C}_{ji}}(\overline{x}_j-p_i) = \sum_{jk\in\overline{I}}\langle\overline{z}^{1*}_{jk}, \overline{x}_j-\overline{x}_k\rangle+\sum_{ji\in\overline{\overline{I}}}\langle\overline{\overline{z}}^{1*}_{ji}, \overline{x}_j-p_i\rangle$,

(iii) $\Big\langle \sum_{jk\in\overline{I}} A^*_{jk}\overline{z}^{1*}_{jk} + \sum_{ji\in\overline{\overline{I}}} B^*_{ji}\overline{\overline{z}}^{1*}_{ji}, \overline{x}\Big\rangle = -\sigma_S\Big(-\sum_{jk\in\overline{I}} A^*_{jk}\overline{z}^{1*}_{jk} - \sum_{ji\in\overline{\overline{I}}} B^*_{ji}\overline{\overline{z}}^{1*}_{ji}\Big)$,

(iv) $\sum_{jk\in\overline{I}} \overline{z}^{0*}_{jk} + \sum_{ji\in\overline{\overline{I}}} \overline{\overline{z}}^{0*}_{ji} \le 1$, $\overline{z}^{0*}_{jk} > 0$, $jk \in \overline{I}$, $\overline{\overline{z}}^{0*}_{ji} > 0$, $ji \in \overline{\overline{I}}$ and $\overline{z}^{0*}_{ef} = 0$,
$ef \in V \setminus \overline{I}$, $\overline{\overline{z}}^{0*}_{ed} = 0$, $ed \in \widetilde{V} \setminus \overline{\overline{I}}$,

(v) $\gamma_{C^0_{jk}}(\overline{z}^{1*}_{jk}) \le \overline{z}^{0*}_{jk}$, $\overline{z}^{1*}_{jk} \in X^*$, $jk \in \overline{I}$ and $\overline{z}^{1*}_{ef} = 0_{X^*}$, $ef \in V \setminus \overline{I}$,

(vi) $\gamma_{\widetilde{C}^0_{ji}}(\overline{\overline{z}}^{1*}_{ji}) \le \overline{\overline{z}}^{0*}_{ji}$, $\overline{\overline{z}}^{1*}_{ji} \in X^*$, $ji \in \overline{\overline{I}}$ and $\overline{\overline{z}}^{1*}_{ed} = 0_{X^*}$, $ed \in \widetilde{V} \setminus \overline{\overline{I}}$.

Condition (ii) yields

$$\sum_{jk\in\overline{I}} [\overline{z}^{0*}_{jk}\gamma_{C_{jk}}(\overline{x}_j - \overline{x}_k) - \langle\overline{z}^{1*}_{jk}, \overline{x}_j - \overline{x}_k\rangle] + \sum_{ji\in\overline{\overline{I}}} [\overline{\overline{z}}^{0*}_{ji}\gamma_{\widetilde{C}_{ji}}(\overline{x}_j - p_i) - \langle\overline{\overline{z}}^{1*}_{ji}, \overline{x}_j - p_i\rangle] = 0 \tag{6}$$

and by (4), (5) and the Young-Fenchel inequality it follows that the brackets in (6) are non-negative and must be equal to zero, i.e.

$$\overline{z}^{0*}_{jk}\gamma_{C_{jk}}(\overline{x}_j - \overline{x}_k) = \langle\overline{z}^{1*}_{jk}, \overline{x}_j - \overline{x}_k\rangle \text{ and } \overline{\overline{z}}^{0*}_{ji}\gamma_{\widetilde{C}_{ji}}(\overline{x}_j - p_i) = \langle\overline{\overline{z}}^{1*}_{ji}, \overline{x}_j - p_i\rangle \tag{7}$$

for $jk \in \overline{I}$ and $ji \in \overline{\overline{I}}$. Combining condition (v) with (7) reveals by using the generalized Cauchy-Schwarz inequality (see (2)) that, for $jk \in \overline{I}$,

$$\overline{z}^{0*}_{jk}\gamma_{C_{jk}}(\overline{x}_j - \overline{x}_k) = \langle\overline{z}^{1*}_{jk}, \overline{x}_j - \overline{x}_k\rangle \le \gamma_{C^0_{jk}}(\overline{z}^{1*}_{jk})\gamma_{C_{jk}}(\overline{x}_j - \overline{x}_k) \le \overline{z}^{0*}_{jk}\gamma_{C_{jk}}(\overline{x}_j - \overline{x}_k),$$

which means that
$$\gamma_{C^0_{jk}}(\overline{z}^{1*}_{jk}) = \overline{z}^{0*}_{jk},\ jk \in \overline{I}. \tag{8}$$

In the same way we get
$$\gamma_{\widetilde{C}^0_{ji}}(\overline{\overline{z}}^{1*}_{ji}) = \overline{\overline{z}}^{0*}_{ji},\ ji \in \overline{\overline{I}}. \tag{9}$$

Moreover, by conditions (i) and (iv) we have

$$
\begin{aligned}
&\max\left\{\left(\gamma_{C_{ef}}(\overline{x}_e-\overline{x}_f)\right)_{ef\in V},\ \left(\gamma_{\widetilde{C}_{ed}}(\overline{x}_e-p_d)\right)_{ed\in\widetilde{V}}\right\} \qquad (10)\\
=\ &\sum_{jk\in\overline{I}}\overline{z}^{0*}_{jk}\gamma_{C_{jk}}(\overline{x}_j-\overline{x}_k)+\sum_{ji\in\overline{\overline{I}}}\overline{\overline{z}}^{0*}_{ji}\gamma_{\widetilde{C}_{ji}}(\overline{x}_j-p_i)\\
\leq\ &\sum_{jk\in\overline{I}}\overline{z}^{0*}_{jk}\max\left\{\left(\gamma_{C_{ef}}(\overline{x}_e-\overline{x}_f)\right)_{ef\in V},\ \left(\gamma_{\widetilde{C}_{ed}}(\overline{x}_e-p_d)\right)_{ed\in\widetilde{V}}\right\}\\
&+\sum_{ji\in\overline{\overline{I}}}\overline{\overline{z}}^{0*}_{ji}\max\left\{\left(\gamma_{C_{ef}}(\overline{x}_e-\overline{x}_f)\right)_{ef\in V},\ \left(\gamma_{\widetilde{C}_{ed}}(\overline{x}_e-p_d)\right)_{ed\in\widetilde{V}}\right\}\\
\leq\ &\max\left\{\left(\gamma_{C_{ef}}(\overline{x}_e-\overline{x}_f)\right)_{ef\in V},\ \left(\gamma_{\widetilde{C}_{ed}}(\overline{x}_e-p_d)\right)_{ed\in\widetilde{V}}\right\}, \qquad (11)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
0=&\sum_{jk\in\overline{I}}\overline{z}^{0*}_{jk}\left[\max\left\{\left(\gamma_{C_{ef}}(\overline{x}_e-\overline{x}_f)\right)_{ef\in V},\ \left(\gamma_{\widetilde{C}_{ed}}(\overline{x}_e-p_d)\right)_{ed\in\widetilde{V}}\right\}-\gamma_{C_{jk}}(\overline{x}_j-\overline{x}_k)\right]\\
&+\sum_{ji\in\overline{\overline{I}}}\overline{\overline{z}}^{0*}_{ji}\left[\max\left\{\left(\gamma_{C_{ef}}(\overline{x}_e-\overline{x}_f)\right)_{ef\in V},\ \left(\gamma_{\widetilde{C}_{ed}}(\overline{x}_e-p_d)\right)_{ed\in\widetilde{V}}\right\}-\gamma_{\widetilde{C}_{ji}}(\overline{x}_j-p_i)\right]
\end{aligned}
$$

and as $\overline{z}^{0*}_{jk}>0,\ jk\in\overline{I}$, and $\overline{\overline{z}}^{0*}_{ji}>0,\ ji\in\overline{\overline{I}}$, it follows that

$$\max\left\{\left(\gamma_{C_{ef}}(\overline{x}_e-\overline{x}_f)\right)_{ef\in V},\left(\gamma_{\widetilde{C}_{ed}}(\overline{x}_e-p_d)\right)_{ed\in\widetilde{V}}\right\}=\gamma_{C_{jk}}(\overline{x}_j-\overline{x}_k),\ ik\in\overline{I}, \qquad (12)$$

and

$$\max\left\{\left(\gamma_{C_{ef}}(\overline{x}_e-\overline{x}_f)\right)_{ef\in V},\left(\gamma_{\widetilde{C}_{ed}}(\overline{x}_e-p_d)\right)_{ed\in\widetilde{V}}\right\}=\gamma_{\widetilde{C}_{ji}}(\overline{x}_j-p_i),\ ji\in\overline{\overline{I}}. \qquad (13)$$

Furthermore, we get by (11) that

$$
\begin{aligned}
&\sum_{jk\in\overline{I}}\overline{z}^{0*}_{jk}\max\left\{\left(\gamma_{C_{ef}}(\overline{x}_e-\overline{x}_f)\right)_{ef\in V},\ \left(\gamma_{\widetilde{C}_{ed}}(\overline{x}_e-p_d)\right)_{ed\in\widetilde{V}}\right\}\\
&+\sum_{ji\in\overline{\overline{I}}}\overline{\overline{z}}^{0*}_{ji}\max\left\{\left(\gamma_{C_{ef}}(\overline{x}_e-\overline{x}_f)\right)_{ef\in V},\ \left(\gamma_{\widetilde{C}_{ed}}(\overline{x}_e-p_d)\right)_{ed\in\widetilde{V}}\right\}\\
&=\max\left\{\left(\gamma_{C_{ef}}(\overline{x}_e-\overline{x}_f)\right)_{ef\in V},\ \left(\gamma_{\widetilde{C}_{ed}}(\overline{x}_e-p_d)\right)_{ed\in\widetilde{V}}\right\},
\end{aligned}
$$

from which it follows that

$$\sum_{jk\in\overline{I}}\overline{z}^{0*}_{jk}+\sum_{ji\in\overline{\overline{I}}}\overline{\overline{z}}^{0*}_{ji}=1. \qquad (14)$$

Combining now the conditions (i)–(vi) with (7), (8), (9), (12), (13) and (14) provides us the desired conclusion.

(b) The calculations made in (a) can also be done in the reverse direction, which completes the proof. □

Remark 3.5. We want to point out that the optimality condition (i) of the previous theorem can be expressed by means of the subdifferential. We have

$$f(y^0, \widetilde{y}^0) = \begin{cases} \max\left\{ \left(y^0_{jk}\right)_{jk\in V},\ \left(\widetilde{y}^0_{ji}\right)_{ji\in\widetilde{V}} \right\}, & \text{if } (y^0,\widetilde{y}^0) \in \mathbb{R}^{|V|}_+ \times \mathbb{R}^{|\widetilde{V}|}_+, \\ +\infty, \text{ otherwise}, \end{cases}$$

and

$$f^*(z^{0*}, \widetilde{z}^{0*}) = \begin{cases} 0, \text{ if } \sum\limits_{jk\in V} z^{0*}_{jk} + \sum\limits_{ji\in\widetilde{V}} \widetilde{z}^{0*}_{ji} \leq 1,\ z^{0*} \in \mathbb{R}^{|V|}_+,\ \widetilde{z}^{0*} \in \mathbb{R}^{|\widetilde{V}|}_+, \\ +\infty, \text{ otherwise}, \end{cases}$$

and by the optimality condition (i) of the previous theorem, we have

$$\begin{aligned} & f\left((\gamma_{C_{ef}}(\overline{x}_e - \overline{x}_f))_{ef\in V}, (\gamma_{\widetilde{C}_{ed}}(\overline{x}_e - p_d))_{ed\in\widetilde{V}} \right) + f^*(\overline{z}^{0*}, \overline{\widetilde{z}}^{0*}) \\ = & \sum_{jk\in\overline{I}} \overline{z}^{0*}_{jk} \gamma_{C_{jk}}(\overline{x}_j - \overline{x}_k) + \sum_{ji\in\overline{\widetilde{I}}} \overline{\widetilde{z}}^{0*}_{ji} \gamma_{\widetilde{C}_{ji}}(\overline{x}_j - p_i), \end{aligned}$$

in other words, the optimality condition (i) can be rewritten as

(i) $\quad (\overline{z}^{0*}, \overline{\widetilde{z}}^{0*}) \in \partial f \left((\gamma_{C_{ef}}(\overline{x}_e - \overline{x}_f))_{ef\in V}, (\gamma_{\widetilde{C}_{ed}}(\overline{x}_e - p_d))_{ed\in\widetilde{V}} \right).$

Moreover, for the optimality conditions (ii), (iv) and (v) we get by analogous considerations

(ii) $\quad -\sum\limits_{jk\in\overline{I}} A^*_{jk}\overline{z}^{1*}_{jk} - \sum\limits_{ji\in\overline{\widetilde{I}}} B^*_{ji}\overline{\widetilde{z}}^{1*}_{ji} \in \partial\delta_S(\overline{x}) = N_S(\overline{x}),$

(iv) $\quad \overline{z}^{1*}_{jk} \in \partial(\overline{z}^{0*}_{jk}\gamma_{C_{jk}})(\overline{x}_j - \overline{x}_k) = \partial(\overline{z}^{0*}_{jk}\gamma_{C_{jk}})(A_{jk}\overline{x})$

$\quad\quad \Leftrightarrow A^*_{jk}\overline{z}^{1*}_{jk} \in A^*_{jk}\partial((\overline{z}^{0*}_{jk}\gamma_{C_{jk}}) \circ A_{jk})(\overline{x}),\ jk \in \overline{I},$

(v) $\quad \overline{\widetilde{z}}^{1*}_{ji} \in \partial(\overline{\widetilde{z}}^{0*}_{ji}\gamma_{\widetilde{C}_{ji}})(\overline{x}_j - p_i) = \partial(\overline{\widetilde{z}}^{0*}_{ji}\gamma_{\widetilde{C}_{ji}})(B_{ji}\overline{x} - p_i)$

$\quad\quad \Leftrightarrow B^*_{ji}\overline{\widetilde{z}}^{1*}_{ji} \in B^*_{ji}\partial\left(((\overline{\widetilde{z}}^{0*}_{ji}\gamma_{\widetilde{C}_{ji}}) \circ B_{ji})(\cdot - p_i) \right)(\overline{x}),\ ji \in \overline{\widetilde{I}}.$

Taking (ii), (iv) and (v) together implies that

$$\begin{aligned} \sum_{jk\in\overline{I}} A^*_{jk}\overline{z}^{1*}_{jk} + \sum_{ji\in\overline{\widetilde{I}}} B^*_{ji}\overline{\widetilde{z}}^{1*}_{ji} \in & \Bigg(\sum_{jk\in\overline{I}} A^*_{jk}\partial((\overline{z}^{0*}_{jk}\gamma_{C_{jk}}) \circ A_{jk})(\overline{x}) \\ & + \sum_{ji\in\overline{\widetilde{I}}} B^*_{ji}\partial\Big(((\overline{\widetilde{z}}^{0*}_{ji}\gamma_{\widetilde{C}_{ji}}) \circ B_{ji})(\cdot - p_i) \Big)(\overline{x}) \Bigg) \bigcap (-N_S(\overline{x})). \end{aligned}$$

Finally, notice that the optimality conditions (iv), (v), (viii) and (ix) of the previous theorem give a detailed characterization of the subdifferentials of the associated gauges. □

Now, we show that the dual problem (D^M) is equivalent to the problem

$$(\widetilde{D}^M) \qquad \sup_{(z^*,\widetilde{z}^*)\in\widetilde{\mathcal{B}}} \Big\{ -\sigma_S\Big(-\sum_{jk\in I} A^*_{jk} z^*_{jk} - \sum_{ji\in\widetilde{I}} B^*_{ji}\widetilde{z}^*_{ji}\Big) - \sum_{ji\in\widetilde{I}} \langle \widetilde{z}^*_{ji}, p_i\rangle \Big\}, \tag{15}$$

where $(z^*,\widetilde{z}^*) = \Big((z^*_{jk})_{jk\in V}, (\widetilde{z}^*_{ji})_{ji\in\widetilde{V}}) \Big)$ and

$$\widetilde{\mathcal{B}} = \left\{ \begin{array}{l} \Big((z^*_{jk})_{jk\in V}, (\widetilde{z}^*_{ji})_{ji\in\widetilde{V}} \Big) \in (X^*)^{|V|} \times (X^*)^{|\widetilde{V}|} : \ I \subseteq V, \ \widetilde{I} \subseteq \widetilde{V}, \\ \sum\limits_{jk\in I} \gamma_{C^0_{jk}}(z^*_{jk}) + \sum\limits_{ji\in\widetilde{I}} \gamma_{\widetilde{C}^0_{ji}}(\widetilde{z}^*_{ji}) \leq 1, \ z^*_{jk} \in X^*, \ jk \in I, \ \widetilde{z}^*_{ji} \in X^*, \\ ji \in \widetilde{I} \text{ and } z^*_{ef} = 0_{X^*}, \ ef \in V\setminus I, \ \widetilde{z}^*_{ed} = 0_{X^*}, \ ed \in \widetilde{V}\setminus\widetilde{I} \end{array} \right\},$$

in the sense of the next theorem, where $v(\widetilde{D}^M)$ denotes the optimal objective value of the problem $(\widetilde{D}^M)$.

Theorem 3.6. *It holds* $v(D^M) = v(\widetilde{D}^M)$.

Proof. Let $(z^*,\widetilde{z}^*)$ be a feasible element to $(\widetilde{D}^M)$ and set

$$z^{1*}_{jk} = z^*_{jk}, \ z^{0*}_{jk} = \gamma_{C^0_{jk}}(z^*_{jk}) \ \text{ for } \ jk \in I, \ z^{1*}_{ef} = 0_{X^*}, \ z^{0*}_{ef} = 0 \ \text{ for } \ ef \in V\setminus I,$$

and

$$\widetilde{z}^{1*}_{ji} = \widetilde{z}^*_{ji}, \ \widetilde{z}^{0*}_{ji} = \gamma_{\widetilde{C}^0_{ji}}(\widetilde{z}^*_{ji}) \ \text{ for } \ ji \in \widetilde{I}, \ \widetilde{z}^{1*}_{ed} = 0_{X^*}, \ \widetilde{z}^{0*}_{ed} = 0 \ \text{ for } \ ed \in \widetilde{V}\setminus\widetilde{I}.$$

Then it is clear that $(z^{0*},\widetilde{z}^{0*},z^{1*},\widetilde{z}^{1*})$ is a feasible element to (D^M). Furthermore,

$$\begin{aligned} &-\sigma_S\Big(-\sum_{jk\in I} A^*_{jk} z^*_{jk} - \sum_{ji\in\widetilde{I}} B^*_{ji}\widetilde{z}^*_{ji}\Big) - \sum_{ji\in\widetilde{I}} \langle \widetilde{z}^*_{ji}, p_i\rangle \\ &\qquad = -\sigma_S\Big(-\sum_{jk\in I} A^*_{jk} z^{1*}_{jk} - \sum_{ji\in\widetilde{I}} B^*_{ji}\widetilde{z}^{1*}_{ji}\Big) - \sum_{ji\in\widetilde{I}} \langle \widetilde{z}^{1*}_{ji}, p_i\rangle \leq v(D^M), \end{aligned}$$

for all $(z^*,\widetilde{z}^*)$ feasible to $(\widetilde{D}^M)$, from which follows that $v(\widetilde{D}^M) \leq v(D^M)$.

Now, let $(z^{0*},\widetilde{z}^{0*},z^{1*},\widetilde{z}^{1*})$ be a feasible element to (D^M). By a careful look at the constraint set $\mathcal{B}$ we get by setting $z^*_{jk} = z^{1*}_{jk}$ for $jk \in I$, $\widetilde{z}^*_{ji} = \widetilde{z}^{1*}_{ji}$ for $ji \in \widetilde{I}$ and $z^*_{ef} = 0_{X^*}$ for $ef \in V\setminus I$, $\widetilde{z}^*_{ed} = 0_{X^*}$ for $ed \in \widetilde{V}\setminus\widetilde{I}$ that

$$\sum_{jk\in I} \gamma_{C^0_{jk}}(z^*_{jk}) + \sum_{ji\in\widetilde{I}} \gamma_{\widetilde{C}^0_{ji}}(\widetilde{z}^*_{ji}) \leq 1.$$

Therefore, $(z^*, \widetilde{z}^*)$ is feasible to $(\widetilde{D}^M)$ and we have

$$\begin{aligned}
&-\sigma_S\Big(-\sum_{jk\in I} A^*_{jk} z^{1*}_{jk} - \sum_{ji\in\widetilde{I}} B^*_{ji}\widetilde{z}^{1*}_{ji}\Big) - \sum_{ji\in\widetilde{I}}\langle \widetilde{z}^{1*}_{ji}, p_i\rangle \\
&\qquad = -\sigma_S\Big(-\sum_{jk\in I} A^*_{jk} z^*_{jk} - \sum_{ji\in\widetilde{I}} B^*_{ji}\widetilde{z}^*_{ji}\Big) - \sum_{ji\in\widetilde{I}}\langle \widetilde{z}^*_{ji}, p_i\rangle \le v(\widetilde{D}^M),
\end{aligned}$$

for all $(z^{0*}, \widetilde{z}^{0*}, z^{1*}, \widetilde{z}^{1*})$ feasible to (D^M), i.e. $v(D^M) \le v(\widetilde{D}^M)$, which completes the proof. $\square$

The next two theorems are direct consequences of Theorem 3.6.

Theorem 3.7. (Strong duality) *Between (P^M) and $(\widetilde{D}^M)$ holds strong duality, i.e. $v(P^M) = v(\widetilde{D}^M)$ and the dual problem has an optimal solution.*

Theorem 3.8. (Optimality conditions)

(a) *Let $\overline{x} \in S$ be an optimal solution to the problem (P^M). Then there exists an optimal solution to $(\widetilde{D}^M)$ $(\overline{z}^*, \overline{\widetilde{z}}^*) \in (X^*)^{|V|} \times (X^*)^{|\widetilde{V}|}$ with the corresponding index sets $\overline{I} \subseteq V$ and $\overline{\widetilde{I}} \subseteq \widetilde{V}$ such that*

(i) $$\begin{aligned}&\max\Big\{\big(\gamma_{C_{ef}}(\overline{x}_e - \overline{x}_f)\big)_{ef\in V},\ \Big(\gamma_{\widetilde{C}_{ed}}(\overline{x}_e - p_d)\Big)_{ed\in\widetilde{V}}\Big\} \\ &\quad = \sum_{jk\in\overline{I}} \gamma_{C^0_{jk}}(\overline{z}^*_{jk})\gamma_{C_{jk}}(\overline{x}_j - \overline{x}_k) + \sum_{ji\in\overline{\widetilde{I}}} \gamma_{\widetilde{C}^0_{ji}}(\overline{\widetilde{z}}^*_{ji})\gamma_{\widetilde{C}_{ji}}(\overline{x}_j - p_i),\end{aligned}$$

(ii) $\Big\langle \sum_{jk\in\overline{I}} A^*_{jk}\overline{z}^*_{jk} + \sum_{ji\in\overline{\widetilde{I}}} B^*_{ji}\overline{\widetilde{z}}^*_{ji}, \overline{x}\Big\rangle = -\sigma_S\Big(\sum_{jk\in\overline{I}} A^*_{jk}\overline{z}^*_{jk} + \sum_{ji\in\overline{\widetilde{I}}} B^*_{ji}\overline{\widetilde{z}}^*_{ji}\Big),$

(iii) $\gamma_{C^0_{jk}}(\overline{z}^*_{jk})\gamma_{C_{jk}}(\overline{x}_j - \overline{x}_k) = \langle \overline{z}^*_{jk}, \overline{x}_j - \overline{x}_k\rangle,\ jk \in \overline{I},$

(iv) $\gamma_{\widetilde{C}^0_{ji}}(\overline{\widetilde{z}}^*_{ji})\gamma_{\widetilde{C}_{ji}}(\overline{x}_j - p_i) = \langle \overline{\widetilde{z}}^*_{ji}, \overline{x}_j - p_i\rangle,\ ji \in \overline{\widetilde{I}},$

(v) $\max\Big\{\big(\gamma_{C_{ef}}(\overline{x}_e - \overline{x}_f)\big)_{ef\in V}, \big(\gamma_{\widetilde{C}_{ed}}(\overline{x}_e - p_d)\big)_{ed\in\widetilde{V}}\Big\} = \gamma_{C_{jk}}(\overline{x}_j - \overline{x}_k),\ jk\in\overline{I},$

(vi) $\max\Big\{\big(\gamma_{C_{ef}}(\overline{x}_e - \overline{x}_f)\big)_{ef\in V}, \big(\gamma_{\widetilde{C}_{ed}}(\overline{x}_e - p_d)\big)_{ed\in\widetilde{V}}\Big\} = \gamma_{\widetilde{C}_{ji}}(\overline{x}_j - p_i),\ ji\in\overline{\widetilde{I}},$

(vii) $\sum_{jk\in\overline{I}} \gamma_{C^0_{jk}}(\overline{z}^*_{jk}) + \sum_{ji\in\overline{\widetilde{I}}} \gamma_{\widetilde{C}^0_{ji}}(\overline{\widetilde{z}}^*_{ji}) = 1,\ \gamma_{C^0_{jk}}(\overline{z}^*_{jk}) > 0,\ jk\in\overline{I},\ \gamma_{\widetilde{C}^0_{ji}}(\overline{\widetilde{z}}^*_{ji}) > 0,$
$ji\in\overline{\widetilde{I}}$, *and* $\overline{z}^*_{ef} = 0_{X^*},\ ef\in V\setminus\overline{I},\ \overline{\widetilde{z}}^*_{ed} = 0_{X^*},\ ed\in\widetilde{V}\setminus\overline{\widetilde{I}}.$

(b) *If there exists $\overline{x} \in S$ such that for some $(\overline{z}^*, \overline{\widetilde{z}}^*)$ and the corresponding index sets $\overline{I}$ and $\overline{\widetilde{I}}$ the conditions (i)-(vii) are fulfilled, then $\overline{x}$ is an optimal solution to (P^M), $(\overline{z}^*, \overline{\widetilde{z}}^*)$ is an optimal solution to $(\widetilde{D}^M)$ and $v(P^M) = v(\widetilde{D}^M)$.*

Proof. (a) Theorem 3.7 implies for an optimal solution $\overline{x} \in S$ of (P^M) the existence of an optimal solution $(\overline{z}^*, \overline{\widetilde{z}}^*) \in (X^*)^{|V|} \times (X^*)^{|\widetilde{V}|}$ to $(\widetilde{D}^M)$ with corresponding index sets $\overline{I}$ and $\overline{\widetilde{I}}$, such that $v(P^M) = v(\widetilde{D}^M)$, i.e.

$$
\begin{aligned}
&\max\Big\{\big(\gamma_{C_{ef}}(\overline{x}_e-\overline{x}_f)\big)_{ef\in V},\ \Big(\gamma_{\widetilde{C}_{ed}}(\overline{x}_e-p_d)\Big)_{ed\in\widetilde{V}}\Big\}\\
&\qquad=-\sigma_S\Big(-\sum_{jk\in\overline{I}}A^*_{jk}\overline{z}^*_{jk}-\sum_{ji\in\overline{\widetilde{I}}}B^*_{ji}\overline{\widetilde{z}}^*_{ji}\Big)-\sum_{ji\in\overline{\widetilde{I}}}\langle\overline{\widetilde{z}}^*_{ji},p_i\rangle\\
&\Leftrightarrow\max\Big\{\big(\gamma_{C_{ef}}(\overline{x}_e-\overline{x}_f)\big)_{ef\in V},\ \Big(\gamma_{\widetilde{C}_{ed}}(\overline{x}_e-p_d)\Big)_{ed\in\widetilde{V}}\Big\}\\
&\qquad+\sigma_S\Big(-\sum_{jk\in\overline{I}}A^*_{jk}\overline{z}^*_{jk}-\sum_{ji\in\overline{\widetilde{I}}}B^*_{ji}\overline{\widetilde{z}}^*_{ji}\Big)+\sum_{ji\in\overline{\widetilde{I}}}\langle\overline{\widetilde{z}}^*_{ji},p_i\rangle=0\\
&\Leftrightarrow\max\Big\{\big(\gamma_{C_{ef}}(\overline{x}_e-\overline{x}_f)\big)_{ef\in V},\ \Big(\gamma_{\widetilde{C}_{ed}}(\overline{x}_e-p_d)\Big)_{ed\in\widetilde{V}}\Big\}\\
&\qquad+\sigma_S\Big(-\sum_{jk\in\overline{I}}A^*_{jk}\overline{z}^*_{jk}-\sum_{ji\in\overline{\widetilde{I}}}B^*_{ji}\overline{\widetilde{z}}^*_{ji}\Big)+\sum_{ji\in\overline{\widetilde{I}}}\langle\overline{\widetilde{z}}^*_{ji},p_i\rangle\\
&\qquad+\sum_{jk\in\overline{I}}[\gamma_{C^0_{jk}}(\overline{z}^*_{jk})\gamma_{C_{jk}}(\overline{x}_j-\overline{x}_k)-\langle\overline{z}^*_{jk},\overline{x}_j-\overline{x}_k\rangle]\\
&\qquad-\sum_{jk\in\overline{I}}[\gamma_{C^0_{jk}}(\overline{z}^*_{jk})\gamma_{C_{jk}}(\overline{x}_j-\overline{x}_k)-\langle\overline{z}^*_{jk},\overline{x}_j-\overline{x}_k\rangle]\\
&\qquad+\sum_{ji\in\overline{\widetilde{I}}}[\gamma_{\widetilde{C}^0_{ji}}(\overline{\widetilde{z}}^*_{ji})\gamma_{\widetilde{C}_{ji}}(\overline{x}_j-p_i)-\langle\overline{\widetilde{z}}^*_{ji},\overline{x}_j\rangle]\\
&\qquad-\sum_{ji\in\overline{\widetilde{I}}}[\gamma_{\widetilde{C}^0_{ji}}(\overline{\widetilde{z}}^*_{ji})\gamma_{\widetilde{C}_{ji}}(\overline{x}_j-p_i)-\langle\overline{\widetilde{z}}^*_{ji},\overline{x}_j\rangle]=0\\
&\Leftrightarrow\Big[\max\Big\{\big(\gamma_{C_{ef}}(\overline{x}_e-\overline{x}_f)\big)_{ef\in V},\ \Big(\gamma_{\widetilde{C}_{ed}}(\overline{x}_e-p_d)\Big)_{ed\in\widetilde{V}}\Big\}\\
&\qquad-\sum_{jk\in\overline{I}}\gamma_{C^0_{jk}}(\overline{z}^*_{jk})\gamma_{C_{jk}}(\overline{x}_j-\overline{x}_k)-\sum_{ji\in\overline{\widetilde{I}}}\gamma_{\widetilde{C}^0_{ji}}(\overline{\widetilde{z}}^*_{ji})\gamma_{\widetilde{C}_{ji}}(\overline{x}_j-p_i)\Big]\\
&\qquad+\sum_{jk\in\overline{I}}[\gamma_{C^0_{jk}}(\overline{z}^*_{jk})\gamma_{C_{jk}}(\overline{x}_j-\overline{x}_k)-\langle\overline{z}^*_{jk},\overline{x}_j-\overline{x}_k\rangle]\\
&\qquad+\sum_{ji\in\overline{\widetilde{I}}}[\gamma_{\widetilde{C}^0_{ji}}(\overline{\widetilde{z}}^*_{ji})\gamma_{\widetilde{C}_{ji}}(\overline{x}_j-p_i)-\langle\overline{\widetilde{z}}^*_{ji},\overline{x}_j-p_i\rangle]\\
&\qquad+\Big[\sigma_S\Big(-\sum_{jk\in\overline{I}}A^*_{jk}\overline{z}^*_{jk}-\sum_{ji\in\overline{\widetilde{I}}}B^*_{ji}\overline{\widetilde{z}}^*_{ji}\Big)+\langle\overline{z}^*_{jk},\overline{x}_j-\overline{x}_k\rangle+\langle\overline{\widetilde{z}}^*_{ji},\overline{x}_j\rangle\Big]=0\\
&\Leftrightarrow\Big[\max\Big\{\big(\gamma_{C_{ef}}(\overline{x}_e-\overline{x}_f)\big)_{ef\in V},\ \Big(\gamma_{\widetilde{C}_{ed}}(\overline{x}_e-p_d)\Big)_{ed\in\widetilde{V}}\Big\}\\
&\qquad-\sum_{jk\in\overline{I}}\gamma_{C^0_{jk}}(\overline{z}^*_{jk})w_{jk}\gamma_{C_{jk}}(\overline{x}_j-\overline{x}_k)-\sum_{ji\in\overline{\widetilde{I}}}\gamma_{\widetilde{C}^0_{ji}}(\overline{\widetilde{z}}^*_{ji})\widetilde{w}_{ji}\gamma_{\widetilde{C}_{ji}}(\overline{x}_j-p_i)\Big]\\
&\qquad+\sum_{jk\in\overline{I}}[\gamma_{C^0_{jk}}(\overline{z}^*_{jk})\gamma_{C_{jk}}(\overline{x}_j-\overline{x}_k)-\langle\overline{z}^*_{jk},\overline{x}_j-\overline{x}_k\rangle]
\end{aligned}
$$

$$+\sum_{ji\in\overline{\widetilde{I}}}[\gamma_{\widetilde{C}^0_{ji}}(\overline{\widetilde{z}}^*_{ji})\gamma_{\widetilde{C}_{ji}}(\overline{x}_j-p_i)-\langle\overline{\widetilde{z}}^*_{ji},\overline{x}_j-p_i\rangle]$$
$$+\Big[\sigma_S\Big(-\sum_{jk\in\overline{I}}A^*_{jk}\overline{z}^*_{jk}-\sum_{ji\in\overline{\widetilde{I}}}B^*_{ji}\overline{\widetilde{z}}^*_{ji}\Big)+\langle A^*_{jk}\overline{z}^*_{jk},\overline{x}\rangle+\langle B^*_{ji}\overline{\widetilde{z}}^*_{ji},\overline{x}\rangle\Big]=0.$$

Lemma 2.12 implies that the first bracket is non-negative, from the generalized Cauchy-Schwarz inequality (see 2) follows that the brackets in the two sums are non-negative and from the Young-Fenchel inequality we get that the last bracket is also non-negative. Hence, the statements (i)-(iv) are proved. Now, we take a careful look at the first bracket

$$\begin{aligned}
&\max\Big\{\big(\gamma_{C_{ef}}(\overline{x}_e-\overline{x}_f)\big)_{ef\in V},\ \Big(\gamma_{\widetilde{C}_{ed}}(\overline{x}_e-p_d)\Big)_{ed\in\widetilde{V}}\Big\}\\
=\ &\sum_{jk\in\overline{I}}\gamma_{C^0_{jk}}(\overline{z}^*_{jk})w_{jk}\gamma_{C_{jk}}(\overline{x}_j-\overline{x}_k)+\sum_{ji\in\overline{\widetilde{I}}}\gamma_{\widetilde{C}^0_{ji}}(\overline{\widetilde{z}}^*_{ji})\widetilde{w}_{ji}\gamma_{\widetilde{C}_{ji}}(\overline{x}_j-p_i)\\
\leq\ &\sum_{jk\in\overline{I}}\gamma_{C^0_{jk}}(\overline{z}^*_{jk})\max\Big\{\big(\gamma_{C_{ef}}(\overline{x}_e-\overline{x}_f)\big)_{ef\in V},\ \Big(\gamma_{\widetilde{C}_{ed}}(\overline{x}_e-p_d)\Big)_{ed\in\widetilde{V}}\Big\}\\
&+\sum_{ji\in\overline{\widetilde{I}}}\gamma_{\widetilde{C}^0_{ji}}(\overline{\widetilde{z}}^*_{ji})\max\Big\{\big(\gamma_{C_{ef}}(\overline{x}_e-\overline{x}_f)\big)_{ef\in V},\ \Big(\gamma_{\widetilde{C}_{ed}}(\overline{x}_e-p_d)\Big)_{ed\in\widetilde{V}}\Big\}\\
\leq\ &\max\Big\{\big(\gamma_{C_{ef}}(\overline{x}_e-\overline{x}_f)\big)_{ef\in V},\ \Big(\gamma_{\widetilde{C}_{ed}}(\overline{x}_e-p_d)\Big)_{ed\in\widetilde{V}}\Big\},
\end{aligned}$$

from which follows on the one hand that

$$\sum_{jk\in I}\gamma_{C^0_{jk}}(z^*_{jk})+\sum_{ji\in\widetilde{I}}\gamma_{\widetilde{C}^0_{ji}}(\widetilde{z}^*_{ji})=1,$$

i.e. condition (vii), and on the other hand that

$$\begin{aligned}
&\sum_{jk\in\overline{I}}\gamma_{C^0_{jk}}(\overline{z}^*_{jk})\max\Big\{\big(\gamma_{C_{ef}}(\overline{x}_e-\overline{x}_f)\big)_{ef\in V},\ \Big(\gamma_{\widetilde{C}_{ed}}(\overline{x}_e-p_d)\Big)_{ed\in\widetilde{V}}\Big\}\\
&+\sum_{ji\in\overline{\widetilde{I}}}\gamma_{\widetilde{C}^0_{ji}}(\overline{\widetilde{z}}^*_{ji})\max\Big\{\big(\gamma_{C_{ef}}(\overline{x}_e-\overline{x}_f)\big)_{ef\in V},\ \Big(\gamma_{\widetilde{C}_{ed}}(\overline{x}_e-p_d)\Big)_{ed\in\widetilde{V}}\Big\}\\
&=\sum_{jk\in\overline{I}}\gamma_{C^0_{jk}}(\overline{z}^*_{jk})\gamma_{C_{jk}}(\overline{x}_j-\overline{x}_k)+\sum_{ji\in\overline{\widetilde{I}}}\gamma_{\widetilde{C}^0_{ji}}(\overline{\widetilde{z}}^*_{ji})\gamma_{\widetilde{C}_{ji}}(\overline{x}_j-p_i)\\
\Leftrightarrow\ &\sum_{jk\in\overline{I}}\gamma_{C^0_{jk}}(\overline{z}^*_{jk})\Big[\max\Big\{\big(\gamma_{C_{ef}}(\overline{x}_e-\overline{x}_f)\big)_{ef\in V},\ \Big(\gamma_{\widetilde{C}_{ed}}(\overline{x}_e-p_d)\Big)_{ed\in\widetilde{V}}\Big\}\\
&\qquad\qquad\qquad-\gamma_{C_{jk}}(\overline{x}_j-\overline{x}_k)\Big]\\
&+\sum_{ji\in\overline{\widetilde{I}}}\gamma_{\widetilde{C}^0_{ji}}(\overline{\widetilde{z}}^*_{ji})\Big[\max\Big\{\big(\gamma_{C_{ef}}(\overline{x}_e-\overline{x}_f)\big)_{ef\in V},\Big(\gamma_{\widetilde{C}_{ed}}(\overline{x}_e-p_d)\Big)_{ed\in\widetilde{V}}\Big\}\\
&\qquad\qquad\qquad-\gamma_{\widetilde{C}_{ji}}(\overline{x}_j-p_i)\Big]=0.
\end{aligned}$$

As $\gamma_{C^0_{jk}}(\overline{z}^*_{jk}) > 0,\ jk \in \overline{I}$, as well as $\gamma_{\widetilde{C}^0_{ji}}(\overline{\widetilde{z}}^*_{ji}) > 0,\ ji \in \overline{\widetilde{I}}$, we obtain that the brackets are non-negative and must therefore be equal to zero, which finally yields the conditions (v) and (vi).

(b) All calculations done within part (a) can also be made in the reverse direction. □

Remark 3.9. Analogously to Remark 3.5 one can determine equivalent formulations of the optimality conditions given in Theorem 3.8 by using the subdifferential, which look like as follows

(i) $$\left(\left(\gamma_{C^0_{jk}}(\overline{z}^*_{jk})\right)_{jk\in V}, \left(\gamma_{\widetilde{C}^0_{ji}}(\overline{\widetilde{z}}^*_{ji})\right)_{ji\in\widetilde{V}}\right) \in \partial f\left((\gamma_{C_{ef}}(\overline{x}_e-\overline{x}_f))_{ef\in V}, (\gamma_{\widetilde{C}_{ed}}(\overline{x}_e-p_d))_{ed\in\widetilde{V}}\right),$$

(ii) $$-\sum_{jk\in\overline{I}} A^*_{jk}\overline{z}^*_{jk} - \sum_{ji\in\overline{\widetilde{I}}} B^*_{ji}\overline{\widetilde{z}}^*_{ji} \in \partial\delta_S(\overline{x}) = N_S(\overline{x}),$$

(iii) $$\overline{z}^*_{jk} \in \partial\left(\gamma_{C^0_{jk}}(\overline{z}^*_{jk})\gamma_{C_{jk}}\right)(\overline{x}_j-\overline{x}_k) = \partial\left(\gamma_{C^0_{jk}}(\overline{z}^*_{jk})\gamma_{C_{jk}}\right)(A_{jk}\overline{x})$$
$$\Leftrightarrow A^*_{jk}\overline{z}^*_{jk} \in A^*_{jk}\partial\left(\left(\gamma_{C^0_{jk}}(\overline{z}^*_{jk})\gamma_{C_{jk}}\right)\circ A_{jk}\right)(\overline{x}),\ jk\in\overline{I},$$

(iv) $$\overline{\widetilde{z}}^*_{ji} \in \partial\left(\gamma_{\widetilde{C}^0_{ji}}(\overline{\widetilde{z}}^*_{ji})\gamma_{\widetilde{C}_{ji}}\right)(\overline{x}_j-p_i) = \partial\left(\gamma_{\widetilde{C}^0_{ji}}(\overline{\widetilde{z}}^*_{ji})\gamma_{\widetilde{C}_{ji}}\right)(B_{ji}\overline{x}-p_i)$$
$$\Leftrightarrow B^*_{ji}\overline{\widetilde{z}}^*_{ji} \in B^*_{ji}\partial\left(\left(\left(\gamma_{\widetilde{C}^0_{ji}}(\overline{\widetilde{z}}^*_{ji})\gamma_{\widetilde{C}_{ji}}\right)\circ B_{ji}\right)(\cdot-p_i)\right)(\overline{x}),\ ji\in\overline{\widetilde{I}}.$$

Combining (ii)–(iv) yields

$$\sum_{jk\in\overline{I}} A^*_{jk}\overline{z}^*_{jk} + \sum_{ji\in\overline{\widetilde{I}}} B^*_{ji}\overline{\widetilde{z}}^*_{ji} \in \Bigg\{\sum_{jk\in\overline{I}} A^*_{jk}\partial\Big(\Big(\gamma_{C^0_{jk}}(\overline{z}^*_{jk})\gamma_{C_{jk}}\Big)\circ A_{jk}\Big)(\overline{x})$$
$$+\sum_{ji\in\overline{\widetilde{I}}} B^*_{ji}\partial\Big(\Big(\Big(\gamma_{\widetilde{C}^0_{ji}}(\overline{\widetilde{z}}^*_{ji})\gamma_{\widetilde{C}_{ji}}\Big)\circ B_{ji}\Big)(\cdot-p_i)\Big)(\overline{x})\Bigg\}\bigcap\left(-N_S(\overline{x})\right), \tag{16}$$

where the optimality conditions (iii) and (iv) of Theorem 3.8 give a detailed characterization of the subdifferentials of the gauges involved in 16. □

While Remark 3.1 presented a method to solve the location problem (P^M) numerically, the next remark discusses a solving technique for the dual one $(\widetilde{D}^M)$.

Remark 3.10. First, we want to underline that the introduced dual problem $(\widetilde{D}^M)$ has less dual variables as well as less constraints compared to (D^M). This fact allows not only to give a more detailed geometrical interpretation of the set of optimal solutions to $(\widetilde{D}^M)$ in the next section (see Example 4.5) but also a new numerical solving method.

Let $S = X^m$, $X = \mathcal{H}$ and set $w_{jk} \geq 0$, $\gamma_{C_{jk}}(\cdot) = w_{jk}\|\cdot\|_{\mathcal{H}}$, $jk \in J$, and $\widetilde{w}_{ji} \geq 0$, $w_{ji}\gamma_{\widetilde{C}_{ji}}(\cdot) = \|\cdot\|_{\mathcal{H}}$, $ji \in \widetilde{J}$. Additionally, we set $V = \{jk \in J : 1 \leq j < k \leq m,$ $w_{jk} > 0\}$ (note that $|V| \leq (m/2)(m-1)$) and $\widetilde{V} = \{ji \in J : w_{ji} > 0\}$, then

$\gamma_{C^0_{jk}}(\cdot) = (1/w_{jk})\|\cdot\|_{\mathcal{H}}$, $jk \in V$, $\gamma_{C^0_{ji}}(\cdot) = (1/\widetilde{w}_{ji})\|\cdot\|_{\mathcal{H}}$, $ji \in \widetilde{V}$, and the dual problem $(\widetilde{D}^M)$ transforms to (according to Remark 3.2, we omit the index sets I and $\widetilde{I}$ in the following formulation)

$$(\widetilde{D}^M) \qquad \max\left\{ -\sum_{ji\in\widetilde{V}}\langle \widetilde{z}^*_{ji}, p_i\rangle : \begin{array}{c} (z^*,\widetilde{z}^*) \in \mathcal{H}^{|V|}\times\mathcal{H}^{|\widetilde{V}|}, \\ \sum\limits_{jk\in V}\frac{1}{w_{jk}}\|z^*_{jk}\|_{\mathcal{H}} + \sum\limits_{ji\in\widetilde{V}}\frac{1}{\widetilde{w}_{ji}}\|\widetilde{z}^*_{ji}\|_{\mathcal{H}} \leq 1, \\ \sum\limits_{jk\in V} A^*_{jk}z^*_{jk} + \sum\limits_{ji\in\widetilde{V}} B^*_{ji}\widetilde{z}^*_{ji} = 0_{\mathcal{H}\times\ldots\times\mathcal{H}} \end{array} \right\}.$$

Further, by rewriting $(\widetilde{D}^M)$ into an unconstrained optimization problem of the form

$$(\widetilde{D}^M) \qquad -\min_{(z^*,\widetilde{z}^*)\in\mathcal{H}^{|V|}\times\mathcal{H}^{|\widetilde{V}|}} \left\{ \sum_{ji\in\widetilde{V}}\langle \widetilde{z}^*_{ji}, p_i\rangle + \delta_{D_1}(z^*,\widetilde{z}^*) + \delta_{D_2}(z^*,\widetilde{z}^*)\right\}, \tag{17}$$

where $D_1 = \left\{(z^*,\widetilde{z}^*)\in\mathcal{H}^{|V|}\times\mathcal{H}^{|\widetilde{V}|} : \sum\limits_{jk\in V} A^*_{jk}z^*_{jk} + \sum\limits_{ji\in\widetilde{V}} B^*_{ji}\widetilde{z}^*_{ji} = 0_{\mathcal{H}\times\ldots\times\mathcal{H}}\right\}$

and $D_2 = \left\{(z^*,\widetilde{z}^*)\in\mathcal{H}^{|V|}\times\mathcal{H}^{|\widetilde{V}|} : \sum\limits_{jk\in V}\frac{1}{w_{jk}}\|z^*_{jk}\|_{\mathcal{H}} + \sum\limits_{ji\in\widetilde{V}}\frac{1}{\widetilde{w}_{ji}}\|\widetilde{z}^*_{ji}\|_{\mathcal{H}} \leq 1\right\}$,

the minimization problem in (17) can be solved numerically by the parallel splitting algorithm [1, Proposition 27.8]. Here, one can apply the formulae given in [28, Lemma 1.1] for the projection onto a unit ball generated by the weighted sum of norms, i.e. D_2. To get the projection operator onto D_1 one can use the formulas presented in [1, Example 28.14], while the proximal operators of the linear functions involved in the objective function in (17) can simply be determined by [1, Example 28.16] and the well-known Moreau's decomposition formula given in [1, Theorem 14.3(ii)].

The procedure for reconstruction of the optimal solution to (P^M) by using an optimal solution to $(\widetilde{D}^M)$ is described in Example 4.5. □

4. Unconstrained multifacility minimax location problem in the Euclidean space

In this section we are interested in a detailed analysis of the situation when $S = X^m$ and $X = \mathbb{R}^d$ and the gauges are defined by the weighted Euclidean norm, while the sets V and $\widetilde{V}$ are given as in Remark 3.10, i.e. $V = \{jk \in J : 1 \leq j < k \leq m,\ w_{jk} > 0\}$ and $\widetilde{V} = \{ji \in J : w_{ji} > 0\}$. In other words, we will explore in the following the location problem

$$(P^M_N) \qquad \inf_{x_i\in\mathbb{R}^d,\ i=1,\ldots,m} \max\left\{(w_{jk}\|x_j - x_k\|)_{jk\in V},\ (\widetilde{w}_{ji}\|x_j - p_i\|)_{ji\in\widetilde{V}}\right\}. \tag{18}$$

For the dual of the location problem (P_N^M) we get by (15)

$$(\widetilde{D}_N^M) \qquad \sup_{(z^*,\widetilde{z}^*)\in\widetilde{\mathcal{B}}_N} \Big\{ -\sum_{ji\in\widetilde{I}} \langle \widetilde{z}^*_{ji}, p_i\rangle \Big\}, \tag{19}$$

where

$$\widetilde{\mathcal{B}}_N = \left\{ \begin{array}{l} (z^*,\widetilde{z}^*) = \Big((z^*_{jk})_{jk\in V}, (\widetilde{z}^*_{ji})_{ji\in\widetilde{V}} \Big) \in (\mathbb{R}^d)^{|V|} \times (\mathbb{R}^d)^{|\widetilde{V}|} :\ I\subseteq V,\ \widetilde{I}\subseteq\widetilde{V}, \\ z^*_{jk}\in\mathbb{R}^d,\ jk\in I,\ \widetilde{z}^*_{ji}\in\mathbb{R}^d,\ ji\in\widetilde{I},\ \sum\limits_{jk\in I}\frac{1}{w_{jk}}\|z^*_{jk}\| + \sum\limits_{ji\in\widetilde{I}}\frac{1}{\widetilde{w}_{ji}}\|\widetilde{z}^*_{ji}\| \le 1, \\ \sum\limits_{jk\in I} A^*_{jk} z^*_{jk} + \sum\limits_{ji\in\widetilde{I}} B^*_{ji}\widetilde{z}^*_{ji} = 0_{\underbrace{\mathbb{R}^d\times\ldots\times\mathbb{R}^d}_{m-\text{times}}}, \\ z^*_{ef} = 0_{\mathbb{R}^d},\ ef\in V\setminus I,\ \widetilde{z}^*_{ed} = 0_{\mathbb{R}^d},\ ed\in\widetilde{V}\setminus\widetilde{I} \end{array} \right\}.$$

The next theorems are direct consequences of the results of the previous section.

Theorem 4.1. (Strong duality) *Between (P_N^M) and $(\widetilde{D}_N^M)$ strong duality holds, i.e. $v(P_N^M) = v(\widetilde{D}_N^M)$ and the dual problem has an optimal solution.*

Theorem 4.2. (Optimality conditions)

(a) *Let $(\overline{x}_1,\ldots,\overline{x}_m)$ be an optimal solution to the problem (P_N^M). Then there exists an optimal solution to $(\widetilde{D}_N^M)$ $(\overline{z}^*,\overline{\widetilde{z}}^*)$ with the corresponding index sets $\overline{I}\subseteq V$ and $\overline{\widetilde{I}}\subseteq\widetilde{V}$ such that*

(i) $\max\Big\{ (w_{jk}\|x_j-x_k\|)_{jk\in V},\ (\widetilde{w}_{ji}\|x_j-p_i\|)_{ji\in\widetilde{V}} \Big\}$
$$= \sum_{jk\in\overline{I}} \|\overline{z}^*_{jk}\|\|\overline{x}_j-\overline{x}_k\| + \sum_{ji\in\overline{\widetilde{I}}} \|\overline{\widetilde{z}}^*_{ji}\|\|\overline{x}_j-p_i\|,$$

(ii) $\sum_{jk\in I} A^*_{jk} z^*_{jk} + \sum_{ji\in\widetilde{I}} B^*_{ji}\widetilde{z}^*_{ji} = 0_{\mathbb{R}^d\times\ldots\times\mathbb{R}^d}$,

(iii) $\|\overline{z}^*_{jk}\|\|\overline{x}_j-\overline{x}_k\| = \langle \overline{z}^*_{jk}, \overline{x}_j-\overline{x}_k\rangle,\ jk\in\overline{I}$,

(iv) $\|\overline{\widetilde{z}}^*_{ji}\|\|\overline{x}_j-p_i\| = \langle \overline{\widetilde{z}}^*_{ji}, \overline{x}_j-p_i\rangle,\ ji\in\overline{\widetilde{I}}$,

(v) $\max\Big\{ (w_{jk}\|x_j-x_k\|)_{jk\in V}, (\widetilde{w}_{ji}\|x_j-p_i\|)_{ji\in\widetilde{V}} \Big\} = w_{jk}\|\overline{x}_j-\overline{x}_k\|,\ jk\in\overline{I}$,

(vi) $\max\Big\{ (w_{jk}\|x_j-x_k\|)_{jk\in V}, (\widetilde{w}_{ji}\|x_j-p_i\|)_{ji\in\widetilde{V}} \Big\} = \widetilde{w}_{ji}\|\overline{x}_j-p_i\|,\ ji\in\overline{\widetilde{I}}$,

(vii) $\sum_{jk\in I}\frac{1}{w_{jk}}\|\overline{z}^*_{jk}\| + \sum_{ji\in\widetilde{I}}\frac{1}{\widetilde{w}_{ji}}\|\overline{\widetilde{z}}^*_{ji}\| = 1,\ \overline{z}^*_{jk}\in\mathbb{R}^d\setminus\{0_{\mathbb{R}^d}\}$ *for* $jk\in\overline{I}$,
$\overline{\widetilde{z}}^*_{ji}\in\mathbb{R}^d\setminus\{0_{\mathbb{R}^d}\}$ *for* $ji\in\overline{\widetilde{I}}$ *and* $\overline{z}^*_{jk} = 0_{\mathbb{R}^d}$ *for* $jk\in V\setminus\overline{I}$,
$\overline{\widetilde{z}}^*_{ji} = 0_{\mathbb{R}^d}$ *for* $ji\in\widetilde{V}\setminus\overline{\widetilde{I}}$.

(b) *If there exists $(\overline{x}_1,\ldots,\overline{x}_m)$ such that for some $(\overline{z}^*,\overline{\widetilde{z}}^*)$ and the corresponding index sets $\overline{I}$ and $\overline{\widetilde{I}}$ the conditions* (i)–(vii) *are fulfilled, then $\overline{x}$ is an optimal solution to (P_N^M), $(\overline{z}^*,\overline{\widetilde{z}}^*)$ is an optimal solution to $(\widetilde{D}_N^M)$ and $v(P_N^M) = v(\widetilde{D}_N^M)$.*

Remark 4.3. The dual problem $(\widetilde{D}_N^M)$ can equivalently be written in the form (see Remark 3.2)

$$(\widetilde{D}_N^M) \qquad \sup_{(z^*,\widetilde{z}^*)\in\widetilde{\mathcal{B}}_N} \Big\{ -\sum_{ji\in\widetilde{V}} \langle \widetilde{z}_{ji}^*, p_i\rangle \Big\},$$

where

$$\widetilde{\mathcal{B}}_N = \left\{ \begin{array}{l} (z^*,\widetilde{z}^*) = \Big((z_{jk}^*)_{jk\in V}, (\widetilde{z}_{ji}^*)_{ji\in\widetilde{V}}\Big) \in (\mathbb{R}^d)^{|V|}\times(\mathbb{R}^d)^{|\widetilde{V}|} : \\ \sum\limits_{jk\in V} \frac{1}{w_{jk}}\|z_{jk}^*\| + \sum\limits_{ji\in\widetilde{V}} \frac{1}{\widetilde{w}_{ji}}\|\widetilde{z}_{ji}^*\| \leq 1, \\ \sum\limits_{jk\in V} A_{jk}^* z_{jk}^* + \sum\limits_{ji\in\widetilde{V}} B_{ji}^*\widetilde{z}_{ji}^* = 0_{\underbrace{\mathbb{R}^d\times\ldots\times\mathbb{R}^d}_{m-\text{times}}} \end{array} \right\}.$$

For its corresponding Lagrange dual problem we obtain

$$\begin{aligned}
(D\widetilde{D}_N^M) \quad & \inf_{\substack{\lambda\geq 0,\\ x=(x_1,\ldots,x_m)\in\mathbb{R}^d\times\ldots\times\mathbb{R}^d}} \sup_{(z^*,\widetilde{z}^*)\in\widetilde{\mathcal{B}}_N} \Big\{ -\sum_{ji\in\widetilde{V}}\langle\widetilde{z}_{ji}^*, p_i\rangle + \\
& \Big\langle x, \sum_{jk\in V} A_{jk}^T z_{jk}^* + \sum_{ji\in\widetilde{V}} B_{ji}^T\widetilde{z}_{ji}^*\Big\rangle - \lambda\Big(\sum_{jk\in V}\frac{1}{w_{jk}}\|z_{jk}^*\| + \sum_{ji\in\widetilde{V}}\frac{1}{\widetilde{w}_{ji}}\|\widetilde{z}_{ji}^*\| - 1\Big)\Big\} \\
= & \inf_{\substack{\lambda\geq 0,\ x_i\in\mathbb{R}^d,\\ i=1,\ldots,m}} \Big\{ \lambda + \sup_{(z^*,\widetilde{z}^*)\in\widetilde{\mathcal{B}}_N} \Big\{ -\sum_{ji\in\widetilde{V}}\langle\widetilde{z}_{ji}^*, p_i\rangle \\
& + \sum_{jk\in V}\langle x, A_{jk}^T z_{jk}^*\rangle + \sum_{ji\in\widetilde{V}}\langle x, B_{ji}^T\widetilde{z}_{ji}^*\rangle - \sum_{jk\in V}\frac{\lambda}{w_{jk}}\|z_{jk}^*\| - \sum_{ji\in\widetilde{V}}\frac{\lambda}{\widetilde{w}_{ji}}\|\widetilde{z}_{ji}^*\|\Big\}\Big\} \\
= & \inf_{\substack{\lambda\geq 0,\ x_i\in\mathbb{R}^d\\ i=1,\ldots,m}} \Big\{ \lambda + \sum_{jk\in V}\sup_{z_{jk}^*\in\mathbb{R}^d}\Big\{\langle A_{jk}x, z_{jk}^*\rangle - \frac{\lambda}{w_{jk}}\|z_{jk}^*\|\Big\} \\
& \qquad + \sum_{ji\in\widetilde{V}}\sup_{\widetilde{z}_{ji}^*\in\mathbb{R}^d}\Big\{\langle B_{ji}x, \widetilde{z}_{ji}^*\rangle - \Big\langle p_i, \widetilde{z}_{ji}^*\Big\rangle\Big\} - \frac{\lambda}{w_{ji}}\|\widetilde{z}_{ji}^*\|\Big\}\Big\} \\
= & \inf_{\substack{\lambda\geq 0,\ x_i\in\mathbb{R}^d\\ i=1,\ldots,m}} \Big\{ \lambda + \sum_{jk\in V}\sup_{z_{jk}^*\in\mathbb{R}^d}\Big\{\langle x_j - x_k, z_{jk}^*\rangle - \frac{\lambda}{w_{jk}}\|z_{jk}^*\|\Big\} \\
& \qquad + \sum_{ji\in\widetilde{V}}\sup_{\widetilde{z}_{ji}^*\in\mathbb{R}^d}\Big\{\langle x_j - p_i, \widetilde{z}_{ji}^*\rangle - \frac{\lambda}{\widetilde{w}_{ji}}\|\widetilde{z}_{ji}^*\|\Big\}\Big\}.
\end{aligned}$$

The case $\lambda = 0$ leads to $x_j - p_i = 0$, $ji \in \widetilde{V}$, and $x_j - x_k = 0$, $jk \in V$, which contradicts our assumption that the given points p_i, $i = 1, \ldots, n$, are distinct, such that we can assume $\lambda > 0$. For this reason we can write for the Lagrange dual problem, or rather, the bidual of the location problem (P_N^M),

$$
\begin{aligned}
(D\widetilde{D}_N^M) \quad & \inf_{\substack{\lambda>0,\\ (x_1,\ldots,x_m)\in\mathbb{R}^d\times\ldots\times\mathbb{R}^d}} \Big\{\lambda + \sum_{jk\in V}\frac{\lambda}{w_{jk}}\sup_{z^*_{jk}\in\mathbb{R}^d}\Big\{\Big\langle \frac{w_{jk}}{\lambda}(x_j-x_k), z^*_{jk}\Big\rangle - \|z^*_{jk}\|\Big\} \\
& \qquad + \sum_{ji\in\widetilde{V}}\frac{\lambda}{\widetilde{w}_{ji}}\sup_{\widetilde{z}^*_{ji}\in\mathbb{R}^d}\Big\{\Big\langle \frac{\widetilde{w}_{ji}}{\lambda}(x_j-p_i), \widetilde{z}^*_{ji}\Big\rangle - \|\widetilde{z}^*_{ji}\|\Big\}\Big\} \\
= & \inf_{\substack{\lambda>0,\ (x_1,\ldots,x_m)\in\mathbb{R}^d\times\ldots\times\mathbb{R}^d,\\ w_{jk}\|x_j-x_k\|\leq\lambda,\ jk\in V,\ \widetilde{w}_{ji}\|x_j-p_i\|\leq\lambda,\ ji\in\widetilde{V}}} \lambda \\
= & \inf_{(x_1,\ldots,x_m)\in\mathbb{R}^d\times\ldots\times\mathbb{R}^d} \max\Big\{\Big(w_{jk}\|x_j-x_k\|\Big)_{jk\in V},\ \Big(\widetilde{w}_{ji}\|x_j-p_i\|\Big)_{ji\in\widetilde{V}}\Big\}.
\end{aligned}
$$

By using the Lagrange dual concept we transformed the dual problem $(\widetilde{D}_N^M)$ back into the multifacility minimax location problem (P_N^M), showing that one has a full symmetry between the location problem (P_N^M), its dual problem $(\widetilde{D}_N^M)$ and the Lagrange dual problem $(D\widetilde{D}_N^M)$. In addition, we see that the Lagrange multiplier associated to the equality constraint can be identified as the optimal solution of the multifacility minimax location problem (P_N^M) and the Lagrange multiplier associated to the inequality constraint as the optimal objective value. A similar fact was stated in [19] for the case of a multifacility minisum location problem.

The described symmetry between the primal, its dual and bidual problem can also be applied to general optimization problems where the underlying space is more general than the finite-dimensional one. In this context, we refer the interested reader to [6] where in addition criteria were given which secure this kind of symmetry. $\square$

The next corollary gives an estimation of the length of the vectors z^*_{jk}, $jk\in V$, and $\widetilde{z}^*_{ji}$, $ji\in\widetilde{V}$, feasible to the dual problem $(\widetilde{D}_N^M)$.

Corollary 4.4. *Let $\overline{w}_s := \max\{(w_{jk})_{jk\in V}, (w_{ji})_{ji\in\widetilde{V}}\}$, then for any feasible solution $(z^*, \widetilde{z}^*)$ of the problem $(\widetilde{D}_N^M)$ we have*

$$
\|z^*_{jk}\| \leq \frac{\overline{w}_s w_{jk}}{\overline{w}_s + w_{jk}} \ \text{ for } \ jk\in V \ \text{ and } \ \|\widetilde{z}^*_{ji}\| \leq \frac{\overline{w}_s w_{ji}}{\overline{w}_s + w_{ji}} \ \text{ for } \ ji\in\widetilde{V}.
$$

Proof. As $(z^*, \widetilde{z}^*)$ is a feasible solution of $(\widetilde{D}_N^M)$, we have

$$
\begin{aligned}
& \sum_{jk\in V} A^*_{jk} z^*_{jk} + \sum_{ji\in\widetilde{V}} B^*_{ji}\widetilde{z}^*_{ji} = 0_{\mathbb{R}^d\times\ldots\times\mathbb{R}^d} \\
\Leftrightarrow \quad & -A^*_{uv}z^*_{uv} = \sum_{\substack{jk\in V,\\ jk\neq uv}} A^*_{jk}z^*_{jk} + \sum_{ji\in\widetilde{V}} B^*_{ji}\widetilde{z}^*_{ji} \\
\Rightarrow \quad & \|A^*_{uv}z^*_{uv}\| = \|\sum_{\substack{jk\in V,\\ jk\neq uv}} A^*_{jk}z^*_{jk} + \sum_{ji\in\widetilde{V}} B^*_{ji}\widetilde{z}^*_{ji}\|
\end{aligned}
$$

$$\begin{aligned}
&\Rightarrow \ \|A^*_{uv} z^*_{uv}\| \leq \sum_{\substack{jk\in V,\\ jk\neq uv}} \|A^*_{jk} z^*_{jk}\| + \sum_{ji\in\widetilde{V}} \|B^*_{ji}\widetilde{z}^*_{ji}\| \\
&\Leftrightarrow \ \sqrt{2}\|z^*_{uv}\| \leq \sum_{\substack{jk\in V,\\ jk\neq uv}} \sqrt{2}\|z^*_{jk}\| + \sum_{ji\in\widetilde{V}} \|\widetilde{z}^*_{ji}\| \\
&\Leftrightarrow \ \|z^*_{uv}\| \leq \sum_{\substack{jk\in V,\\ jk\neq uv}} \|z^*_{jk}\| + \frac{1}{\sqrt{2}}\sum_{ji\in\widetilde{V}} \|\widetilde{z}^*_{ji}\| \\
&\Rightarrow \ \|z^*_{uv}\| \leq \sum_{\substack{jk\in V,\\ jk\neq uv}} \|z^*_{jk}\| + \sum_{ji\in\widetilde{V}} \|\widetilde{z}^*_{ji}\|, \ uv \in V,
\end{aligned}$$

and more than that, it holds

$$\begin{aligned}
1 \ &\geq \ \sum_{jk\in V}\frac{1}{w_{jk}}\|z^*_{jk}\| + \sum_{ji\in\widetilde{V}}\frac{1}{\widetilde{w}_{ji}}\|\widetilde{z}^*_{ji}\| = \frac{1}{w_{uv}}\|z^*_{uv}\| + \sum_{\substack{jk\in V,\\ jk\neq uv}}\frac{1}{w_{jk}}\|z^*_{jk}\| + \sum_{ji\in\widetilde{V}}\frac{1}{\widetilde{w}_{ji}}\|\widetilde{z}^*_{ji}\| \\
&\geq \ \frac{1}{w_{uv}}\|z^*_{uv}\| + \frac{1}{\overline{w}_s}\Bigg(\sum_{\substack{jk\in V,\\ jk\neq uv}}\|z^*_{jk}\| + \sum_{ji\in\widetilde{V}}\|\widetilde{z}^*_{ji}\|\Bigg) \geq \frac{1}{w_{uv}}\|z^*_{uv}\| + \frac{1}{\overline{w}_s}\|z^*_{uv}\| \\
&= \ \frac{\overline{w}_s + w_{uv}}{\overline{w}_s w_{uv}}\|z^*_{uv}\|,
\end{aligned}$$

which means that $$\|z^*_{jk}\| \leq \frac{\overline{w}_s w_{jk}}{\overline{w}_s + w_{jk}}, \ \ jk \in V.$$

In the same way, we get $$\|\widetilde{z}^*_{ji}\| \leq \frac{\overline{w}_s w_{ji}}{\overline{w}_s + w_{ji}}, \ \ ji \in \widetilde{V}.$$ □

Example 4.5. We consider the existing facilities $p_1 = (0,0)^T$, $p_2 = (-2,3)^T$ and $p_3 = (5,8)^T$ $(t=3)$. We want to locate two new facilities $(m=2)$ in the plane $(d=2)$. The weights are given by $w_{12} = \widetilde{w}_{11} = \widetilde{w}_{13} = \widetilde{w}_{21} = \widetilde{w}_{22} = 1$ and $\widetilde{w}_{12} = \widetilde{w}_{23} = 0$. We define the following multifacility minimax location problem:

$$(P^M_N) \quad \inf_{(x_1,x_2)\in\mathbb{R}^2\times\mathbb{R}^2} \max\left\{\|x_1 - x_2\|, \|x_1 - p_1\|, \|x_1 - p_3\|, \|x_2 - p_1\|, \|x_2 - p_2\|\right\},$$

i.e. $V = \{12\}$, $|V| = 1$, $\widetilde{V} = \{11, 13, 21, 22\}$ and $|\widetilde{V}| = 4$. From the Matlab Optimization Toolbox we obtained the following solution $\overline{x}_1 = (2.5, 4)$ and $\overline{x}_2 = (0,0)^T$. The corresponding objective value was $v(P^M_N) = 4.72$.

The dual problem (see Remark 3.2)

$$(\widetilde{D}^M_N) \quad \max_{(z^*_{12},\widetilde{z}^*_{11},\widetilde{z}^*_{13},\widetilde{z}^*_{21},\widetilde{z}^*_{22})\in\widetilde{\mathcal{B}}_N} \left\{\langle\widetilde{z}^*_{11} + \widetilde{z}^*_{21}, p_1\rangle + \langle\widetilde{z}^*_{22}, p_2\rangle + \langle\widetilde{z}^*_{13}, p_3\rangle\right\},$$

where

$$\widetilde{\mathcal{B}}_N = \left\{\begin{array}{c} (z^*_{12}, \widetilde{z}^*_{11}, \widetilde{z}^*_{13}, \widetilde{z}^*_{21}, \widetilde{z}^*_{22}) \in \mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2 : \\ z^*_{12} + \widetilde{z}^*_{11} + \widetilde{z}^*_{13} = 0_{\mathbb{R}^2}, \ \widetilde{z}^*_{21} + \widetilde{z}^*_{22} = 0_{\mathbb{R}^2}, \\ \|z^*_{12}\| + \|\widetilde{z}^*_{11}\| + \|\widetilde{z}^*_{21}\| + \|\widetilde{z}^*_{22}\| + \|\widetilde{z}^*_{13}\| \leq 1 \end{array}\right\},$$

was also solved by the Matlab Optimization Toolbox. The following solution was obtained

$$\overline{z}^*_{12} = \overline{\overline{z}}^*_{11} = (0.13, 0.21)^T,\ \overline{\overline{z}}^*_{13} = (-0.26, -0.42)^T,\ \overline{\overline{z}}^*_{21} = \overline{\overline{z}}^*_{22} = (0, 0)^T,$$

with the corresponding objective value $v(\widetilde{D}^M_N) = 4.72 = v(P^M_N)$, i.e. $\overline{I} = \{12\} \subseteq V$ and $\overline{\overline{I}} = \{11, 13\} \subseteq \widetilde{V}$.

In the situation when we have only the solution of the dual problem one can reconstruct the optimal solution of the primal problem in a recursive way by using the necessary and sufficient optimality conditions given in Theorem 4.2. By condition (iv) we know that there exists $\widetilde{\alpha}_{11} > 0$ such that

$$\overline{\overline{z}}^*_{11} = \widetilde{\alpha}_{11}(\overline{x}_1 - p_1),\ \text{i.e.}\ \|\overline{\overline{z}}^*_{11}\| = \widetilde{\alpha}_{11}\|\overline{x}_1 - p_1\|, \tag{20}$$

and as, by condition (vi) it holds

$$v(\widetilde{D}^M_N) = v(P^M_N) = \|\overline{x}_1 - p_1\| = \frac{\|\overline{\overline{z}}^*_{11}\|}{\widetilde{\alpha}_{11}}, \tag{21}$$

we get by combining (20) and (21) that

$$\overline{\overline{z}}^*_{11} = \frac{\|\overline{\overline{z}}^*_{11}\|}{v(\widetilde{D}^M_N)}(\overline{x}_1 - p_1) \ \Leftrightarrow\ \overline{x}_1 = \frac{v(\widetilde{D}^M_N)}{\|\overline{\overline{z}}^*_{11}\|}\overline{\overline{z}}^*_{11} + p_1 = \frac{4.72}{0.25}(0.13, 0.21)^T = (2.5, 4)^T.$$

More than that, by condition (iii) there exists $\alpha_{12} > 0$ such that

$$\overline{z}^*_{12} = \alpha_{12}(\overline{x}_1 - \overline{x}_2),\ \text{i.e.}\ \|\overline{z}^*_{12}\| = \alpha_{12}\|\overline{x}_1 - \overline{x}_2\|, \tag{22}$$

and therefore, we derive from condition (v) that

$$v(\widetilde{D}^M_N) = v(P^M_N) = \|\overline{x}_1 - \overline{x}_2\| = \frac{\|\overline{z}^*_{12}\|}{\alpha_{12}}. \tag{23}$$

Finally, taking (22) and (23) together yields

$$\begin{aligned} \overline{z}^*_{12} &= \frac{\|\overline{z}^*_{12}\|}{v(\widetilde{D}^M_N)}(\overline{x}_1 - \overline{x}_2) \\ \Leftrightarrow \quad \overline{x}_2 = \overline{x}_1 - \frac{v(\widetilde{D}^M_N)}{\|\overline{z}^*_{12}\|}\overline{z}^*_{12} &= (2.5, 4)^T - \frac{4.72}{0.25}(0.13, 0.21)^T = (0, 0)^T. \end{aligned}$$

For a geometrical illustration see Figure 4.1. $\square$

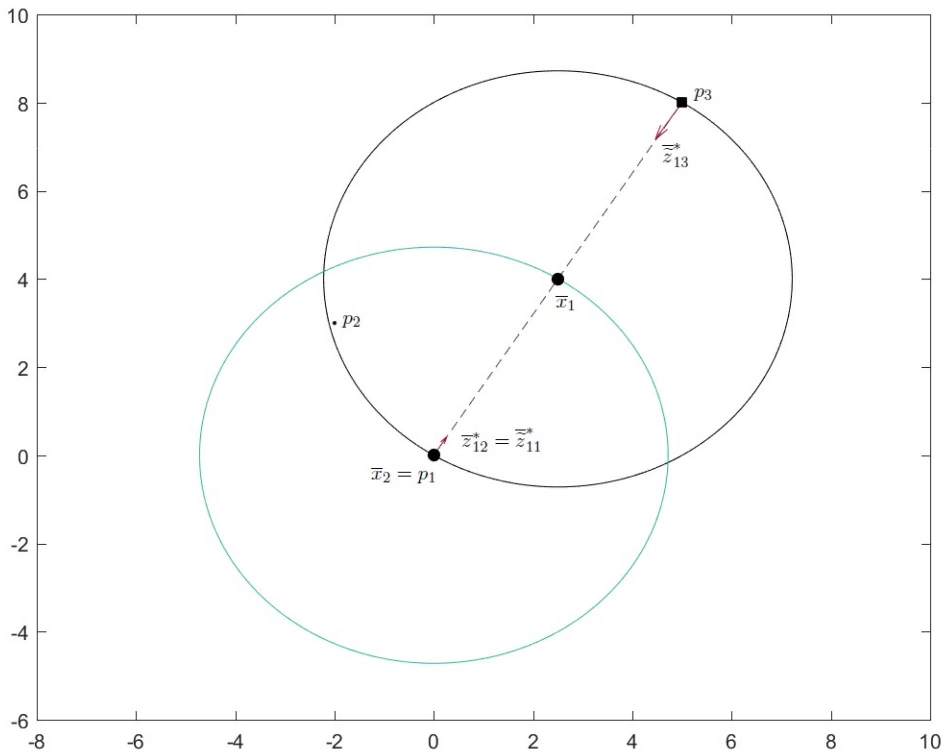

Figure 4.1: Illustration of Example 4.5

Geometrical interpretation

In the following we provide a geometrical characterization of the set of optimal solutions of the dual problem by Theorem 4.2. By the conditions (iii) and (iv) it is clear that for $jk \in \overline{I}$ and $ji \in \overline{\overline{I}}$ the vectors $\overline{z}^*_{jk}$ and $\overline{\overline{z}}^*_{ji}$ are parallel to the vectors $\overline{x}_j - \overline{x}_k$ and $\overline{x}_j - p_i$ directed to $\overline{x}_j$, respectively. In addition, if we take into account the conditions (v), (vi) and (vii), then it is also evident that $jk \in \overline{I}$ and $ji \in \overline{\overline{I}}$, i.e. $\overline{z}^*_{jk} \neq 0_{\mathbb{R}^d}$ and $\overline{\overline{z}}^*_{ji} \neq 0_{\mathbb{R}^d}$, if the points $\overline{x}_k$ and p_i are lying on the border of the minimum covering ball with radius $v(P^M_N)$ centered in $\overline{x}_j$, respectively. Vice versa, if $jk \in V \setminus \overline{I}$ and $ji \in \widetilde{V} \setminus \overline{\overline{I}}$, then $\overline{z}^*_{jk} = 0_{\mathbb{R}^d}$ and $\overline{\overline{z}}^*_{ji} = 0_{\mathbb{R}^d}$, which is exactly the case when the corresponding weights are zero or the points $\overline{x}_k$ and p_i are lying inside the minimum covering ball centered in $\overline{x}_j$, respectively. Therefore, analogously to the geometrical interpretation presented in [27] for single minimax location problems, one can identify the vectors $\overline{z}^*_{jk}$, $jk \in \overline{I}$, and $\overline{\overline{z}}^*_{ji}$, $ji \in \overline{\overline{I}}$, as force vectors, which pull the points lying on the borders of the minimum covering balls inside the balls in direction to the their corresponding centers, the gravity points $\overline{x}_j$ (see Figure 4.1).

Acknowledgements. The authors are grateful to the anonymous referee for comments and suggestions that have improved the quality of the article.

References

[1] H. H. Bauschke, P. L. Combettes: *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, CMS Books in Mathematics, Springer, New York (2011).

[2] R. I. Boţ, S.-M. Grad, G. Wanka: *Duality in Vector Optimization*, Springer-Verlag, Berlin (2009).

[3] R. I. Boţ, E. Vargyas, G. Wanka: *Duality for location problems with unbounded unit balls*, Eur. J. Oper. Research 179 (2007) 1252–1265.

[4] R. I. Boţ, G. Wanka: *Duality for composed convex functions with applications in location theory*, in: Multi-Criteria- und Fuzzy-Systeme in Theorie und Praxis, Deutscher Universitäts-Verlag, Wiesbaden (2003) 1–28.

[5] R. Durier, C. Michelot: *Geometrical properties of the Fermat-Weber problem*, Eur. J. Oper. Research 20 (1985) 332–343.

[6] I. Ekeland, R. Temam: *Convex Analysis and Variational Problems*, North-Holland, Amsterdam (1976).

[7] J. Elzinga, D. W. Hearn: *Geometrical solutions for some minimax location problems*, Transportation Science 6 (1972) 379–394.

[8] J. Elzinga, D. W. Hearn: *The minimum covering sphere problem*, Management Science 19 (1972) 96–104.

[9] J. Elzinga, D. Hearn, W. Randolph: *Minimax multifacility location with Euclidean distances*, Transportation Science 10 (1976) 321–336.

[10] J. Fliege: *Coincidence conditions in multifacility location problems with positive and negative weights*, Eur. J. Oper. Research 104 (1998) 310–320.

[11] R. L. Francis: *Some aspects of a minimax location problem*, Operations Research 15 (1967) 1163–1169.

[12] S.-M. Grad, G. Wanka, O. Wilfer: *Duality and ε-optimality conditions for multi-composed optimization problems with applications to fractional and entropy optimization*, Pure Appl. Functional Analysis 2 (2017) 43–63.

[13] S.-M. Grad, O. Wilfer: *A proximal method for solving nonlinear minmax location problems with perturbed minimal time functions via conjugate duality*, J. Global Optimization 74(1) (2019) 121–160.

[14] Y. Hinojosa, J. Puerto: *Single facility location problems with unbounded unit balls*, Math. Methods Oper. Research 58 (2003) 87–104.

[15] H. Idrissi, O. Lefebvre, C. Michelot: *Duality constrained multifacility location problems with mixed norms and applications*, Annals Oper. Research 18 (1989) 71–92.

[16] T. R. Jefferson, S. Jorjani, C. H. Scott: *Quality locations for the constrained minimax location model*, Int. J. Systems Science 24 (1993) 1009–1016.

[17] T. R. Jefferson, C. H. Scott: *Duality for minmax programs*, J. Math. Analysis Appl. 100 (1984) 385–392.

[18] H. Juel, R. F. Love: *Duality in constrained location problems*, Operations Res. Letters 6 (1987) 281–284.

[19] H. Juel, R. F. Love: *On the dual of the linearly constrained multi-facility location problem with arbitrary norms*, Transportation Science 15 (1981) 329–337.

[20] H. Juel, R. F. Love: *The dual of a generalized minimax location problem*, Annals Oper. Research 40 (1992) 261–264.

[21] H. W. Kuhn: *A note on Fermat's problem*, Math. Programming 4 (1973) 98–107.

[22] T. Matutomi, H. Ishii: *Minimax location problem with A-distance*, J. Oper. Res. Soc. Japan 41 (1998) 181–185.

[23] C. Michelot, F. Plastria: *An extended multifacility minimax location problem revisited*, Annals Oper. Research 111 (2002) 167–179.

[24] B. A. Murtagh, C. H. Scott, E. Sirri: *Solution of constrained minmax location with Euclidean distances via conjugate duality*, New Zealand Oper. Research 13 (1985) 61–67.

[25] G. Wanka, U. Krallert: *Duality for optimal control-approximation problems with gauges*, J. Analysis Appl. 18 (1999) 491–504.

[26] G. Wanka, O. Wilfer: *A Lagrange duality approach for multi-composed optimization problems*, TOP 25 (2017) 288–313.

[27] G. Wanka, O. Wilfer: *Duality results for nonlinear single minimax location problems via multi-composed optimization*, Math. Methods Oper. Research 86 (2017) 401–439.

[28] G. Wanka, O. Wilfer: *Formulae of epigraphical projection for solving minimax location problems*, Preprint 2017-5, Faculty of Mathematics, Chemnitz University of Technology (2017).