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Sion's Minimax Theorem and Nash Equilibria of Symmetric Multi-Players Zero-Sum Games with Continuous Strategies

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About a symmetric multi-players zero-sum game with continuous strategies we will show the following results.

(1) A modified version of Sion's minimax theorem with the coincidence of the maximin strategy and the minimax strategy are proved by the existence of a symmetric Nash equilibrium.

(2) The existence of a symmetric Nash equilibrium is proved by the modified version of Sion's minimax theorem with the coincidence of the maximin strategy and the minimax strategy.

Thus, they are equivalent. If a zero-sum game is asymmetric, maximin strategies and minimax strategies of players may not correspond to Nash equilibrium strategies. However, if it is symmetric, the maximin strategies and the minimax strategies constitute a Nash equilibrium.

Keywords: Multi-players zero-sum game, Nash equilibrium, Sion's minimax theorem, Cournot oligopoly.

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1. Introduction

We consider the relation between Sion's minimax theorem for a continuous function and the existence of Nash equilibrium in a symmetric multi-players zero-sum game with continuous strategies. A zero-sum game is a game in which the sum of the payoffs of all players is zero (or constant). Meaning of a symmetric game is explained in Section 3. It is a game in which all players are treated equal.

We will show the following results.

1. A modified version of Sion's minimax theorem with the coincidence of the maximin strategy and the minimax strategy are proved by the existence of a symmetric Nash equilibrium.

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2. The existence of a symmetric Nash equilibrium is proved by the modified version of Sion's minimax theorem with the coincidence of the maximin strategy and the minimax strategy.

Thus, they are equivalent. A symmetric Nash equilibrium is a Nash equilibrium of a game in which all players choose the same strategy.

An example of such a game is a relative profit maximization game in a Cournot oligopoly. Suppose that there are $n \geq 3$ firms in an oligopolistic industry. Let $\bar{\pi}_i$ be the absolute profit of the i-th firm. Then, its relative profit is

$$\pi_i = \bar{\pi}_i - \frac{1}{n-1} \sum_{j=1, j \neq i}^{n} \bar{\pi}_j.$$

We see

$$\sum_{i=1}^{n} \pi_i = \sum_{i=1}^{n} \bar{\pi}_i - \frac{1}{n-1}(n-1) \sum_{j=1}^{n} \bar{\pi}_j = 0.$$

Thus, the relative profit maximization game in a Cournot oligopoly is a zero-sum game[1]. If the oligopoly is asymmetric because the demand function is not symmetric (in a case of differentiated goods) or firms have different cost functions (in both homogeneous and differentiated goods cases), maximin strategies and minimax strategies of firms may not correspond to Nash equilibrium strategies. However, if the demand function is symmetric and the firms have the same cost function, the maximin strategies and the minimax strategies constitute a Nash equilibrium.

In Section 2 we present an example of a three-firms relative profit maximizing oligopoly. In Section 3 the model and the minimax theorem are described, and in Section 4 we show the main results. In Section 5 we mention the problem about a non-zero sum game.

2. Example of a multi-players zero-sum game

Consider a three-players game. Suppose that the payoff function of Player i, where $i = 1, 2, 3$, is

$$\pi_i = \left(a - \sum_i s_i\right) s_i - c_i s_i^2 - \frac{1}{2} \sum_{j \neq i} \left[\left(a - \sum_i s_i\right) s_j - c_j s_j^2\right].$$

This is a model of relative profit maximization in a three firms Cournot oligopoly with quadratic cost functions producing a homogeneous good. s_i is the output of the firm. If $c_1 = c_2 = c_3$, this game is symmetric. It is a zero-sum game because

$$\sum_i \pi_i = 0.$$

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[1] About relative profit maximization under imperfect competition please see [4], [5], [6], [7], [10], [11] and [12]

The condition for maximization of $\pi_i,\ i=1,2,3$, is

$$\frac{\partial \pi_i}{\partial s_i} = a - 2(1+c_i)s_i - \frac{1}{2}\sum_{j\neq i} s_j = 0.$$

The Nash equilibrium strategy is

$$s_i = \frac{a\prod_{j\neq i}(4c_j+3)}{32\prod_i c_i + 32\sum_i \prod_{j\neq i} c_j + 30\sum_i c_i + 27}.$$

If $c_2 = c_3 = c_1$, then $s_1 = s_2 = s_3 = \frac{a}{2c_1+3}$, and we have a *symmetric Nash equilibrium.*

We consider maximin and minimax strategy about Players i and j. The condition for minimization of π_i with respect to s_j is $\frac{\partial \pi_i}{\partial sj} = 0$. Denote s_j which satisfies this condition by $s_j(s_i, s_k)$, $k \neq i, j$, and substitute it into π_i. Then, the condition for maximization of π_i with respect to s_i given $s_j(s_i, s_k)$ and s_k is

$$\frac{\partial \pi_i}{\partial s_i} + \frac{\partial \pi_i}{\partial s_j}\frac{\partial s_j}{\partial s_i} = 0.$$

We call the strategy of Player i obtained from these conditions the *maximin strategy* of Player i to Player j. It is denoted by $\arg\max_{s_i}\min_{s_j}\pi_i$. The condition for maximization of π_i with respect to s_i is $\frac{\partial \pi_i}{\partial s_i} = 0$. Denote s_i which satisfies this condition by $s_i(s_j, s_k)$, and substitute it into π_i. Then, the condition for minimization of π_i with respect to s_j given $s_i(s_j, s_k)$ is

$$\frac{\partial \pi_i}{\partial s_j} + \frac{\partial \pi_i}{\partial s_i}\frac{\partial s_i}{\partial s_j} = 0.$$

We call the strategy of Player j obtained from these conditions the *minimax strategy* of Player j to Player i. It is denoted by $\arg\min_{s_j}\max_{s_i}\pi_i$. In our example we obtain

$$\arg\max_{s_i}\min_{s_j}\pi_i = \frac{3a-2c_js_k+4ac_j}{8c_ic_j+8c_i+8c_j+9},\quad \arg\min_{s_j}\max_{s_i}\pi_i = \frac{6a-8c_is_k-9s_k+4ac_i}{8c_ic_j+8c_i+8c_j+9}.$$

If the game is fully asymmetric, that is, $c_1 \neq c_2 \neq c_3$, then

$$\arg\max_{s_i}\min_{s_j}\pi_i \neq \arg\max_{s_i}\min_{s_k}\pi_i,$$

that is, the maximin strategy for Player i to Player j is not equivalent to the maximin strategy for Player i to Player k. Also we have

$$\arg\min_{s_j}\max_{s_i}\pi_i \neq \arg\min_{s_k}\max_{s_i}\pi_i,$$

that is, the minimax strategy for Player j to Player i is not equivalent to the minimax strategy for Player k to Player i. Therefore, at least one of maximin strategies and one of minimax strategies are not equivalent to the Nash equilibrium strategy[2].

If the game is symmetric, and $c_2 = c_3 = c_1$,

$$\arg\max_{s_i}\min_{s_j}\pi_i = \frac{3a-2c_1s_k+4ac_1}{8c_1^2+16c_1+9},\quad \arg\min_{s_j}\max_{s_i}\pi_i = \frac{6a-8c_1s_k-9s_k+4ac_1}{8c_1^2+16c_1+9}.$$

Let

$$s_i = \arg\max_{s_i}\min_{s_j}\pi_i.$$

From the above equations, for all combinations of $i,\ j,\ k$, we obtain

$$\arg\max_{s_i}\min_{s_j}\pi_i = \frac{a}{2c_1+3}.$$

Similarly, we get $\arg\min_{s_j}\max_{s_i}\pi_i = \frac{a}{2c_1+3}$. Therefore, the maximin strategies, the minimax strategies and the Nash equilibrium strategies for all players are equal.

We have got an inspiration for the main results of this paper from these analyses of relative profit maximization. In the following sections we examine under a more general model whether the same results as this example hold.

3. The model and Sion's minimax theorem

Consider a symmetric n-players zero-sum game with $n \geq 3$ as follows. There are n players, 1, 2, ..., n. The set of players is denoted by N. A vector of strategic variables is $(s_1, s_2, \dots, s_n) \in S_1 \times S_2 \times \dots \times S_n$. S_i is a convex and compact set in a linear topological space for each $i \in N$. The payoff function of each player is $u_i(s_1, s_2, \dots, s_n)$ for $i \in N$. Since this game is a zero-sum game, the payoff functions satisfy

$$\sum_{i=1}^{n} u_i(s_1, s_2, \dots, s_n) = 0,$$

for any $(s_1, s_2, \dots, s_n)$. We also assume

> u_i for each $i \in N$ is continuous on $S_1 \times S_2 \times \dots \times S_n$, *strictly quasi-concave* on S_i for each $s_j \in S_j,\ j \in N,\ j \neq i$, and *strictly quasi-convex* on S_j for $j \in N,\ j \neq i$ for each $s_i \in S_i$.

Symmetry of the game means the following facts.

1. If $s_1 = s_2 = \dots = s_n$, then $u_1(s_1, s_2, \dots, s_n) = u_2(s_1, s_2, \dots, s_n) = \dots = u_n(s_1, s_2, \dots, s_n)$.
2. In the payoff function of each Player i, Players j and k, $j, k \neq i$, are interchangeable, for example,

$$u_i(s_1, \dots, s_j, s_k, \dots, s_n) = u_i(s_1, \dots, s_k, s_j, \dots, s_n).$$

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[2] If a game is partially asymmetric, for example, $c_1 = c_2 \neq c_3$, maximin and minimax strategies may correspond to Nash equilibrium strategies in some sense ([8]).

Sion's minimax theorem ([2], [3], [9]) for a continuous function can be stated as follows.

Lemma 3.1. *Let X and Y be non-void convex and compact subsets of two linear topological spaces, and let $f\colon X \times Y \to \mathbb{R}$ be a function that is continuous and quasi-concave in the first variable and continuous and quasi-convex in the second variable. Then*

$$\max_{x\in X}\min_{y\in Y} f(x,y) = \min_{y\in Y}\max_{x\in X} f(x,y).$$

We follow the description of this theorem in [2].

Suppose that $s_k \in S_k$ for all $k \in N$ other then i and j, $j \neq i$, are given. Denote a vector of such s_k's by $s_{-i,j}$. Then, $u_i(s_1, s_2, \dots, s_n)$ is written as $u_i(s_i, s_j, s_{-i,j})$, and it is a function of s_i and s_j. We can apply Lemma 3.1 to such a situation, and get the following equation.

$$\max_{s_i\in S_i}\min_{s_j\in S_j} u_i(s_i, s_j, s_{-i,j}) = \min_{s_j\in S_j}\max_{s_i\in S_i} u_i(s_i, s_j, s_{-i,j}). \tag{1}$$

We assume strict quasi-concavity and strict quasi-convexity of payoff functions instead of quasi-concavity and quasi-convexity. They mean the uniqueness of $\arg\max_{s_i\in S_i}\min_{s_j\in S_j} u_i(s_i, s_j, s_{-i,j})$ and $\arg\min_{s_j\in S_j}\max_{s_i\in S_i} u_i(s_i, s_j, s_{-i,j})$, that is, they are single-valued for any pair of i and j. By the maximum theorem they are continuous in $s_{-i,j}$. Also the best response of each player in any situation is unique. Since we consider a symmetric game, we can assume that when $s_{-i,j} = s_{-k,l}$ (1) means

$$\begin{aligned}&\max_{s_i\in S_i}\min_{s_j\in S_j} u_i(s_i, s_j, s_{-i,j}) = \max_{s_k\in S_k}\min_{s_l\in S_l} u_k(s_k, s_l, s_{-k,l})\\ &= \min_{s_l\in S_l}\max_{s_k\in S_k} u_k(s_k, s_l, s_{-k,l}) = \min_{s_j\in S_j}\max_{s_i\in S_i} u_i(s_i, s_j, s_{-i,j}),\end{aligned}$$

$$\arg\max_{s_i\in S_i}\min_{s_j\in S_j} u_i(s_i, s_j, s_{-i,j}) = \arg\max_{s_k\in S_k}\min_{s_l\in S_l} u_k(s_k, s_l, s_{-k,l}),$$

$$\arg\min_{s_l\in S_l}\max_{s_k\in S_k} u_k(s_k, s_l, s_{-k,l}) = \arg\min_{s_j\in S_j}\max_{s_i\in S_i} u_i(s_i, s_j, s_{-i,j}) \text{ for } i,j,k,l \in N.$$

4. The main results

First we prove the existence of a symmetric Nash equilibrium in a symmetric game.

Lemma 4.1. *In a symmetric game there exists a symmetric Nash equilibrium.*

Proof. Consider a symmetric n-players game. There are Players 1, 2, ..., n. The strategies, the set of strategies and the payoff functions of the players are the same as those above defined.

Suppose a symmetric situation. Assume $(s_1, s_2, \dots, s_n) = (s, s, \dots, s)$, and let $\mathbf{s}_{-i}$ be a vector of $s_j,\ j \in N,\ j \neq i$ such that $s_j = s$. Consider the following function;

$$s \to \arg\max_{s_i \in S_i} u_i(s_i, \mathbf{s}_{-i}).$$

Since u_i is continuous and S_i is compact, this function is also continuous. Thus, by the Glicksberg fixed point theorem ([1]) there exists a fixed point. Denote it by s^*. It satisfies

$$s^* = \arg\max_{s_i \in S} u_i(s_i, \mathbf{s}^*_{-i}), \quad i \in \{1, 2, \dots, n\}. \tag{2}$$

Here, $\mathbf{s}^*_{-i}$ is a vector of $s_j,\ j \in N,\ j \neq i$ such that $s_j = s^*$. Equation (2) means that $(s_1, s_2, \dots, s_n) = (s^*, s^*, \dots, s^*)$ is a symmetric Nash equilibrium. □

We use the following result.

Theorem 4.2. (Modified version of Sion's minimax theorem) *Let $j \neq i$, and S_i and S_j be non-void convex and compact subsets of two linear topological spaces, let $u_i \colon S_i \times S_j \to \mathbb{R}$ given $\mathbf{s}_{-i,j}$ be a function that is continuous on $S_1 \times S_2 \times \cdots \times S_n$, strictly quasi-concave on S_i and strictly quasi-convex on S_j. Then, there exists $\tilde{s}$ such that*

$$\max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}) = \min_{s_j \in S_j} \max_{s_i \in S_i} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}) \text{ for any } i, j,$$

and $$\arg\max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}) = \arg\min_{s_j \in S_j} \max_{s_i \in S_i} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}) = \tilde{s}.$$

The following Theorem 4.3 shows that the existence of a symmetric Nash equilibrium means this modified version of Sion's minimax theorem, and Theorem 4.4 shows that this result with the coincidence of the maximin strategy and the minimax strategy imply the existence of a symmetric Nash equilibrium.

Theorem 4.3. *The existence of a symmetric Nash equilibrium in a symmetric multi-players zero-sum game implies the modified version of Sion's minimax theorem with the coincidence of the maximin strategy and the minimax strategy at the symmetric Nash equilibrium.*

Proof. Let $(s^*, s^*, \dots, s^*)$ be a symmetric Nash equilibrium of an n-players zero-sum game. Then,

$$u_i(s^*, s^*, \mathbf{s}^*_{-i,j}) = \max_{s_i \in S_i} u_i(s_i, s^*, \mathbf{s}^*_{-i,j}) \geq u_i(s_i, s^*, \mathbf{s}^*_{-i,j}). \tag{3}$$

$\mathbf{s}^*_{-i,j}$ is a vector of $s_k,\ k \in N,\ k \neq i, j$ such that $s_k = s^*$. Since the game is zero-sum,

$$u_i(s_i, s^*, \mathbf{s}^*_{-i,j}) + (n-1)u_j(s_i, s^*, \mathbf{s}^*_{-i,j}) = 0,\ j \neq i$$

implies

$$u_i(s_i, s^*, \mathbf{s}^*_{-i,j}) = -(n-1)u_j(s_i, s^*, \mathbf{s}^*_{-i,j}).$$

This equation holds for any s_i. Thus,

$$\arg\max_{s_i\in S_i} u_i(s_i, s^*, \mathbf{s}^*_{-i,j}) = \arg\min_{s_i\in S_i} u_j(s_i, s^*, \mathbf{s}^*_{-i,j}) = s^*. \tag{4}$$

By the assumption of uniqueness of the best responses, they are unique. (4) does not hold in non zero-sum games. By symmetry of the game,

$$\arg\max_{s_i\in S_i} u_i(s_i, s^*, \mathbf{s}^*_{-i,j}) = \arg\min_{s_j\in S_j} u_i(s^*, s_j, \mathbf{s}^*_{-i,j}) = s^*.$$

Therefore,

$$u_i(s^*, s^*, \mathbf{s}^*_{-i,j}) = \min_{s_j\in S_j} u_i(s^*, s_j, \mathbf{s}^*_{-i,j}) \le u_i(s^*, s_j, \mathbf{s}^*_{-i,j}).$$

With (3), we get

$$\max_{s_i\in S_i} u_i(s_i, s^*, \mathbf{s}^*_{-i,j}) = u_i(s^*, s^*, \mathbf{s}^*_{-i,j}) = \min_{s_j\in S_j} u_i(s^*, s_j, \mathbf{s}^*_{-i,j}).$$

This means

$$\begin{aligned} \min_{s_j\in S_j}\max_{s_i\in S_i} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) &\le \max_{s_i\in S_i} u_i(s_i, s^*, \mathbf{s}^*_{-i,j}) \\ = \min_{s_j\in S_j} u_i(s^*, s_j, \mathbf{s}^*_{-i,j}) &\le \max_{s_i\in S_i}\min_{s_j\in S_j} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}). \end{aligned} \tag{5}$$

On the other hand, since

$$\min_{s_j\in S_j} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) \le u_i(s_i, s_j, \mathbf{s}^*_{-i,j}),$$

we have

$$\max_{s_i\in S_i}\min_{s_j\in S_j} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) \le \max_{s_i\in S_i} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}).$$

This inequality holds for any s_j. Thus,

$$\max_{s_i\in S_i}\min_{s_j\in S_j} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) \le \min_{s_j\in S_j}\max_{s_i\in S_i} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}).$$

With (5), we obtain

$$\max_{s_i\in S_i}\min_{s_j\in S_j} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) = \min_{s_j\in S_j}\max_{s_i\in S_i} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}). \tag{6}$$

(5) implies

$$\max_{s_i\in S_i}\min_{s_j\in S_j} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) = \max_{s_i\in S_i} u_i(s_i, s^*, \mathbf{s}^*_{-i,j}),$$

and

$$\min_{s_j\in S_j}\max_{s_i\in S_i} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) = \min_{s_j\in S_j} u_i(s^*, s_j, \mathbf{s}^*_{-i,j}).$$

From

$$\min_{s_j\in S_j} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) \le u_i(s_i, s^*, \mathbf{s}^*_{-i,j}),$$

and

$$\max_{s_i\in S_i}\min_{s_j\in S_j} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) = \max_{s_i\in S_i} u_i(s_i, s^*, \mathbf{s}^*_{-i,j}),$$

we have

$$\arg\max_{s_i\in S_i}\min_{s_j\in S_j} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) = \arg\max_{s_i\in S_i} u_i(s_i, s^*, \mathbf{s}^*_{-i,j}) = s^*.$$

Also, from

$$\max_{s_j\in S_j} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) \ge u_i(s^*, s_j, \mathbf{s}^*_{-i,j}),$$

and

$$\min_{s_j\in S_j}\max_{s_i\in S_i} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) = \min_{s_j\in S_j} u_i(s^*, s_j, \mathbf{s}^*_{-i,j}),$$

we get

$$\arg\min_{s_j\in S_j}\max_{s_i\in S_i} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) = \arg\min_{s_j\in S_j} u_i(s^*, s_j, \mathbf{s}^*_{-i,j}) = s^*.$$

Therefore,

$$\arg\max_{s_i\in S_i}\min_{s_j\in S_j} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) = \arg\min_{s_j\in S_j}\max_{s_i\in S_i} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) = s^*. \tag{7}$$

(6) and (7) are equivalent to Theorem 4.2. □

Next we show the following theorem.

Theorem 4.4. *The modified version of Sion's minimax theorem with the coincidence of the maximin strategy and the minimax strategy imply the existence of a symmetric Nash equilibrium.*

Proof. Let $\tilde{s}$ be a value of s such that

$$\tilde{s} = \arg\max_{s_i\in S_i}\min_{s_j\in S_j} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}) = \arg\min_{s_j\in S_j}\max_{s_i\in S_i} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}),$$

and

$$\begin{aligned}\max_{s_i\in S_i}\min_{s_j\in S_j} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}) = \min_{s_j\in S_j} u_i(\tilde{s}, s_j, \tilde{\mathbf{s}}_{-i,j}) &= \min_{s_j\in S_j}\max_{s_i\in S_i} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}) \\ &= \max_{s_i\in S_i} u_i(s_i, \tilde{s}, \tilde{\mathbf{s}}_{-i,j}).\end{aligned}$$

$\tilde{\mathbf{s}}_{-i,j}$ is a vector of $s_k,\ k\in N,\ k\neq i,j$ such that $s_k=\tilde{s}$. Since

$$u_i(s_i, \tilde{s}, \tilde{\mathbf{s}}_{-i,j}) \ge \min_{s_j\in S_j} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}),$$

and

$$\max_{s_i \in S_i} u_i(s_i, \tilde{s}, \tilde{\mathbf{s}}_{-i,j}) = \max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}),$$

we obtain

$$\arg\max_{s_i \in S_i} u_i(s_i, \tilde{s}, \tilde{\mathbf{s}}_{-i,j}) = \arg\max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}) = \tilde{s}.$$

Therefore,

$$u_i(\tilde{s}, \tilde{s}, \tilde{\mathbf{s}}_{-i,j}) \geq u_i(s_i, \tilde{s}, \tilde{\mathbf{s}}_{-i,j}),$$

and so $(\tilde{s}, \tilde{s}, \tilde{\mathbf{s}}_{-i,j}) = (\tilde{s}, \tilde{s}, \dots, \tilde{s})$ is a symmetric Nash equilibrium of an n-players zero-sum game. ☐

5. Note on a non zero-sum game

Consider a simple two-players non zero-sum game with Players 1 and 2. Their strategies are s_1 and s_2, and their payoff functions are $u_1(s_1, s_2)$ and $u_2(s_1, s_2)$. The condition for minimization of $u_1(s_1, s_2)$ with respect to s_2 is $\frac{\partial u_1(s_1,s_2)}{\partial s_2}$. Denote s_2 which satisfies this condition by $s_2(s_1)$, and substitute it into u_1, we have $u_1(s_1, s_2(s_1))$. The condition for the maximin strategy for Player 1 to Player 2 is

$$\frac{\partial u_1(s_1, s_2(s_1))}{\partial s_1} + \frac{\partial u_1(s_1, s_2(s_1))}{\partial s_2}\frac{\partial s_2}{\partial s_1} = 0. \tag{8}$$

Similarly, the condition for the maximin strategy for Player 1 to Player 2 is written as

$$\frac{\partial u_2(s_1(s_2), s_2)}{\partial s_2} + \frac{\partial u_2(s_1(s_2), s_2)}{\partial s_1}\frac{\partial s_1}{\partial s_2} = 0. \tag{9}$$

The conditions for the Nash equilibrium are

$$\frac{\partial u_1(s_1, s_2)}{\partial s_1} = 0, \frac{\partial u_2(s_1, s_2)}{\partial s_2} = 0.$$

They are not equivalent to (8) and (9) in general.

If the game is zero-sum, $u_1(s_1, s_2) = -u_2(s_1, s_2)$. Therefore, (8) and (9) are reduced to

$$\frac{\partial u_1(s_1, s_2(s_1))}{\partial s_1} - \frac{\partial u_2(s_1, s_2(s_1))}{\partial s_2}\frac{\partial s_2}{\partial s_1} = 0,$$

$$\frac{\partial u_2(s_1(s_2), s_2)}{\partial s_2} - \frac{\partial u_1(s_1(s_2), s_2)}{\partial s_1}\frac{\partial s_1}{\partial s_2} = 0.$$

From these two equations we obtain

$$\frac{\partial u_1(s_1, s_2(s_1))}{\partial s_1} = \frac{\partial u_2(s_1, s_2(s_1))}{\partial s_2} = 0 \quad \text{if} \quad \frac{\partial s_1}{\partial s_2} \neq 1 \text{ and } \frac{\partial s_2}{\partial s_1} \neq 1.$$

Thus, the conditions for maximin and minimax strategies are equivalent to those for the Nash equilibrium.

6. Concluding Remark

In this paper we have shown that a modified version of Sion's minimax theorem with the coincidence of the maximin strategy and the minimax strategy is equivalent to the existence of a symmetric Nash equilibrium in a symmetric multi-players zero-sum game.

In a future research we will consider the relation between the mini-max theorem and Nash equilibrium in an asymmetric zero-sum game.

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