

# Erratum to “Quasi-Equilibrium Problems and Fixed Point Theorems of the Sum of l.s.c. and u.s.c. Mappings”

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The purpose of this note is to correct the proof of Theorem 3.1 in our paper “Quasi-equilibrium problems and fixed point theorems of the sum of l.s.c. and u.s.c. mappings”, *Minimax Theory and its Applications* 3(1) (2018) 57–72. We made a mistake in the use of Sion’s Minimax Theorem.

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Let  $X, Y$  and  $Z$  be Hausdorff locally convex topological vector spaces over reals,  $D \subset X$ ,  $K \subset Z$  be nonempty subsets. Given multi-valued mappings  $P: D \times K \rightarrow 2^D$ ,  $Q: D \times K \rightarrow 2^K$  and  $F: D \times K \rightarrow 2^Y$ , we are interested in the problem of finding  $(\bar{x}, \bar{y}) \in D \times K$  such that

$$\bar{x} \in P(\bar{x}, \bar{y}), \quad \bar{y} \in Q(\bar{x}, \bar{y}), \quad \text{and } 0 \in F(\bar{x}, \bar{y}).$$

This problem is called “generalized quasi-equilibrium problem” (in short, (GQEP)), in which the multivalued mappings  $P$  and  $Q$  are constraint mappings and  $F$  is a utility multivalued mapping.

We shall give a new proof for Theorem 3.1 for the case  $F = G + H: D \times K \rightarrow 2^X$ .

In (3.5) p. 64 in [1], we used

$$\begin{aligned} & \inf_{u \in \text{co}G(\bar{x}, \bar{y}) + H(\bar{x}, \bar{y}) \cap T_{P(\bar{x}, \bar{y})}(\bar{x})} \max_{i \in I(\bar{x}, \bar{y})} p_{j(i)}(u) \\ &= \max_{i \in I(\bar{x}, \bar{y})} \left\{ \inf_{v+w \in \text{co}G(\bar{x}, \bar{y}) + H(\bar{x}, \bar{y}) \cap T_{P(\bar{x}, \bar{y})}(\bar{x})} (p_{j(i)}(v) + p_{j(i)}(w)) \right\}. \end{aligned}$$

But the correct use the Sion' minimax Theorem should have been be

$$\begin{aligned} & \inf_{u \in \text{co}G(\bar{x}, \bar{y}) + H(\bar{x}, \bar{y}) \cap T_{P(\bar{x}, \bar{y})}(\bar{x})} \max_{p_{j(i)} \in M} p_{j(i)}(u) \\ &= \max_{p_{j(i)} \in M} \left\{ \inf_{v+w \in \text{co}G(\bar{x}, \bar{y}) + H(\bar{x}, \bar{y}) \cap T_{P(\bar{x}, \bar{y})}(\bar{x})} (p_{j(i)}(v) + p_{j(i)}(w)) \right\}. \end{aligned}$$

Here,  $\{p_{j(i)}\}_{i \in I(\bar{x}, \bar{y})} \neq M = \overline{\text{co}}\{p_{j(1)}, \dots, p_{j(s)}\}$ . Therefore, the proof of this theorem in [1] is not correct.

We therefore give here new proof of Theorem 3.1 in [1]. In the conclusion of Theorem 3.1, we forgot to write  $0 \in \text{co}G(\bar{x}, \bar{y}) + H(\bar{x}, \bar{y})$ , but only  $0 \in G(\bar{x}, \bar{y}) + H(\bar{x}, \bar{y})$ .

We first reformulate this theorem as follows:

**Theorem.** (Theorem 3.1 in [1]). *We assume the following conditions to hold:*

- (i)  $D, K$  are nonempty convex compact sets;
- (ii)  $P: D \times K \rightarrow 2^D$  is a continuous multivalued mapping with nonempty closed convex values,  $Q: D \times K \rightarrow 2^K$  is a u.s.c multivalued mapping with nonempty closed convex values;
- (iii)  $G: D \times K \rightarrow 2^X$  is a l.s.c multivalued mapping with nonempty values;
- (iv)  $H: D \times K \rightarrow 2^X$  is a u.s.c multivalued mapping with nonempty closed convex values;
- (v) for any  $(x, y) \in P(x, y) \times Q(x, y)$ ,  $H(x, y)$  has nonempty convex closed values and  $\emptyset \neq G(x, y) + H(x, y) \cap T_{P(x, y)}(x) \subset T_{P(x, y)}(x)$ .

Then there exists  $(\bar{x}, \bar{y}) \in D \times K$  such that

$$(1) \bar{x} \in P(\bar{x}, \bar{y}), \quad (2) \bar{y} \in Q(\bar{x}, \bar{y}), \quad (3) 0 \in \text{co}G(\bar{x}, \bar{y}) + H(\bar{x}, \bar{y}).$$

**Proof.** We set  $B = \{(x, y) \in D \times K \mid x \in P(x, y), y \in Q(x, y)\}$ .

Since the multivalued mapping  $T: D \times K \rightarrow 2^{D \times K}$ , defined by

$$T(x, y) = P(x, y) \times Q(x, y), \quad (x, y) \in D \times K,$$

is upper semi-continuous with nonempty convex and closed values, by using the Ky Fan fixed point Theorem, we conclude that  $T$  has a fixed point in  $D \times K$ . Therefore,  $B$  is a nonempty set. The upper semi-continuity and the closedness of values of  $T$  imply that  $B$  is a closed and then a compact set.

Assume that we have for any  $(x, y) \in B$ ,  $0 \notin \text{co}G(x, y) + H(x, y)$ . Take a point  $\bar{v} \in \text{co}G(x, y)$ . Since  $H(x, y)$  is a nonempty closed convex, so is  $\bar{v} + H(x, y)$  and  $0 \notin (\bar{v} + H(x, y))$ . By the Hahn -Banach Theorem, there exists  $p \in X^*$  such that

$$p(\bar{v}) + \sup_{w \in H(x,y)} p(w) < 0.$$

It follows

$$\inf_{v \in \text{co}(G(x,y))} p(v) + \sup_{w \in H(x,y)} p(w) < 0.$$

Further, we define functions  $c_p^1(.,.), c_p^2(.,.): D \times K \rightarrow R \cup \{-\infty\}$  by

$$c_p^1(x', y') = \inf_{v \in \text{co}(G(x',y'))} p(v) \quad \text{and} \quad c_p^2(x', y') = \sup_{w \in H(x',y')} p(w).$$

By Proposition 2.2 in [1] the functions  $c_p^1$  and  $c_p^2$  are u.s.c on  $D \times K$ ; therefore the set

$$U_p(x, y) = \{(x', y') \in D \times K | c_p^1(x', y') + c_p^2(x', y') < 0\},$$

is open. Since  $(x, y) \in U_p(x, y)$ , then is a nonempty open neighborhood of  $(x, y)$ .

Thus, for any  $(x, y) \in B$  there is  $p \in X^*$  such that

$$U_p(x, y) = \{(x', y') \in D \times K | c_p^1(x', y') + c_p^2(x', y') < 0\}$$

is nonempty and open and hence  $\{U_p\}_{p \in X^*}$  is an open cover of  $B$ . Since  $B$  is compact, there exist finite  $p_1, \dots, p_s \in X^*$ , such that

$$B \subseteq \bigcup_{j=1}^s U_{p_j}.$$

Further, since  $B$  is closed in  $D \times K$ ,  $U_{p_0} = D \times K \setminus B$  is open in  $D \times K$  and hence  $\{U_{p_0}, U_{p_1}, \dots, U_{p_s}\}$  is an open cover of the compact set  $D \times K$ . By Proposition 2.9 in Section 2 in [1], there exist continuous functions  $\psi_i: D \times K \rightarrow R$ , ( $i = 0, 1, \dots, s$ ), such that

- (i)  $0 \leq \psi_i(x, y) \leq 1$ ,
- (ii)  $\sum_{i=1}^s \psi_i(x, y) = 1$ , for all  $(x, y) \in D \times K$ , and
- (iii) for any  $i \in \{0, 1, \dots, s\}$ , there exists  $j(i) \in \{0, \dots, s\}$  such that  $\text{supp } \psi_{j(i)} \subset U_{p_{j(i)}}$ .

We define the function  $\phi: K \times D \times D \rightarrow R$  by

$$\phi(y, x, t) = \sum_{i=0}^s \psi_{j(i)}(x, y) \cdot p_{j(i)}(t - x), \quad \text{for any } t, x \in D, y \in K.$$

Then,  $\phi$  is a continuous function on  $K \times D \times D$ . Moreover, for every fixed  $(x, y) \in D \times K$ ,  $\phi(y, x, .): D \rightarrow R$  is an affine function and  $\phi(y, x, x) = 0$  for all  $(x, y) \in D \times K$ . Further, we define the multivalued mapping  $N: D \times K \rightarrow 2^D$  by

$$N(x, y) = \{z \in P(x, y) | \phi(y, x, z) \leq \phi(y, x, t), \text{ for all } t \in P(x, y)\}.$$

Since  $\phi$  is a continuous function and  $P(x, y)$  is a compact set,  $\phi((y, x, \cdot)): D \rightarrow R$  is an affine continuous function, we conclude that  $N(x, y)$  is a nonempty convex and closed subset, for any fixed  $(x, y) \in D \times K$ . It is also easy to prove that  $N$  is a closed mapping and then is a upper semi-continuous multivalued mapping with nonempty convex compact values.

Therefore, the multivalued mapping  $M: D \times K \rightarrow 2^{D \times K}$ , defined by

$$M(x, y) = N(x, y) \times Q(x, y), (x, y) \in D \times K,$$

is a u.s.c multivalued mapping with nonempty convex compact values. Applying the Ky Fan fixed point Theorem again, we conclude that there exists  $(\bar{x}, \bar{y}) \in D \times K$  such that  $(\bar{x}, \bar{y}) \in N(\bar{x}, \bar{y}) \times Q(\bar{x}, \bar{y})$ . This follows  $\phi(\bar{y}, \bar{x}, t) \geq 0$ , for all  $t \in P(\bar{x}, \bar{y})$ . This gives

$$\sum_{i=0}^s \psi_i(\bar{x}, \bar{y}) \cdot p_{j(i)}(t - \bar{x}) \geq 0, \text{ for all } t \in P(\bar{x}, \bar{y}). \quad (1)$$

Setting  $p^* = \sum_{i=0}^s \psi_i(\bar{x}, \bar{y}) \cdot p_{j(i)}$ , we get from (1),  $p^*(t - \bar{x}) \geq 0$ , for all  $t \in P(\bar{x}, \bar{y})$  and hence  $p^*(u) \geq 0$ , for all  $u \in T_{P(\bar{x}, \bar{y})}(\bar{x})$ . By assumption (v),

$$\emptyset \neq G(\bar{x}, \bar{y}) + H(\bar{x}, \bar{y}) \cap T_{P(\bar{x}, \bar{y})}(\bar{x}) \subset T_{P(\bar{x}, \bar{y})}(\bar{x}),$$

and we conclude that

$$p^*(v) + p^*(w) \geq 0, \text{ for all } v \in \text{co}G(\bar{x}, \bar{y}), w \in H(\bar{x}, \bar{y}) \cap T_{P(\bar{x}, \bar{y})}(\bar{x}).$$

This yields

$$\inf_{v+w \in \text{co}G(\bar{x}, \bar{y}) + H(\bar{x}, \bar{y}) \cap T_{P(\bar{x}, \bar{y})}(\bar{x})} p^*(v+w) \geq 0. \quad (2)$$

Since

$$p^*(\text{co}G(\bar{x}, \bar{y}) + H(\bar{x}, \bar{y}) \cap T_{P(\bar{x}, \bar{y})}(\bar{x})) = p^*(\text{co}G(\bar{x}, \bar{y})) + p^*(H(\bar{x}, \bar{y}) \cap T_{P(\bar{x}, \bar{y})}(\bar{x})),$$

we get

$$\begin{aligned} \inf_{v+w \in \text{co}G(\bar{x}, \bar{y}) + H(\bar{x}, \bar{y}) \cap T_{P(\bar{x}, \bar{y})}(\bar{x})} \{p^*(v) + p^*(w)\} &= \inf_{v \in \text{co}G(\bar{x}, \bar{y})} p^*(v) + \inf_{w \in H(\bar{x}, \bar{y}) \cap T_{P(\bar{x}, \bar{y})}(\bar{x})} p^*(w) \\ &\leq \inf_{v \in \text{co}G(\bar{x}, \bar{y})} p^*(v) + \sup_{w \in H(\bar{x}, \bar{y}) \cap T_{P(\bar{x}, \bar{y})}(\bar{x})} p^*(w). \end{aligned} \quad (3)$$

Now, we assert that for any convex subset  $A \subset D$ , the following equality holds

$$p^*(A) = \sum_{i=0}^s \psi_i(\bar{x}, \bar{y}) \cdot p_{j(i)}(A). \quad (4)$$

Indeed, for any  $v \in A$ ,  $p^*(v) = \sum_{i=0}^s \psi_i(\bar{x}, \bar{y}) \cdot p_{j(i)}(v)$ . This shows

$$p^*(A) \subseteq \sum_{i=0}^s \psi_i(\bar{x}, \bar{y}) \cdot p_{j(i)}(A).$$

We need to verify 
$$\sum_{i=0}^s \psi_i(\bar{x}, \bar{y}) \cdot p_{j(i)}(A) \subseteq p^*(A).$$

Take  $w \in \sum_{i=0}^s \psi_i(\bar{x}, \bar{y}) \cdot p_{j(i)}(A)$ , we can write  $w = \sum_{i=0}^s \psi_i(\bar{x}, \bar{y}) \cdot p_{j(i)}(v_i)$  with  $v_i \in A$ , for  $i = 0, \dots, s$ . By the definition of  $p^*$ , we have

$$p^*(\psi_m(\bar{x}, \bar{y})v_m) = \sum_{i=0}^s \psi_i(\bar{x}, \bar{y}) \cdot p_{j(i)}(\psi_m(\bar{x}, \bar{y})v_m), m = 0, \dots, s.$$

Taking the sum of the both sides of these equalities, we obtain

$$p^*\left(\sum_{m=0}^s \psi_m(\bar{x}, \bar{y})v_m\right) = \sum_{m=0}^s \psi_m(\bar{x}, \bar{y})w = w.$$

Setting  $v = \sum_{m=0}^s \psi_m(\bar{x}, \bar{y})v_m$  and using the convexity of  $A$ , we conclude that  $w = p^*(v) \in p^*(A)$ . Thus, we proved (4).

Inserting  $A = \text{co}G(\bar{x}, \bar{y}) + H(\bar{x}, \bar{y}) \cap T_{P(\bar{x}, \bar{y})}(\bar{x})$  in (4), we deduce

$$\begin{aligned} p^*(\text{co}G(\bar{x}, \bar{y}) + H(\bar{x}, \bar{y}) \cap T_{P(\bar{x}, \bar{y})}(\bar{x})) &= p^*(\text{co}G(\bar{x}, \bar{y})) + p^*(H(\bar{x}, \bar{y}) \cap T_{P(\bar{x}, \bar{y})}(\bar{x})) \\ &= \sum_{i=0}^s \psi_i(\bar{x}, \bar{y}) \cdot p_{j(i)}(\text{co}G(\bar{x}, \bar{y})) + \sum_{i=0}^s \psi_i(\bar{x}, \bar{y}) \cdot p_{j(i)}(H(\bar{x}, \bar{y}) \cap T_{P(\bar{x}, \bar{y})}(\bar{x})) \end{aligned}$$

This gives

$$\begin{aligned} &\inf_{v+w \in (\text{co}G(\bar{x}, \bar{y}) + H(\bar{x}, \bar{y}) \cap T_{P(\bar{x}, \bar{y})}(\bar{x}))} p^*(v+w) \\ &= \sum_{i=0}^s \psi_i(\bar{x}, \bar{y}) \left\{ \inf_{v \in \text{co}G(\bar{x}, \bar{y})} p_{j(i)}(v) + \inf_{w \in H(\bar{x}, \bar{y}) \cap T_{P(\bar{x}, \bar{y})}(\bar{x})} p_{j(i)}(w) \right\} \\ &\leq \sum_{i=0}^s \psi_i(\bar{x}, \bar{y}) \cdot p_{j(i)} \left\{ \inf_{v \in \text{co}G(\bar{x}, \bar{y})} p_{j(i)}(v) + \sup_{w \in H(\bar{x}, \bar{y}) \cap T_{P(\bar{x}, \bar{y})}(\bar{x})} p_{j(i)}(w) \right\}. \quad (5) \end{aligned}$$

Put  $I(\bar{x}, \bar{y}) = \{i \in \{0, 1, \dots, s\} \mid \psi_i(\bar{x}, \bar{y}) > 0\}$ . Since  $\psi_i(\bar{x}, \bar{y}) \geq 0$  and since  $\sum_{i=0}^s \psi_i(\bar{x}, \bar{y}) = 1$ , we deduce that  $I(\bar{x}, \bar{y}) \neq \emptyset$ . Therefore, for any  $i \in I(\bar{x}, \bar{y})$ , we have  $(\bar{x}, \bar{y}) \in \text{supp}\psi_{j(i)} \subset U_{p_{j(i)}}$ . This yields

$$\inf_{v \in \text{co}G(\bar{x}, \bar{y})} p_{j(i)}(v) + \sup_{w \in H(\bar{x}, \bar{y}) \cap T_{P(\bar{x}, \bar{y})}(\bar{x})} p_{j(i)}(w) = c_{p_{j(i)}}^1(\bar{x}, \bar{y}) + c_{p_{j(i)}}^2(\bar{x}, \bar{y}) < 0,$$

and then

$$\max_{i \in I(\bar{x}, \bar{y})} \{c_{p_{j(i)}}^1(\bar{x}, \bar{y}) + c_{p_{j(i)}}^2(\bar{x}, \bar{y})\} < 0.$$

Therefore, together with (5), we have

$$\begin{aligned}
\inf_{v+w \in \text{co}G(\bar{x}, \bar{y}) + H(\bar{x}, \bar{y}) \cap T_{P(\bar{x}, \bar{y})}(\bar{x})} p^*(v+w) &\leq \inf_{v \in \text{co}G(\bar{x}, \bar{y})} p^*(v) + \sup_{w \in H(\bar{x}, \bar{y}) \cap T_{P(\bar{x}, \bar{y})}(\bar{x})} p^*(w) \\
&\leq \sum_{i \in I(\bar{x}, \bar{y})} \psi_i(\bar{x}, \bar{y}) \left\{ \inf_{v \in (\text{co}G(\bar{x}, \bar{y}))} p_{j(i)}(v) + \sup_{w \in H(\bar{x}, \bar{y}) \cap T_{P(\bar{x}, \bar{y})}(\bar{x})} p_{j(i)}(w) \right\} \\
&\leq \max_{i \in I(\bar{x}, \bar{y})} \{c_{p_{j(i)}}^1(\bar{x}, \bar{y}) + c_{p_{j(i)}}^2(\bar{x}, \bar{y})\} < 0. \tag{6}
\end{aligned}$$

A combination of (2) and (6) gives a contradiction. This completes the proof of the theorem.  $\square$

**Remark.** (1) In [1], we forgot to state explicitly  $0 \in \text{co}G(\bar{x}, \bar{y}) + H(\bar{x}, \bar{y})$  as assumption (we assumed  $0 \in G(\bar{x}, \bar{y}) + H(\bar{x}, \bar{y})$  only), making Theorem 3.1 false. This can be seen by the following simple example: Take  $D = K = \langle -1, 1 \rangle \subset \mathbb{R}$ ,  $P(x, y) = Q(x, y) = \langle -1, 1 \rangle$ ,  $G(x, y) = D \setminus \{x\}$  and  $H(x, y) = \{x\}$ . We can easily verify that all assumptions of Theorem 3.1 as formulated in [1] are satisfied. But,  $0 \notin G(x, y) + H(x, y)$  for all  $x, y \in D$ .

(2) We must replace  $G$  by  $\text{co}G$  in other results in [1], or assume, (v): For any  $(x, y) \in P(x, y) \times Q(x, y)$ ,  $H(x, y)$  has nonempty convex closed values, and the condition  $\emptyset \neq G(x, y) + H(x, y) \cap T_{P(x, y)}(x) \subset T_{P(x, y)}(x)$  is replaced by (v'): For any  $(x, y) \in P(x, y) \times Q(x, y)$ ,  $G(x, y)$  is convex and  $H(x, y)$  is convex closed and  $\emptyset \neq G(x, y) + H(x, y) \cap T_{P(x, y)}(x) \subset T_{P(x, y)}(x)$ .

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## References

- [1] N. X. Tan, N. Q. Hoa, N. B. Minh: *Quasi-equilibrium problems and fixed point theorems of the sum of l.s.c. and u.s.c. mappings*, Minimax Theory Appl. 3(1) (2018) 57–72.