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Infinitely Many Solutions for Semilinear Δ_γ-Differential Equations in $\mathbb{R}^N$ without the Ambrosetti-Rabinowitz Condition

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We study the existence of infinitely many nontrivial solutions of the semilinear Δ_γ-differential equations in $\mathbb{R}^N$

$$-\Delta_\gamma u + b(x)u = f(x,u) \quad \text{ in } \mathbb{R}^N,$$

where Δ_γ is the subelliptic operator of the type

$$\Delta_\gamma := \sum_{j=1}^{N} \partial_{x_j}\left(\gamma_j^2 \partial_{x_j}\right), \quad \partial_{x_j} := \frac{\partial}{\partial x_j}, \quad \gamma := (\gamma_1, \gamma_2, ..., \gamma_N),$$

and the potential $b(x)$ and nonlinearity $f(x,u)$ are not assumed to be continuous, moreover f may not satisfy the Ambrosetti–Rabinowitz (AR) condition. Under some growth conditions on b and f, we show that there are infinitely many solutions to the problem.

Keywords: Δ_γ-Laplace problems, Cerami condition, variational method, weak solutions, Mountain Pass Theorem.

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1. Introduction

In the last years, the semilinear Schrödinger equation

$$-\Delta u + b(x)u = f(x,u), \quad x \in \mathbb{R}^N, \qquad u \in H^1(\mathbb{R}^N), \tag{1}$$

has been studied by several authors. It has attracted much interest not only because of its importance in applications, but also as it provides a good model for developing mathematical methods. With the aid of variational methods, the existence and multiplicity of nontrivial solutions for the problem (1) have been extensively investigated in the literature over the past decades. See, e.g., [1]–[6], [10, 11], [15]–[18], [21] and the references therein.

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The function $f(x,\xi)$ is said to satisfy the *Ambrosetti-Rabinowitz condition* ((AR) condition for short) if there exists a constant $\theta > 2$ such that

$$\theta F(x,\xi) \leq f(x,\xi)\xi, \quad \forall\, \xi \neq 0, \quad \forall\, x \in \mathbb{R}^N, \quad \text{where } F(x,\xi) = \int_0^\xi f(x,\tau)\mathrm{d}\tau.$$

This condition not only ensures that the Euler-Lagrange functional associated to the problem (1) has a mountain pass geometry, but also guarantees that Palais-Smale sequences of the Euler-Lagrange functional are bounded. With this (AR) condition, one can use the classical version of the Mountain Pass Theorem of Ambrosetti and Rabinowitz to study the existence of solutions (see e.g. [2, 3, 6, 10, 11, 15, 16, 17]). Although (AR) condition is quite natural and important, but there are many problems where the nonlinear term $f(x,\xi)$ does not satisfy the (AR) condition, for example,

$$f(x,\xi) = \xi \ln(1 + |\xi|).$$

In this paper, we study the existence and multiplicity of nontrivial weak solutions to the following problem

$$-\Delta_\gamma u + b(x)u = f(x,u) \quad \text{ in } \mathbb{R}^N, \tag{2}$$

where Δ_γ is a subelliptic operator of the form

$$\Delta_\gamma := \sum_{j=1}^N \partial_{x_j}\left(\gamma_j^2 \partial_{x_j}\right), \quad \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_N) : \mathbb{R}^N \to \mathbb{R}^N.$$

The Δ_γ-operator was considered by Kogoj and Lanconelli in [9]. This operator has the same form as in [7], however the functions $\gamma(x)$ in [9] are more generalized than those considered in [7]. The Δ_γ-operator contains many degenerate elliptic operators such as the Grushin-type operator (see [8])

$$G_\alpha := \Delta_x + |x|^{2\alpha}\Delta_y, \quad \alpha \geq 0, \quad (x,y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2},$$

and the operator

$$P_{\alpha,\beta} := \Delta_x + \Delta_y + |x|^{2\alpha}|y|^{2\beta}\Delta_z, \quad (x,y,z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3},$$

where α, β are nonnegative real numbers (see [19, 20]).

To study the problem (2), we make the following assumptions:

(B1) $b : \mathbb{R}^N \to \mathbb{R}$ such that $b \in L^1_{loc}(\mathbb{R}^N)$ and

$$\mu_0 = \operatorname*{ess\,inf}_{x \in \mathbb{R}^N} b(x) := \sup\{\mu \in \mathbb{R} : Vol(\{x \in \mathbb{R}^N, b(x) < \mu\}) = 0\} > 0;$$

(B2) There exists a constant $d_0 > 0$ such that

$$\lim_{|y| \to +\infty} \text{meas}\{x \in \mathbb{R}^N : |x - y| \leq d_0, b(x) \leq M\} = 0, \quad \forall M > 0,$$

where meas$\{\cdot\}$ denotes the Lebesgue measure of a set in $\mathbb{R}^N$;

(B3) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

$$|f(x,\xi)| \leq f_1(x)\,|\xi| + f_2(x)\,|\xi|^{p-1} \text{ for almost every } (x,\xi) \in \mathbb{R}^N \times \mathbb{R},$$

where $f_1, f_2 : \mathbb{R}^N \to \mathbb{R}$ are nonnegative and $f_2(x) \in L^{p_2}(\mathbb{R}^N) \bigcap L^{p_3}(\mathbb{R}^N)$, $f_1(x) \in L^{p_1}(\mathbb{R}^N) \bigcap L^{p_3}(\mathbb{R}^N) \bigcap L^{\frac{2^*_\gamma p_3}{p_3(p-1)+2^*_\gamma}}(\mathbb{R}^N)$, $2p_1/(p_1-1) < 2^*_\gamma := \frac{2\widetilde{N}}{\widetilde{N}-2}$ (where $\widetilde{N}$ defined by formula (3)) $pp_2/(p_2-1) < 2^*_\gamma$, $p \in (2, 2^*_\gamma)$, $p_1, p_2 > 1$, $p_3 \geq \frac{2^*_\gamma}{2^*_\gamma - p}$, $p_3(2^*_\gamma - 2p + 2) \leq 2.2^*_\gamma$;

(B4) $\lim\limits_{|\xi|\to+\infty} \frac{F(x,\xi)}{\xi^2} = +\infty$, a.e. $\xi \in \mathbb{R}^N$, and $F(x,\xi) \geq 0$, for all $(x,\xi) \in \mathbb{R}^N \times \mathbb{R}$;

(B5) $\mathcal{F}(x,\xi) := \frac{1}{2} f(x,\xi)\xi - F(x,\xi) \geq 0$, and there exists $c_0 > 0, r_0 \geq 0$ and $\kappa > \max\{1, \frac{\widetilde{N}}{2}\}$ such that

$$|F(x,\xi)|^\kappa \leq c_0 |\xi|^{2\kappa} \mathcal{F}(x,\xi), \quad \forall (x,\xi) \in \mathbb{R}^N \times \mathbb{R},\ |\xi| \geq r_0;$$

(B6) There exist $\mu > 2$ and $\varrho > 0$ such that

$$\mu F(x,\xi) \leq \xi f(x,\xi) + \varrho \xi^2, \quad \forall (x,\xi) \in \mathbb{R}^N \times \mathbb{R};$$

(B7) $f(x,-\xi) = -f(x,\xi)$, for all $(x,\xi) \in \mathbb{R}^N \times \mathbb{R}$.

Now, we are ready to state the main results of this paper.

Theorem 1.1. *Assume that b and f satisfy* (B1)–(B5), *and* (B7). *Then problem* (2) *possesses infinitely many nontrivial solutions.*

Theorem 1.2. *Assume that b and f satisfy* (B1)–(B4), (B6) *and* (B7). *Then problem* (2) *possesses infinitely many nontrivial solutions.*

Example 1.3. When $\gamma_1 = \gamma_2 = \cdots = \gamma_N \equiv 1$, $b(x) = n+1$ for $n \leq |x| < n+1$, for every $n \in \mathbb{N}$ and $f(x,\xi) = f_1(x)\xi^3 - f_2(x)\xi$ where $f_1, f_2 : \mathbb{R}^3 \to \mathbb{R}$ are positive

$$f_1 \in (L^2(\mathbb{R}^3) \cap L^3(\mathbb{R}^3) \cap L^{\frac{6}{5}}(\mathbb{R}^3)) \backslash (C(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)),$$
$$f_2 \in L^3(\mathbb{R}^3) \cap L^4(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3), f_1(x) > 2f_2(x)$$

for all $x \in \mathbb{R}^3$, it is easy to check that $f(x,\xi), b(x)$ satisfy (B1)–(B7). By Theorem 1.1, problem (2) possesses infinitely many nontrivial solutions. It is easy to verify that f does not satisfy the condition (A3) of [18, 21], meaning $f \in C(\mathbb{R}^N, \mathbb{R})$, and there exist constants $c_1, c_2 > 0$ and $q \in (2, 2^* := \frac{2N}{N-2})$ such that

$$|f(x,\xi)| \leq c_1|\xi| + c_2|\xi|^{q-1}, \quad \forall (x,\xi) \in \mathbb{R}^N \times \mathbb{R}.$$

Example 1.4. When $b(x) = n+1$ for $n \leq |x| < n+1$, for every $n \in \mathbb{N}$ and

$$\Delta_\gamma \equiv \frac{\partial^2}{\partial x^2} + x^2 \frac{\partial^2}{\partial y^2}.$$

Let $$f(x,y,\xi) = 2f_1(x,y)\xi\left[\ln(1+|\xi|^{\frac{1}{3}}) + \frac{|\xi|^{\frac{1}{3}}}{6(1+|\xi|^{\frac{1}{3}})}\right],$$

where $f_1 : \mathbb{R}^2 \to \mathbb{R}$ are positive $f_1(x) \in L^{p_1}(\mathbb{R}^2) \bigcap L^{p_2}(\mathbb{R}^2) \bigcap L^{\frac{18p_2}{4p_2+18}}(\mathbb{R}^2)$, $p_1 > \frac{18}{11}$, $p_2 \in (\frac{18}{11}, \frac{36}{17})$ and exist constant $C > 0$ such that $f_1(x,y) \le C$ for all $(x,y) \in \mathbb{R}^2$. By direct computation we obtain

$$F(x,y,\xi) = f_1(x,y)\xi^2\ln(1+|\xi|^{\frac{1}{3}}), \mathcal{F}(x,y,\xi) = \frac{f_1(x,y)\,|\xi|^{\frac{7}{3}}}{6(1+|\xi|^{\frac{1}{3}})}.$$

Thus all conditions of Theorem 1.1 are satisfied with

$$\widetilde{N} = 3;\ 2^*_\gamma = 6;\ p = \tfrac{7}{3},\ \kappa = 2.$$

By Theorem 1.1, problem (2) possesses infinitely many nontrivial solutions. It is easy to verify that f does not satisfy (AR) which was requried in [13].

The paper is organized as follows. In Section 2 for convenience of the readers, we recall some function spaces, embedding theorems and associated functional settings. Section 3 is devoted to the proofs of Theorems 1.1 and 1.2.

2. Preliminary results

2.1. Function spaces and embedding theorems

We recall the functional setting in [9, 12]. We consider the operator of the form

$$\Delta_\gamma := \sum_{j=1}^{N} \partial_{x_j}\left(\gamma_j^2 \partial_{x_j}\right), \quad \partial_{x_j} := \frac{\partial}{\partial x_j},\ j = 1, 2, \ldots, N.$$

Here, the functions $\gamma_j : \mathbb{R}^N \to \mathbb{R}$ are assumed to be continuous, different from zero and of class C^1 in $\mathbb{R}^N \backslash \Pi$, where

$$\Pi := \left\{ x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : \prod_{j=1}^{N} x_j = 0 \right\}.$$

Moreover, we assume the following properties:

(i) There exists a semigroup of dilations $\{\delta_t\}_{t>0}$ such that

$$\delta_t : \mathbb{R}^N \longrightarrow \mathbb{R}^N, \quad (x_1, \ldots, x_N) \longmapsto \delta_t(x_1, \ldots, x_N) = (t^{\varepsilon_1}x_1, \ldots, t^{\varepsilon_N}x_N),$$

where $1 = \varepsilon_1 \le \varepsilon_2 \le \cdots \le \varepsilon_N$, and γ_j is δ_t-homogeneous of degree $\varepsilon_j - 1$, i.e.,

$$\gamma_j(\delta_t(x)) = t^{\varepsilon_j - 1}\gamma_j(x), \forall x \in \mathbb{R}^N, \forall t > 0,\ j = 1, \ldots, N.$$

The number
$$\widetilde{N} := \sum_{j=1}^{N} \varepsilon_j \tag{3}$$

is called the *homogeneous dimension* of $\mathbb{R}^N$ with respect to $\{\delta_t\}_{t>0}$.

(ii) $\gamma_1 = 1, \gamma_j(x) = \gamma_j(x_1, x_2, \ldots, x_{j-1}),\ j = 2, \ldots, N.$

(iii) There exists a constant $\rho \geq 0$ such that

$$0 \leq x_k \partial_{x_k} \gamma_j(x) \leq \rho \gamma_j(x), \forall k \in \{1, 2, \dots, j-1\}, \forall j = 2, \dots, N,$$

and for every $x \in \overline{\mathbb{R}}_+^N := \left\{(x_1, \dots, x_N) \in \mathbb{R}^N : x_j \geq 0, \forall j = 1, 2, \dots, N\right\}$.

(iv) The equalities $\gamma_j(x) = \gamma_j(x^*)$ $(j = 1, 2, \dots, N)$ are satisfied for every $x \in \mathbb{R}^N$, where

$$x^* = (|x_1|, \dots, |x_N|) \text{ if } x = (x_1, x_2, \dots, x_N).$$

Definition 2.1. By $S^p_\gamma(\mathbb{R}^N)$ $(1 \leq p < +\infty)$ we will denote the set of all functions $u \in L^p(\mathbb{R}^N)$ such that $\gamma_j \partial_{x_j} u \in L^p(\mathbb{R}^N)$ for all $j = 1, \dots, N$. We define the norm in this space as follows

$$\|u\|_{S^p_\gamma(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} (|u|^p + |\nabla_\gamma u|^p) \,\mathrm{d}x\right)^{\frac{1}{p}},$$

where $\nabla_\gamma u = (\gamma_1 \partial_{x_1} u, \gamma_2 \partial_{x_2} u, \dots, \gamma_N \partial_{x_N} u)$. If $p = 2$ we can also define the scalar product in $S^2_\gamma(\mathbb{R}^N)$ as follows

$$(u, v)_{S^2_\gamma(\mathbb{R}^N)} = (u, v)_{L^2(\mathbb{R}^N)} + (\nabla_\gamma u, \nabla_\gamma v)_{L^2(\mathbb{R}^N)}. \qquad \square$$

Define $$S^2_{\gamma, b(x)}(\mathbb{R}^N) = \left\{u \in S^2_\gamma(\mathbb{R}^N) : \int_{\mathbb{R}^N} \left(|\nabla_\gamma u|^2 + b(x) u^2\right) \mathrm{d}x < +\infty\right\}$$

with $b(x)$ satisfying conditions (B1),(B2) then $S^2_{\gamma, b(x)}(\mathbb{R}^N)$ is a Hilbert space with the norm

$$\|u\|_{S^2_{\gamma, b(x)}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} \left(|\nabla_\gamma u|^2 + b(x) u^2\right) \mathrm{d}x\right)^{\frac{1}{2}}.$$

From Lemma 2.2 in [13], we have

Proposition 2.2. *Assume that* $b(x)$ *satisfies* (B1) *and* (B2). *Then the embedding map from* $S^2_{\gamma, b(x)}(\mathbb{R}^N)$ *into* $L^q(\mathbb{R}^N)$ *is compact for* $2 \leq q < 2^*_\gamma$.

Define the Euler-Lagrange functional associated with the problem (2) as follows

$$\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla_\gamma u|^2 + b(x) u^2\right) \mathrm{d}x - \int_{\mathbb{R}^N} F(x, u) \mathrm{d}x.$$

From Lemma 2.3 in [13] and f satisfies (B3), $b(x)$ satisfies (B1), we have Φ is well defined on $S^2_{\gamma, b(x)}(\mathbb{R}^N)$ and $\Phi \in C^1(S^2_{\gamma, b(x)}(\mathbb{R}^N), \mathbb{R})$ with

$$\Phi'(u)(v) = \int_{\mathbb{R}^N} (\nabla_\gamma u \cdot \nabla_\gamma v + b(x) u v) \,\mathrm{d}x - \int_{\mathbb{R}^N} f(x, u) \, v \mathrm{d}x$$

for all $v \in S^2_{\gamma, b(x)}(\mathbb{R}^N)$. One can also check that the critical points of Φ are weak solutions of problem (2).

2.2. Mountain Pass Theorem

Definition 2.3. Let $\mathbb{X}$ be a real Banach space with its dual space $\mathbb{X}^*$ and $\Phi \in C^1(\mathbb{X}, \mathbb{R})$. For $c \in \mathbb{R}$ we say that Φ satisfies the $(C)_c$ *condition*, i.e. the *Cerami condition* at level c, if for any sequence $\{x_m\}_{n=1}^{\infty} \subset \mathbb{X}$ with

$$\Phi(x_m) \to c \text{ and } (1 + \|x_m\|_{\mathbb{X}}) \|\Phi'(x_m)\|_{\mathbb{X}^*} \to 0,$$

then there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ that converges strongly in $\mathbb{X}$. If Φ satisfies the $(C)_c$ condition for all $c > 0$ then we say that Φ satisfies the Cerami condition. $\square$

We will use the following version of the Mountain Pass Theorem.

Lemma 2.4. ([14]) *Let $\mathbb{X}$ be an infinite dimensional Banach space, $\mathbb{X} = \mathbb{Y} \bigoplus \mathbb{Z}$, where $\mathbb{Y}$ is finite dimensional and let $\Phi \in C^1(\mathbb{X}, \mathbb{R})$ satisfy the $(C)_c$ condition for all $c > 0$, $\Phi(0) = 0$, $\Phi(-u) = \Phi(u)$ for all $u \in \mathbb{X}$, and*

(i) *There are constants $\rho, \alpha > 0$ such that $\Phi(u) \geq \alpha$ for all $u \in \mathbb{Z}$ such that $\|u\|_{\mathbb{X}} = \rho$;*

(ii) *For any finite dimensional subspace $\widehat{\mathbb{X}} \subset \mathbb{X}$, there is $R = R(\widehat{\mathbb{X}}) > 0$ such that $\Phi(u) \leq 0$ on $\widehat{\mathbb{X}} \backslash B_R$.*

Then Φ possesses an unbounded sequence of critical values.

3. Proof of the theorems

We prove Theorems 1.1 and 1.2 by verifying that all conditions of Lemma 2.4 are satisfied. First, we check the Cerami condition in this lemma

Lemma 3.1. *Assume that $b(x)$ and $f(x)$ satisfy* (B1)–(B4) *and* (B5). *Then Φ satisfies the $(C)_c$ condition for all $c > 0$ on $S^2_{\gamma,b(x)}(\mathbb{R}^N)$.*

Proof. Let $\{u_m\}_{m=1}^{\infty}$ be a sequence in $S^2_{\gamma,b(x)}(\mathbb{R}^N)$ such that

$$\left(1 + \|u_m\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}\right) \|\Phi'(u_m)\|_{(S^2_{\gamma,b(x)}(\mathbb{R}^N))^*} \to 0 \text{ and } \Phi(u_m) \to c > 0 \tag{4}$$

as $m \to \infty$, hence

$$\Phi'(u_m)(u_m) \to 0 \quad \text{and} \quad \frac{1}{2} \|u_m\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} F(x, u_m) \mathrm{d}x \to c \text{ as } m \to \infty. \tag{5}$$

We first show that $\{u_m\}_{m=1}^{\infty}$ is bounded in $S^2_{\gamma,b(x)}(\mathbb{R}^N)$ by a contradiction argument. Indeed, suppose that

$$\|u_m\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} \to \infty \quad \text{as} \quad m \to \infty. \tag{6}$$

Setting $\quad v_m = \dfrac{u_m}{\|u_m\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}}, \quad$ then $\quad \|v_m\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} = 1.$

When n is large enough, we have

$$c+1 \geq \Phi(u_m) - \frac{1}{2}\langle \Phi'(u_m), u_m\rangle = \int_{\mathbb{R}^N} \mathcal{F}(x, u_m)\mathrm{d}x, \tag{7}$$

where $\mathcal{F}(x,u_m) = \frac{1}{2}f(x,u_m)u_m - F(x,u_m) \geq 0$. By condition (B3), we obtain

$$|F(x,\xi)| \leq \frac{f_1(x)}{2}|\xi|^2 + \frac{f_2(x)}{p}|\xi|^p. \tag{8}$$

For $0 \leq a < b$, let $\quad \Omega_m(a,b) = \{x \in \mathbb{R}^N : a \leq |u_m(x)| < b\}. \quad (9)$

Going if necessary to a subsequence, we may assume that

$$\begin{aligned}
&v_m \rightharpoonup \quad v \text{ weakly in } S^2_{\gamma,b(x)}(\mathbb{R}^N) \ \text{ as } m \to \infty,\\
&v_m \to v \quad\ \text{a.e. in } \mathbb{R}^N \ \text{ as } m \to \infty.\\
&v_m \to v \quad\ \text{strongly in } L^q(\mathbb{R}^N) \ \text{ as } m \to \infty, 2 \leq q < 2^*_\gamma.
\end{aligned} \tag{10}$$

Now, we consider two possible cases: $v = 0$ or $v \neq 0$.

Case 1: if $v = 0$, then $v_m \to v$ in $L^r(\mathbb{R}^N)$, $2 \leq r < 2^*_\gamma$, and $v_m \to 0$ a.e. on $\mathbb{R}^N$. From (5), implies that

$$\lim_{m\to\infty} \int_{\mathbb{R}^N} \frac{F(x,u_m)}{\|u_m\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}}\mathrm{d}x = \frac{1}{2}. \tag{11}$$

On the other hand, by (8), we obtain, as $m \to \infty$:

$$\begin{aligned}
\int_{\Omega_m(0,r_0)} \frac{|F(x,u_m)|}{\|u_m\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}}\mathrm{d}x &= \int_{\Omega_m(0,r_0)} \frac{|F(x,u_m)|}{|u_m|^2}\frac{|u_m|^2}{\|u_m\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}}\mathrm{d}x\\
= \int_{\Omega_m(0,r_0)} \frac{|F(x,u_m)|}{|u_m|^2}|v_m|^2\mathrm{d}x &\leq C\int_{\Omega_m(0,r_0)} \left(|f_1(x)| + |f_2(x)|\right)|v_m|^2\,\mathrm{d}x\\
\leq C\int_{\mathbb{R}^N} \left(|f_1(x)| + |f_2(x)|\right)|v_m|^2\,\mathrm{d}x&\\
\leq C\Big(\|f_1\|_{L^{p_1}(\mathbb{R}^N)}\|v_m\|^2_{L^{\frac{2p_1}{p_1-1}}(\mathbb{R}^N)} &+ \|f_2\|_{L^{p_2}(\mathbb{R}^N)}\|v_m\|^2_{L^{\frac{2p_2}{p_2-1}}(\mathbb{R}^N)}\Big) \to 0.
\end{aligned} \tag{12}$$

Set $\kappa' = \kappa/(\kappa-1)$, $\kappa > \max\{1, \widetilde{N}/2\}$, then $2\kappa' \in [2, 2^*_\gamma)$. Hence, from (B5), (7) and (10), we have

$$\begin{aligned}
\int_{\Omega_m(r_0,+\infty)} \frac{|F(x,u_m)|}{\|u_m\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}}\mathrm{d}x &= \int_{\Omega_m(r_0,+\infty)} \frac{|F(x,u_m)|}{|u_m|^2}|v_m|^2\mathrm{d}x\\
\leq \Big[\int_{\Omega_m(r_0,+\infty)} \Big(\frac{|F(x,u_m)|}{|u_m|^2}\Big)^\kappa \mathrm{d}x\Big]^{1/\kappa} &\Big[\int_{\Omega_m(r_0,+\infty)} |v_m|^{2\kappa'}\mathrm{d}x\Big]^{1/\kappa'}
\end{aligned} \tag{13}$$

$$\leq c_0{}^{1/\kappa}\Big[\int_{\Omega_m(r_0,+\infty)}\mathcal{F}(x,u_m)\mathrm{d}x\Big]^{1/\kappa}\Big[\int_{\Omega_m(r_0,+\infty)}|v_m|^{2\kappa'}\mathrm{d}x\Big]^{1/\kappa'}$$
$$\leq [c_0(c+1)]^{1/\kappa}\Big[\int_{\Omega_m(r_0,+\infty)}|v_m|^{2\kappa'}\mathrm{d}x\Big]^{1/\kappa'} \leq [c_0(c+1)]^{1/\kappa}\,\|v_m\|^2_{L^{2\kappa'}(\mathbb{R}^N)} \to 0$$

as $m\to\infty$. Combining (12) with (13), we have

$$\int_{\mathbb{R}^N}\frac{|F(x,u_m)|}{\|u_m\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}}\mathrm{d}x$$
$$= \int\limits_{\Omega_m(0,r_0)}\frac{|F(x,u_m)|}{|u_m|^2}|v_m|^2\mathrm{d}x + \int\limits_{\Omega_m(r_0,+\infty)}\frac{|F(x,u_m)|}{|u_m|^2}|v_m|^2\mathrm{d}x \to 0, \quad \text{as } m\to\infty,$$

which contradicts (11).

Case 2: $v\neq 0$. Set $A:=\{x\in\mathbb{R}^N : v(x)\neq 0\}$, then $\mathrm{meas}(A)>0$. We have

$$\lim_{m\to\infty} u_m(x) = \lim_{m\to\infty}\|u_m\|^2_{S^2_{1,0}(\Omega)}\,v_m(x) = \infty, \text{ a.e. in } A.$$

It follows from (B3), (B4), (11) and Fatou's Lemma that

$$\frac{1}{2} = \lim_{m\to\infty}\int\limits_{\mathbb{R}^N}\frac{F(x,u_m)}{\|u_m\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}}\mathrm{d}x \geq \liminf_{m\to\infty}\int\limits_{A_v}\frac{F(x,u_m)}{\|u_m\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}}\mathrm{d}x$$
$$\geq \int\limits_{A_v}\liminf_{m\to\infty}\frac{F(x,u_m)}{\|u_m\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}}\mathrm{d}x = \int\limits_{A_v}\liminf_{m\to\infty}\frac{F(x,u_m)}{|u_m|^2}v_m^2\mathrm{d}x = +\infty, \tag{14}$$

where $A_v\subset A, \mathrm{meas}(A\backslash A_v)=0$, which is a contradiction. Thus $\{u_m\}_{m=1}^{\infty}$ is bounded in $S^2_{\gamma,b(x)}(\mathbb{R}^N)$.

Because of the above result, without loss of generality, we can suppose that

$$\begin{aligned} u_m &\rightharpoonup u \quad \text{in } S^2_{\gamma,b(x)}(\mathbb{R}^N) \ \text{ as } m\to\infty \\ u_m &\to u \quad \text{in } L^q(\mathbb{R}^N) \ \text{ as } m\to\infty, 2\leq q<2^*_\gamma. \end{aligned} \tag{15}$$

By (B3), we obtain

$$\Bigg|\int\limits_{\mathbb{R}^N}f(x,u_m)(u_m-u)\mathrm{d}x\Bigg| \leq \int\limits_{\mathbb{R}^N}|f_1(x)|\,|u_m|\,|u_m-u|\,\mathrm{d}x + \int\limits_{\mathbb{R}^N}|u_m-u|\,|u_m|^{p-1}|f_2(x)|\,\mathrm{d}x$$
$$\leq \|u_m-u\|_{L^{\frac{2p_1}{p_1-1}}(\mathbb{R}^N)}\,\|u_m\|_{L^{\frac{2p_1}{p_1-1}}(\mathbb{R}^N)}\,\|f_1\|_{L^{p_1}(\mathbb{R}^N)}$$
$$+\,\|u_m-u\|_{L^{\frac{pp_2}{p_2-1}}(\mathbb{R}^N)}\,\|u_m\|^{p-1}_{L^{\frac{pp_2}{p_2-1}}(\mathbb{R}^N)}\,\|f_2\|_{L^{p_2}(\mathbb{R}^N)}\,.$$

Since (15), we can conclude that

$$\int_{\mathbb{R}^N} f(x,u_m)(u_m-u)\mathrm{d}x \to 0 \quad \text{as} \quad m\to\infty. \tag{16}$$

Therefore
$$\int_{\mathbb{R}^N} [f(x,u_m)-f(x,u)]\,(u_m-u)\mathrm{d}x \to 0 \text{ as } m\to\infty. \tag{17}$$

Observe that

$$\|u_m-u\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} = \langle \Phi'(u_m)-\Phi'(u), u_m-u\rangle + \int_{\mathbb{R}^N} [f(x,u_m)-f(x,u)]\,(u_m-u)\mathrm{d}x.$$

It is clear that
$$\langle \Phi'(u_m)-\Phi'(u), u_m-u\rangle \to 0 \text{ as } m\to\infty. \tag{18}$$

From (17)–(18), we deduce that

$$\|u_m-u\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} \to 0 \text{ as } m\to\infty.$$

Therefore, we conclude that $u_m\to u$ strongly in $S^2_{\gamma,b(x)}(\mathbb{R}^N)$. The proof of Lemma 3.1 is complete. $\square$

Lemma 3.2. *Assume that b and f satisfy* (B1)–(B4) *and* (B6). *Then Φ satisfies the $(C)_c$ condition for all $c>0$ on $S^2_{\gamma,b(x)}(\mathbb{R}^N)$.*

Proof. Let $\{u_m\}_{m=1}^{\infty}$ be a sequence in $S^2_{\gamma,b(x)}(\mathbb{R}^N)$ such that

$$\left(1+\|u_m\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}\right)\|\Phi'(u_m)\|_{(S^2_{\gamma,b(x)}(\mathbb{R}^N))^*} \to 0 \text{ and } \Phi(u_m)\to c>0 \text{ as } m\to\infty,$$

hence

$$\Phi'(u_m)(u_m)\to 0 \quad\text{and}\quad \frac{1}{2}\|u_m\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} F(x,u_m)\mathrm{d}x \to c \text{ as } m\to\infty.$$

We first show that $\{u_m\}_{m=1}^{\infty}$ is bounded in $S^2_{\gamma,b(x)}(\mathbb{R}^N)$ by a contradiction argument. Indeed, suppose that

$$\|u_m\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} \to \infty \text{ as } m\to\infty. \tag{19}$$

Setting $v_m = \dfrac{u_m}{\|u_m\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}}$, then $\|v_m\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}=1$.

When m is large enough, we have

$$\begin{aligned} c+1 &\ge \Phi(u_m) - \frac{1}{\mu}\langle \Phi'(u_m), u_m\rangle \\ &= \frac{\mu-2}{2\mu}\|u_m\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} + \int_{\mathbb{R}^N}\left[\frac{1}{\mu}f(x,u_m)u - F(x,u_m)\right]\mathrm{d}x \\ &\ge \frac{\mu-2}{2\mu}\|u_m\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} - \frac{\varrho}{\mu}\|u_m\|^2_{L^2(\mathbb{R}^N)}, \end{aligned}$$

which implies
$$1 \le \frac{2\varrho}{\mu - 2} \limsup_{m\to\infty} \|v_m\|^2_{L^2(\mathbb{R}^N)}. \tag{20}$$

Hence, it follows from (20) that $v \neq 0$. By a similar process as in (14), we can conclude a contradiction. Thus, $\{u_m\}_{m=1}^{\infty}$ is bounded in $S^2_{\gamma,b(x)}(\mathbb{R}^N)$. The rest proof is the same as that in Lemma 3.2. □

Lemma 3.3. *Assume that b and f satisfy* (B1)- (B3) *and* (B4)*.Then for any finite dimensional subspace $\widehat{\mathbb{X}} \subset S^2_{\gamma,b(x)}(\mathbb{R}^N)$, there is $R = R(\widehat{\mathbb{X}}) > 0$ such that*

$$\Phi(u) \le 0, \quad \forall u \in \widehat{\mathbb{X}}, \ \ \|u\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} \ge R.$$

Proof. Arguing by contradiction, suppose that for some sequence $\{u_m\}_{m=1}^{\infty} \subset \widehat{\mathbb{X}}$ with $\|u_m\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} \to \infty$, there is $M > 0$ such that $\Phi(u_m) \ge -M$ for all $m \in \mathbb{N}$.

Set $\quad v_m(x) = \dfrac{u_m}{\|u_m\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}}, \quad$ then $\|v_m\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} = 1.$

Therefore, we can (by passing to a subsequence if necessary) suppose that

$$\begin{aligned}
v_m &\rightharpoonup v \quad \text{weakly in } S^2_{\gamma,b(x)}(\mathbb{R}^N) \ \text{ as } m \to \infty,\\
v_m &\to v \quad \text{a.e. in } \mathbb{R}^N \ \text{ as } m \to \infty.\\
v_m &\to v \quad \text{strongly in } L^q(\mathbb{R}^N) \ \text{ as } m \to \infty, 2 \le q < 2^*_\gamma.
\end{aligned}$$

Since $\widehat{\mathbb{X}}$ is finite dimensional, then

$$v_m \to v \text{ strongly in } \widehat{\mathbb{X}} \ \text{ as } m \to \infty$$

and $v \in \widehat{\mathbb{X}}$, $\|v\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} = 1$. Therefore, it follows from (14) that

$$\begin{aligned}
0 &= \lim_{m\to\infty} \frac{-M}{\|u_m\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}} \le \lim_{m\to\infty} \frac{\Phi(u_m)}{\|u_m\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}}\\
&\le C_1 - \int_{\mathbb{R}^N} \liminf_{m\to\infty} \frac{F(x,u_m)}{|u_m|^2} v_m^2 \mathrm{d}x = -\infty.
\end{aligned}$$

Hence, we arrive at a contradiction. So, there is $R = R(\widehat{\mathbb{X}}) > 0$ such that $\Phi(u) \le 0$ for $u \in \widehat{\mathbb{X}}$ and $\|u\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} \ge R$. □

Let $\{e_j\}_{j=1}^{\infty}$ be a total orthonormal basis of $S^2_{\gamma,b(x)}(\mathbb{R}^N)$ and define $\mathbb{X}_j = \mathbb{R}e_j$,

$$\mathbb{Y}_k = \bigoplus_{j=1}^{k} \mathbb{X}_j, \quad \mathbb{Z}_k = \bigoplus_{j=k+1}^{\infty} \mathbb{X}_j, \quad k \in \mathbb{N}.$$

Let
$$\beta_k = \sup_{\substack{u\in\mathbb{Z}_k\\ \|u\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}=1}} \|u\|_{L^q(\mathbb{R}^N)}, 2 \le q < 2^*_\gamma \tag{21}$$

then $\beta_k \to 0$ as $k \to \infty$. From Lemma 3.3 in [13], we have

Lemma 3.4. *Assume that b and f satisfy* (B1), (B2) *and* (B3). *Then there exists constants $\rho, \alpha, k > 0$ such that $\Phi(u) \geq \alpha$ for all $u \in \mathbb{Z}_k$ such that $\|u\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} = \rho$.*

Proof of Theorem 1.1. Let $\mathbb{X} \equiv S^2_{\gamma,b(x)}(\mathbb{R}^N), \mathbb{Y} \equiv \mathbb{Y}_k, \mathbb{Z} \equiv \mathbb{Z}_k$. By Lemmas 3.1, 3.3 and 3.4 all conditions of Lemma 2.4 are satisfied. Thus, problem (2) possesses infinitely many nontrivial solutions. $\square$

Proof of Theorem 1.2. Let $\mathbb{X} \equiv S^2_{\gamma,b(x)}(\mathbb{R}^N), \mathbb{Y} \equiv \mathbb{Y}_k, \mathbb{Z} \equiv \mathbb{Z}_k$. By Lemmas 3.2, 3.3 and 3.4 all conditions of Lemma 2.4 are satisfied. Thus, problem (2) possesses infinitely many nontrivial solutions. $\square$

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