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Basic Positive Semi-Definite Hankel Tensors

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Some classes of positive semi-definite Hankel tensors which are not strong Hankel tensors were recently introduced by Q. Wang, G. Li, L. Qi and Y. Xu [*New classes of positive semi-definite Hankel tensors*, Minimax Theory and its Applications 2 (2017) 231-248]. In this paper, we continue the study of such tensors. We introduce a subclass of Hankel tensors called basic positive semi-definite Hankel tensors and intend to find some low-rank basic PSD non-strong Hankel tensors. We show that rank-1 even order strong Hankel tensors are equivalent to rank-1 basic PSD Hankel tensors, and all even order strong Hankel tensors with rank larger than 1 can be expressed as the sum of rank-1 basic PSD Hankel tensors. Thus, the study of non-strong PSD Hankel tensors is reduced to the study of basic PSD Hankel tensors with rank larger than 1. In this paper, we prove that (1) there are no rank-2 basic PSD Hankel tensors, (2) rank-3 basic PSD Hankel tensors with dimension no less than 3 do not exist. Furthermore, an example of low-rank basic PSD Hankel tensor whose rank equals 3 or 4 is provided.

Keywords: Hankel tensors, basic positive semi-definite Hankel tensors, symmetric rank, Vandermonde decomposition.

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1. Introduction

The concept of Hankel tensors was first introduced by Luque and Thibon [15] to our best knowledge. Papy, De Lathauwer, and Van Huffel [19] initially employed Hankel tensors in the harmonic retrieval problem, which is at the heart of many signal processing problems. Hankel tensors have been widely applied in signal processing [1, 3, 4, 7], automatic control [27], and geophysics [18, 28]. Particularly, the positive semi-definiteness of Hankel tensors can be a criterion for the solvability of multidimensional moment problems [2, 12, 22].

It was proved by Hilbert [10] that for homogeneous polynomials, only in the following three cases, a positive semi-definite (PSD) polynomial is definitely a

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sum-of-squares (SOS) polynomial: (1) $n = 2$; (2) $m = 2$; (3) $m = 4$ and $n = 3$, where m is the degree of the polynomial and n is the number of variables. Hilbert proved that in all the other possible combinations of n and even m, there are PSD non-SOS (short for PNS as in [6]) homogeneous polynomials. The most well-known PNS homogeneous polynomial is the Motzkin function [16] with $m = 6$ and $n = 3$. A homogeneous polynomial is uniquely corresponding to a symmetric tensor [20]. For symmetric tensors, m is the order and n is the dimension. A Hankel tensor is clearly a symmetric tensor.

In [21], it was shown that an mth-order n-dimensional tensor is a Hankel tensor if and only if it has a Vandermonde decomposition. Two classes of PSD Hankel tensors were identified: even order strong Hankel tensors and even order complete Hankel tensors. It was proved in [14] that complete Hankel tensors are strong Hankel tensors, and even order strong Hankel tensors are SOS tensors. There were also some examples of SOS Hankel tensors and PSD Hankel tensors which are not strong Hankel tensors. Thus, a question was raised in [14]: Are all PSD Hankel tensors SOS tensors? If there are no PNS Hankel tensors, the problem for determining a given even order Hankel tensor is PSD or not can be answered by solving a semi-definite linear programming problem. The problem raised by the above question is called the Hilbert-Hankel problem, which is the first one of three open problems on Hankel tensors [26].

Generalized anti-circulant tensors [13] were studied, which are one special class of Hankel tensors. The necessary and sufficient conditions for positive semi-definiteness of even order generalized anti-circulant tensors in some cases were given, and the tensors are strong Hankel tensors and SOS tensors in these cases. An inheritance property was established in [21] for strong Hankel tensors, and this property was then extended to general Hankel tensors in [8], which means that if a lower-order Hankel tensor is positive semi-definite (or positive definite, or negative semi-definite, or negative definite, or SOS), then its associated higher-order Hankel tensor with the same generating vector, where the higher order is a multiple of the lower order, is also positive semi-definite (or positive definite, or negative semi-definite, or negative definite, or SOS, respectively). In addition, the SOS decomposition of strong Hankel tensors was also given in [8]. Other discussions about PSD Hankel tensors, SOS Hankel tensors and PNS Hankel tensors and some regions where PNS Hankel tensors do not exist were given in [5]. More properties of the above tensors are introduced in [23, 24]. An algorithm for computing Vandermonde rank decompositions for all Hankel tensors was given in [17] and it was also proved that for a generic Hankel tensor of order even or three, the CP rank, symmetric rank, border rank, symmetric border rank and Vandermonde rank are all the same.

Ding, Qi, and Wei [8] proved that a Hankel tensor is a strong Hankel tensor if and only if it admits a Vandermonde decomposition with positive coefficients or an augmented Vandermonde decomposition with positive coefficients. Thus, the decomposition of strong Hankel tensors has been settled. However, still little is known for non-strong Hankel tensors. Some non-strong PSD Hankel tensors were characterized in [29], yet a systematic investigation on non-strong Hankel tensors needs to be conducted.

In this paper, we continue to study positive semi-definite Hankel tensors that are not strong. A new subclass of Hankel tensors called basic PSD Hankel tensors is introduced. We show that a rank-1 even order Hankel tensor is a strong Hankel tensor if and only if it is a basic PSD Hankel tensor, and even order strong Hankel tensors with rank higher than 1 can be represented as the sum of rank-1 basic PSD Hankel tensors. Therefore, the study of non-strong PSD Hankel tensors is converted to the study of basic PSD Hankel tensors with rank > 1. The properties of basic PSD Hankel tensors and decomposition of non-basic PSD Hankel tensors will help us find solutions to the three open problems on Hankel tensors in further research, since all of the open problems are in the context of PSD Hankel tensors.

The remainder of this paper is organized as follows. Some preliminaries are given in Section 2. Basic positive semi-definite Hankel tensor is defined in Section 3 and the relationships between strong Hankel tensors and basic PSD Hankel tensors are also given. In Section 4, it is proved that within mth-order n-dimensional PSD Hankel tensors, rank-2 basic PSD Hankel tensors do not exist. In Section 5, the existence of rank-3 basic PSD Hankel tensors whose dimensions are not smaller than 3 is disproved. Finally in Section 6, we present an example of 4th-order 2-dimensional basic PSD Hankel tensor whose rank is 3 or 4.

2. Preliminaries

Denote $[n] := \{1, 2, \cdots, n\}$. An mth-order n-dimensional *tensor* $\mathcal{A} = (a_{i_1 \cdots i_m})$ consists of n^m entries $a_{i_1 \cdots i_m} \in \mathbb{C}$, where $i_j \in [n]$ for $j = 1, \cdots, m$ [20].

For a vector $\mathbf{x} \in \mathbb{C}^n$, we use x_i to denote its ith component, and $\mathbf{x}^{[m]}$ to denote a vector in $\mathbb{C}^n$ such that $(\mathbf{x}^{[m]})_i = x_i^m$ for all $i \in [n]$. For $\mathbf{x} \in \mathbb{C}^n$, $\mathcal{A}\mathbf{x}^{m-1}$ denotes a vector in $\mathbb{C}^n$, whose ith component is

$$\left(\mathcal{A}\mathbf{x}^{m-1}\right)_i = \sum_{i_2, \cdots, i_m \in [n]} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}. \tag{1}$$

A complex number λ is called an *eigenvalue* [20] of $\mathcal{A}$ if it and a nonzero vector $\mathbf{x} \in \mathbb{C}^n$ are solutions of the following homogeneous polynomial system:

$$\mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]}, \tag{2}$$

and the solution vector $\mathbf{x}$ is called an *eigenvector* of $\mathcal{A}$ associated with the eigenvalue λ. If an eigenvalue λ has a real eigenvector, then we say that it is an *H-eigenvalue.*

The multiplication of a tensor $\mathcal{A}$ and a matrix $M = (m_{i_k j_k})$ along kth mode [9] is defined by

$$\left(\mathcal{A} \times_k M\right)_{i_1 \cdots i_{k-1} j_k i_{k+1} \cdots i_m} := \sum_{i_k = 1}^{n} a_{i_1 i_2 \cdots i_k \cdots i_m} m_{i_k j_k}. \tag{3}$$

The tensor $\mathcal{A}$ is said to be a *symmetric tensor* if its entries $a_{i_1\cdots i_m}$ are invariant under any index permutation. Denote the set of all real symmetric tensors of order m and dimension n by $S_{m,n}$. All symmetric tensors $\mathcal{A} \in S_{m,n}$ have such symmetric CP decompositions of the form

$$\mathcal{A} = \textstyle\sum_{i=1}^{r} \alpha_i \mathbf{v}_i^{\otimes m},\ \mathbf{v}_i \in \mathbb{R}^n,\ \mathbf{v}_i^{\otimes m} = \underbrace{\mathbf{v}_i \otimes \mathbf{v}_i \otimes \cdots \otimes \mathbf{v}_i}_{m}$$

is a rank-one tensor, where the outer product $\otimes$ is defined by

$$(\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \cdots \otimes \mathbf{v}_m)_{i_1 i_2 \cdots i_m} = (\mathbf{v}_1)_{i_1} (\mathbf{v}_2)_{i_2} \cdots (\mathbf{v}_m)_{i_m}.$$

The *symmetric rank* of $\mathcal{A}$ is defined by

$$\mathrm{rank}(\mathcal{A}) = \min\{r : \mathcal{A} = \textstyle\sum_{i=1}^{r} \alpha_i \mathbf{v}_i^{\otimes m}, \alpha_i \in \mathbb{R}, \mathbf{v}_i \in \mathbb{R}^n\}.$$

For $\mathcal{A} \in S_{m,n}$ and $\mathbf{x} \in \mathbb{R}^n$, we have a homogeneous polynomial $f(\mathbf{x})$ of n variables and degree m,

$$f(\mathbf{x}) = \mathcal{A}\mathbf{x}^m \equiv \sum_{i_1,\cdots,i_m \in [n]} a_{i_1\cdots i_m} x_{i_1} \cdots x_{i_m}. \tag{4}$$

Note that there is a one-to-one correspondence between homogeneous polynomials and symmetric tensors. For even order tensor $\mathcal{A} \in S_{m,n}$, if $f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$, then the homogeneous polynomial $f(\mathbf{x})$ and symmetric tensor $\mathcal{A}$ are called *positive semi-definite* (PSD). If $f(\mathbf{x}) > 0$ for all nonzero $\mathbf{x} \in \mathbb{R}^n$, then $f(\mathbf{x})$ and $\mathcal{A}$ are called *positive definite* (PD). The concepts of positive semi-definite and positive definite symmetric tensors were introduced in [20]. The problem of determining whether a given even order symmetric tensor is positive semi-definite or not has important applications in engineering and science [7, 11, 25]. For a tensor whose order m is odd, if it has no negative H-eigenvalues, we call it a *generalized positive semi-definite* (generalized PSD) tensor.

Let $\mathbf{v} = (v_1, \cdots, v_{(n-1)m+1})^\top$. Define $\mathcal{A} = (a_{i_1\cdots i_m}) \in S_{m,n}$ by

$$a_{i_1\cdots i_m} = v_{i_1+\cdots+i_m-m+1}, \tag{5}$$

for $i_1, \cdots, i_m \in [n]$. Then $\mathcal{A}$ is called a *Hankel tensor* [14, 21] and $\mathbf{v}$ is called the *generating vector* of $\mathcal{A}$. If $\mathcal{A}$ is a Hankel tensor, then homogeneous polynomial $f(\mathbf{x}) = \mathcal{A}\mathbf{x}^m$ is called a Hankel polynomial.

Let $A = (a_{ij})$ be an $\lceil \frac{(n-1)m+2}{2} \rceil \times \lceil \frac{(n-1)m+2}{2} \rceil$ matrix with $a_{ij} \equiv v_{i+j-1}$, where $v_{2\lceil \frac{(n-1)m}{2} \rceil}$ is an additional number which can be arbitrarily selected when $(n-1)m$ is odd. Such a matrix A is called a Hankel matrix, associated with the Hankel tensor $\mathcal{A}$. When $(n-1)m$ is even, the associated Hankel matrix is unique. Recall from [21] that $\mathcal{A}$ is called a *strong Hankel tensor* if there exists an associated Hankel matrix A which is positive semi-definite.

Let $g(\mathbf{y}) = \mathbf{y}^\top A \mathbf{y}$, where $\mathbf{y} = \left(y_1, \cdots, y_{\frac{(n-1)m+2}{2}}\right)^\top$ and A is an associated Hankel matrix of $\mathcal{A}$. Then, $\mathcal{A}$ is a strong Hankel tensor if and only if g is PSD for at least one associated Hankel matrix A of $\mathcal{A}$.

Following [8], if $\mathbf{v}^{\otimes m}$ is a rank-one Hankel tensor, then $\mathbf{v} = \alpha(1, \xi, \cdots, \xi^{n-1})^\top$ or $\alpha \mathbf{e}_n = \alpha(0, 0, \cdots, 0, 1)^\top$; here the vectors $(1, \xi, \cdots, \xi^{n-1})^\top$ and $\mathbf{e}_n$ are called *Vandermonde vectors*. Let $\mathcal{A}$ be an mth-order n-dimensional Hankel tensor and the rank of its associated Hankel matrix be r. $\mathcal{A}$ is a strong Hankel tensor if and only if it admits a Vandermonde decomposition with positive coefficients:

$$\mathcal{A} = \sum_{k=1}^{r} \alpha_k \mathbf{v}_k^{\otimes m}, \; \alpha_k > 0, \; \mathbf{v}_k \text{ are Vandermonde vectors.} \tag{6}$$

Also, there are many PSD Hankel tensors that are not strong Hankel tensors. For instance, consider the Hankel tensors $\mathcal{A}$ generated by

$$\mathbf{v} = \left(v_0, 0, \cdots, 0, v_{\frac{(n-1)m}{2}}, 0, \cdots, 0, v_{(n-1)m}\right)^\top,$$

where n is odd. Such Hankel tensors are called *truncated Hankel tensors* in [29]. In this case $f(\mathbf{x})$ and $g(\mathbf{y})$ have a simple form:

$$\begin{aligned} f(\mathbf{x}) = {} & v_0 x_1^m + v_{(n-1)m} x_n^m + \\ & + v_{\frac{(n-1)m}{2}} \sum \Big\{ \tbinom{m}{t_1} \tbinom{m-t_1}{t_2} \cdots \tbinom{m-t_1-t_2-\cdots-t_{n-2}}{t_{n-1}} x_1^{t_1} x_2^{t_2} \cdots x_n^{m-t_1-t_2-\cdots-t_{n-1}} \\ & : (n-1)t_1 + (n-2)t_2 + \cdots + t_{n-1} = \tfrac{(n-1)m}{2} \Big\}, \quad \text{and} \end{aligned} \tag{7}$$

$$\begin{aligned} g(\mathbf{y}) = {} & v_0 y_1^2 + v_{(n-1)m} y^2_{\frac{(n-1)m+2}{2}} + \\ & + v_{\frac{(n-1)m}{2}} \Big(y^2_{\frac{(n-1)m}{4}+1} + \sum_{i \neq j} \Big\{ y_i y_j : i + j = \tfrac{(n-1)m}{2} + 2 \Big\} \Big). \end{aligned} \tag{8}$$

Since we are only concerned about PSD Hankel tensors, we may assume that $v_0, v_{\frac{(n-1)m}{2}}$, and $v_{(n-1)m}$ are all nonnegative. If $v_{\frac{(n-1)m}{2}} = 0$, then the truncated Hankel tensor $\mathcal{A}$ is a strong Hankel tensor, and furthermore an SOS Hankel tensor if m is even. If $v_{\frac{(n-1)m}{2}} > 0$, then $\mathcal{A}$ is not a strong Hankel tensor [29].

3. Basic PSD Hankel tensors

Since Hankel tensors are symmetric tensors, it is best to describe the rank of Hankel tensors by symmetric rank [24]. In the following, we use rank($\mathcal{A}$) to express the symmetric rank of Hankel tensor $\mathcal{A}$. Let $\mathcal{A}$ be an mth-order n-dimensional PSD Hankel tensor and its rank be r. Then $\mathcal{A}$ is called a *basic PSD Hankel tensor*, if there is no nonzero PSD Hankel tensor $\mathcal{B}$ with rank($\mathcal{B}$) $< r$ such that $\mathcal{A} - \mathcal{B}$ is PSD. From the definition, we can derive the following lemma straightforwardly.

Lemma 3.1. *Given $\mathcal{A}$ is a rank-1 even order Hankel tensor. $\mathcal{A}$ is a strong Hankel tensor if and only if $\mathcal{A}$ is a basic PSD Hankel tensor.*

Proof. This lemma can be easily derived from the Vandermonde decomposition of strong Hankel tensors. ☐

Theorem 3.2. *All even order strong Hankel tensors with rank larger than 1 are not basic PSD Hankel tensors.*

Proof. Assume that an even order strong Hankel tensor $\mathcal{A}$ with rank $r \geq 2$ is basic, then from (6), $\mathcal{A} = \sum_{k=1}^{r} \alpha_k \mathbf{v}_k^{\otimes m}$, $\alpha_k > 0$, $\mathbf{v}_k$ are Vandermonde vectors. Let $\mathcal{B} = \alpha_1 \mathbf{v}_1^{\otimes m}$, then $\mathcal{B}$ is a positive semi-definite Hankel tensor, while $\mathcal{A} - \mathcal{B}$ is still positive semi-definite, which is a contradiction. □

Corollary 3.3. *All even order strong Hankel tensors with rank larger than 1 can be expressed as the sum of rank-1 basic PSD Hankel tensors.*

Clearly, PSD non-strong Hankel tensors with rank larger than 1 do exist. For example [29], for PSD truncated Hankel tensor $\mathcal{A}$ when $v_{\frac{(n-1)m}{2}} > 0$, consider vector $\bar{\mathbf{y}} = \mathbf{e}_i - \mathbf{e}_j$ where $i + j = \frac{(n-1)m}{2} + 2$, $i \neq j$ and $i \neq 1$ or $\frac{(n-1)m+2}{2}$. We see that $g(\bar{\mathbf{y}}) = -2v_{\frac{(n-1)m}{2}} < 0$, hence $\mathcal{A}$ is not a strong Hankel tensor. Therefore basic PSD Hankel tensors with rank larger than 1 also exist. What we concern is the smallest symmetric rank of non-strong basic PSD Hankel tensors.

Lemma 3.4. *If PSD Hankel tensor $\mathcal{A}$ has the following Vandermonde decomposition*

$$\mathcal{A} = \sum_{k=1}^{r} \alpha_k \mathbf{v}_k^{\otimes m},$$

where $\mathbf{v}_k \in \mathbb{R}^n$ are mutually distinct Vandermonde vectors, $r \leq n$, then $\alpha_k > 0$, $k = 1, 2, \cdots, r$. Thus, $\mathcal{A}$ is a strong Hankel tensor.

Proof. $\mathcal{A}$ can be also expressed as

$$\mathcal{A} = \mathcal{D} \times_1 V^\top \times_2 V^\top \cdots \times_m V^\top,$$

where $\mathcal{D}$ is a diagonal tensor with diagonal entries $\alpha_1, \alpha_2, \cdots, \alpha_r$ and matrix $V = (\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r)$ is of full column rank. Without loss of generality, assume that $\alpha_1 < 0$, then there exists a unique $\mathbf{x}$ satisfying $V^T\mathbf{x} = \mathbf{e}_1$ such that $f(\mathbf{x}) = \mathcal{A}\mathbf{x}^m = \mathcal{D}\mathbf{e}_1^m = \alpha_1 < 0$, which is a contradiction to the positive semi-definiteness of $\mathcal{A}$. Therefore we have all $\alpha_k \geq 0$ for $k = 1, 2, \cdots, r$, thus $\mathcal{A}$ is a strong Hankel tensor. □

4. Rank-2 Basic PSD Hankel Tensors

We begin with the rank-2 case and we shall shortly see that there are no rank-2 basic PSD Hankel tensors. For mth-order n-dimensional PSD Hankel tensor $\mathcal{A}$, $m \geq 4$, if $\mathcal{A}$ is basic and its rank is 2, then it has the following form

$$\mathcal{A} = \alpha \mathbf{x}^{\otimes m} + \beta \mathbf{y}^{\otimes m}, \tag{9}$$

where $\alpha, \beta \neq 0$, $\mathbf{x} = (x_1, x_2, ..., x_n)^\top \in \mathbb{R}^n$, $\mathbf{y} = (y_1, y_2, ..., y_n)^\top \in \mathbb{R}^n$, $\mathbf{x} \neq \mathbf{y}$. As $\mathcal{A}$ is positive semi-definite, at least one of α and β is positive. Without loss of generality, assume $\beta = 1$, i.e.,

$$\mathcal{A} = \alpha \mathbf{x}^{\otimes m} + \mathbf{y}^{\otimes m}.$$

Theorem 4.1. *Rank-2 basic PSD Hankel tensors do not exist.*

Proof. If $n = 2$, from Lemma 2, $\mathcal{A}$ is not a basic PSD Hankel tensor. If $n \geq 3$, we classify the decomposition into the next four cases.

Case 1. $\mathbf{x} = (1, x_2, x_3, \cdots, x_n)^\top, \mathbf{y} = (1, y_2, y_3, \cdots, y_n)^\top$.

From the definition of Hankel tensors, $a_{11\ldots122} = a_{11\ldots13}$, then

$$\alpha(x_2^2 - x_3) = y_3 - y_2^2. \tag{10}$$

(a) If $x_2^2 \neq x_3$ and $y_2^2 \neq y_3$, then similarly we have $\alpha x_i(x_2^2 - x_3) = y_i(y_3 - y_2^2)$ for $2 \leq i \leq n$. By dividing this equation by (10) on both sides, we get $x_i = y_i$ for $2 \leq i \leq n$, hence $\mathbf{x} = \mathbf{y}$, the rank of $\mathcal{A}$ is actually 1, which is a contradiction.

(b) If $x_2^2 = x_3$, then $y_2^2 = y_3$. From the definition of Hankel tensors, we have

$$\begin{cases} \alpha x_1^{m-3}x_2^3 + y_1^{m-3}y_2^3 = \alpha x_1^{m-2}x_4 + y_1^{m-2}y_4, \\ \alpha x_1^{m-1}x_5 + y_1^{m-1}y_5 = \alpha x_1^{m-2}x_2x_4 + y_1^{m-2}y_2y_4, \\ \alpha x_1^{m-2}x_3^2 + y_1^{m-2}y_3^2 = \alpha x_1^{m-1}x_5 + y_1^{m-1}y_5. \end{cases}$$

Processing these equations, we have $\alpha(x_2^4 - x_5) = x_2y_4 - x_2y_2^3 + y_5 - y_2y_4 = y_5 - y_2^4$. Thus $x_2 = y_2$ or $y_4 = y_2^3$. If $x_2 = y_2$, then $x_3 = y_3$ and $\alpha = 1$, hence $x_4 = y_4, ..., x_n = y_n$, $\mathbf{x} = \mathbf{y}$, the rank of $\mathcal{A}$ is 1, which is a contradiction. If $y_4 = y_2^3$, then $y_5 = y_2^4$, ..., $y_n = y_2^{n-1}$, and $x_4 = y_2^3$, ..., $x_n = x_2^{n-1}$, $\mathbf{x}$ and $\mathbf{y}$ are both Vandermonde vectors. By Lemma 2, it is a contradiction.

Case 2. $\mathbf{x} = (1, x_2, x_3, \cdots, x_n)^\top, \mathbf{y} = (0, \cdots, 0, 1, y_{k+1}, \cdots, y_n)^\top$, $2 \leq k \leq n$.

If $k = n$, i.e., $\mathbf{y} = \mathbf{e}_n$, $x_j = x_2^{j-1}$ for $2 \leq j \leq n$, it is a Vandermonde decomposition, which is a contradiction. If $2 \leq k \leq n-1$, obviously $x_j = x_2^{j-1}$ for $2 \leq j \leq n$, then (i) $\alpha x_k^m + y_k^m = \alpha x_{k-1}^{\frac{m}{2}} x_{k+1}^{\frac{m}{2}} + y_{k-1}^{\frac{m}{2}} y_{k+1}^{\frac{m}{2}}$ for m is even, (ii) $\alpha x_k^m + y_k^m = \alpha x_{k-1}^{\frac{m-1}{2}} x_k x_{k+1}^{\frac{m-1}{2}} + y_{k-1}^{\frac{m-1}{2}} y_k y_{k+1}^{\frac{m-1}{2}}$ for m is odd, i.e., $\alpha x_k^m + 1 = \alpha x_k^m$, which is also a contradiction.

Case 3. $\mathbf{x} = (0, \cdots, 0, 1, x_{k+1}, \cdots, x_n)^\top$, $\mathbf{y} = (0, \cdots, 0, 1, y_{l+1}, \cdots, y_n)^\top$, where $2 \leq k \leq n$, $2 \leq l \leq n-1$, $k \neq l$.

Without loss of generality, assume $k > l$. We have

(i) $\alpha x_k^m + y_k^m = \alpha x_{k-1}^{\frac{m}{2}} x_{k+1}^{\frac{m}{2}} + y_{k-1}^{\frac{m}{2}} y_{k+1}^{\frac{m}{2}}$ for m is even,

(ii) $\alpha x_k^m + y_k^m = \alpha x_{k-1}^{\frac{m-1}{2}} x_k x_{k+1}^{\frac{m-1}{2}} + y_{k-1}^{\frac{m-1}{2}} y_k y_{k+1}^{\frac{m-1}{2}}$ for m is odd,

i.e., $\alpha = 0$, which is a contradiction.

Case 4. $\mathbf{x} = (0, \cdots, 0, 1, x_{k+1}, \cdots, x_n)^\top, \mathbf{y} = (0, \cdots, 0, 1, y_{k+1}, \cdots, y_n)^\top$, where $2 \leq k \leq n$.

If $k = n$, the rank of $\mathcal{A}$ is 1, which is a contradiction. If $2 \leq k \leq n-1$, we have

(i) $\alpha x_k^m + y_k^m = \alpha x_{k-1}^{\frac{m}{2}} x_{k+1}^{\frac{m}{2}} + y_{k-1}^{\frac{m}{2}} y_{k+1}^{\frac{m}{2}}$ for m is even,

(ii) $\alpha x_k^m + y_k^m = \alpha x_{k-1}^{\frac{m-1}{2}} x_k x_{k+1}^{\frac{m-1}{2}} + y_{k-1}^{\frac{m-1}{2}} y_k y_{k+1}^{\frac{m-1}{2}}$ for m is odd.

From both situations we get $\alpha = -1$, then

(i) $\alpha x_k^{m-1}x_{k+1} + y_k^{m-1}y_{k+1} = \alpha x_{k-1}^{\frac{m}{2}-1}x_k x_{k+1}^{\frac{m}{2}} + y_{k-1}^{\frac{m}{2}-1}y_k y_{k+1}^{\frac{m}{2}}$ for m is even,

(ii) $\alpha x_k^{m-1}x_{k+1} + y_k^{m-1}y_{k+1} = \alpha x_{k-1}^{\frac{m-1}{2}}x_{k+1}^{\frac{m+1}{2}} + y_{k-1}^{\frac{m-1}{2}}y_{k+1}^{\frac{m+1}{2}}$ for m is odd.

Thus $x_{k+1} = y_{k+1}$, ..., $x_n = y_n$, i.e., $\mathcal{A} = \mathbf{0}$, which is a contradiction. □

5. Rank-3 Basic PSD Hankel tensors

For an mth-order n-dimensional PSD Hankel tensor $\mathcal{A}$, $m \geq 4$, $n \geq 3$, if $\mathcal{A}$ is a basic PSD Hankel tensor and its rank is 3, then it can be expressed as

$$\mathcal{A} = \alpha \mathbf{x}^{\otimes m} + \beta \mathbf{y}^{\otimes m} + \gamma \mathbf{z}^{\otimes m}, \tag{11}$$

where $\alpha, \beta, \gamma \neq 0$, $\mathbf{x} = (x_1, x_2, ..., x_n)^\top \in \mathbb{R}^n$, $\mathbf{y} = (y_1, y_2, ..., y_n)^\top \in \mathbb{R}^n$, $\mathbf{z} = (z_1, z_2, ..., z_n)^\top \in \mathbb{R}^n$, $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are mutually distinct. Similar to the previous chapter, at least one of α, β and γ is positive. Without loss of generality, let $\gamma = 1$, then

$$\mathcal{A} = \alpha \mathbf{x}^{\otimes m} + \beta \mathbf{y}^{\otimes m} + \mathbf{z}^{\otimes m}.$$

Theorem 5.1. *Rank-3 basic PSD Hankel tensors with dimension no less than 3 do not exist.*

Proof. The decomposition can be classified into the next four cases.

Case 1. $\mathbf{x} = (1, x_2, x_3, \cdots, x_n)^\top, \mathbf{y} = (1, y_2, y_3, \cdots, y_n)^\top, \mathbf{z} = (1, z_2, z_3, \cdots, z_n)^\top$.

Similar to the rank-2 situation, from the definition of Hankel tensors,

$$\alpha x_1^{m-2}x_2^2 + \beta y_1^{m-2}y_2^2 + z_1^{m-2}z_2^2 = \alpha x_1^{m-1}x_3 + \beta y_1^{m-1}y_3 + z_1^{m-1}z_3,$$

so

$$z_3 - z_2^2 = \alpha(x_2^2 - x_3) + \beta(y_2^2 - y_3). \tag{12}$$

Similarly, we have $\begin{cases} \alpha x_2^3 + \beta y_2^3 + z_2^3 = \alpha x_2 x_3 + \beta y_2 y_3 + z_2 z_3, \\ \alpha x_2^4 + \beta y_2^4 + z_2^4 = \alpha x_2^2 x_3 + \beta y_2^2 y_3 + z_2^2 z_3. \end{cases}$

By substituting (12) into the above equations, we obtain

$$\alpha(x_2 - z_2)(x_2^2 - x_3) = -\beta(y_2 - z_2)(y_2^2 - y_3), \tag{13}$$

$$\alpha(x_2^2 - z_2^2)(x_2^2 - x_3) = -\beta(y_2^2 - z_2^2)(y_2^2 - y_3). \tag{14}$$

Next, we discuss whether the factors in equation (13) are zero or not, and classify it into the next four situations (a)–(d).

(a) $x_2 \neq z_2$, $y_2 \neq z_2$, $x_2^2 \neq x_3$, $y_2^2 \neq y_3$.

Divide (14) by (13) on both sides, we get $x_2 = y_2$. Then, similarly

$$\alpha(x_k - z_k)(x_2^2 - x_3) = -\beta(y_k - z_k)(y_2^2 - y_3) \text{ for } 3 \leq k \leq n,$$

hence $\mathbf{x} = \mathbf{y}$, which is a contradiction.

(b) $x_2 = z_2$, $y_2 = z_2$. For x_3, y_3, z_3, we have

$$\alpha(z_3 - x_3)(x_2^2 - x_3) = -\beta(y_3 - z_3)(y_2^2 - y_3), \tag{15}$$

$$\alpha(z_3^2 - x_3^2)(x_2^2 - x_3) = -\beta(y_3^2 - z_3^2)(y_2^2 - y_3). \tag{16}$$

We get two similar equations about x_3, y_3, z_3, and also discuss the factors in the situations (i)–(iv).

(i) If $x_3 \neq z_3$, $y_3 \neq z_3$, $x_2^2 \neq x_3$, $y_2^2 \neq y_3$, divide (16) by (15), and we get $x_3 = y_3$.

(ii) If $x_3 = z_3$, $y_3 = z_3$, obviously $x_3 = y_3$.

(iii) If $x_3 = x_2^2$, $y_3 = y_2^2$, we obtain $x_3 = y_3$, because $x_2 = y_2$.

(iv) If $x_3 = z_3$, $y_3 = y_2^2$ (or $y_3 = z_3$, $x_3 = x_2^2$), substitute the two equations into (12), we have $x_3 = x_2^2 = y_2^2 = y_3$. Then $x_3 = y_3 = z_3$ for (i)–(iv). Similarly, we can prove $x_k = y_k$ for $4 \leq k \leq n$, hence $\mathbf{x} = \mathbf{y}$, which is a contradiction.

(c) $x_2^2 = x_3$, $y_2 = z_2$.

(i) If $y_2^2 \neq y_3$, we have $\alpha x_2^2 + \beta y_2^2 + y_2^2 = \alpha x_3 + y_3 + z_3$, hence $\beta(y_2^2 - y_3) = z_3 - y_2^2$. Also, for $3 \leq i \leq n$, $\beta y_i(y_2^2 - y_3) = z_i(z_3 - y_2^2)$, hence we get $y_3 = z_3$, ..., $y_n = z_n$, i.e., $\mathbf{y} = \mathbf{z}$. Thus, the rank is 2, which is a contradiction.

(ii) If $y_2^2 = y_3$, we will find $\mathbf{x}$ is a Vandermonde vector, then by (b) of Case 1 in the previous chapter, $\mathbf{y}$ and $\mathbf{z}$ are also Vandermonde vectors. The situation is similar for $y_2^2 = y_3$, $x_2 = z_2$.

(d) $x_2^2 = x_3$, $y_2^2 = y_3$.

It is included in (b) and (c) if at least one of $x_2 = z_2$ and $y_2 = z_2$ is satisfied. So we discuss $x_2 \neq z_2$ and $y_2 \neq z_2$ here. Obviously $z_3 = z_2^2$, if $n = 3$, $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are all Vandermonde vectors. If $n \geq 3$, then from the definition of Hankel tensors, $a_{11\ldots 14} = a_{11\ldots 1222}$, which implies

$$\alpha x_4 + \beta y_4 + z_4 = \alpha x_2^3 + \beta y_2^3 + z_2^3. \tag{17}$$

From $a_{11\ldots 124} = a_{11\ldots 1223}$ and $a_{11\ldots 134} = a_{11\ldots 1233}$, we obtain the two equations

$$\begin{cases} \alpha x_2 x_4 + \beta y_2 y_4 + z_2 z_4 = \alpha x_2^2 x_3 + \beta y_2^2 y_3 + z_2^2 z_3, \\ \alpha x_3 x_4 + \beta y_3 y_4 + z_3 z_4 = \alpha x_2 x_3^2 + \beta y_2 y_3^2 + z_2 z_3^2. \end{cases}$$

Substitute (17) into the above equations and we have

$$\alpha(x_2 - z_2)(x_2^3 - x_4) + \beta(y_2 - z_2)(y_2^3 - y_4) = 0, \tag{18}$$

$$\alpha(x_2^2 - z_2^2)(x_2^3 - x_4) + \beta(y_2^2 - z_2^2)(y_2^3 - y_4) = 0. \tag{19}$$

(i) If $x_2^3 \neq x_4$, $y_2^3 \neq y_4$, then divide (19) by (18), we have $x_2 = y_2$, hence $x_3 = y_3$. Similarly $\alpha(x_k - z_k)(x_2^3 - x_4) + \beta(y_k - z_k)(y_2^3 - y_4) = 0$ and $\alpha(x_k^2 - z_k^2)(x_2^3 - x_4) + \beta(y_k^2 - z_k^2)(y_2^3 - y_4) = 0$ for $4 \leq k \leq n$. We get $x_k = y_k = z_k$ or $x_k = y_k$, thus $\mathbf{x} = \mathbf{y}$, which is a contradiction.

(ii) If $x_2^3 = x_4$, $y_2^3 = y_4$, obviously $z_4 = z_2^3$, then repeat (d) for $x_2^4 = x_5$, $y_2^4 = y_5$, ..., $x_2^{n-1} = x_n$, $y_2^{n-1} = y_n$, and we find $\mathbf{x} = \mathbf{y}$, which is a contradiction, or $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are Vandermonde vectors.

Case 2. $\mathbf{x} = (1, x_2, x_3, \dots, x_n)^\top$, $\mathbf{y} = (1, y_2, y_3, \dots, y_n)^\top$,
$\mathbf{z} = (0, \dots, 0, 1, z_{k+1}, \dots, z_n)^\top$, $2 \le k \le n$.

If $k = n$, i.e., $\mathbf{z} = \mathbf{e}_n$, then similar to the first case of rank 2, $\mathbf{x}$ and $\mathbf{y}$ are Vandermonde vectors, and $\mathbf{z}$ is also a Vandermonde vector. If $2 \le k \le n-1$, $\mathbf{x}$ and $\mathbf{y}$ are Vandermonde vectors, then $\alpha x_k^m + \beta y_k^m + 1 = \alpha x_k^m + \beta y_k^m$, which is a contradiction.

Case 3. $\mathbf{x} = (1, x_2, x_3, \dots, x_n)^\top$, $\mathbf{y} = (0, \dots, 0, 1, y_{k+1}, \dots, y_n)^\top$,
$\mathbf{z} = (0, \dots, 0, 1, z_{l+1}, \dots, z_n)^\top$, $2 \le k \le n-1$, $2 \le l \le n$, $k \ne l$.

From the definition of Hankel tensors,

$$\alpha x_1^{m-1} x_{i+1} + \beta y_1^{m-1} y_{i+1} + z_1^{m-1} z_{i+1} = \alpha x_1^{m-2} x_2 x_i + \beta y_1^{m-1} y_i + z_1^{m-1} z_i$$

for $i = 2, 3, ..., n-1$. Since $y_1 = z_1 = 0$, we have $x_{i+1} = x_2 x_i$, hence $\mathbf{x}$ is a Vandermonde vector. Without loss of generality, assume $k < l$, we have

(i) $\alpha x_k^m + \beta = \alpha x_{k-1}^{\frac{m}{2}} x_{k+1}^{\frac{m}{2}}$ for m is even,

(ii) $\alpha x_k^m + \beta = \alpha x_{k-1}^{\frac{m-1}{2}} x_k x_{k+1}^{\frac{m-1}{2}}$ for m is odd, i.e., $\beta = 0$, which is a contradiction.

Case 4. $\mathbf{x} = (0, \dots, 0, 1, x_{j+1}, \dots, x_n)^\top$, $\mathbf{y} = (0, \dots, 0, 1, y_{k+1}, \dots, y_n)^\top$,
$\mathbf{z} = (0, \cdots, 0, 1, z_{l+1}, \cdots, z_n)^\top$, $2 \le j, k, l \le n$, $l \le k \le j$.

If $j < k \le l$, then $\alpha = 0$, which is a contradiction.

If $j = k < l$, the situation is the same as Case 4 of rank 2, which is a contradiction.

If $j = k = l$, obviously j, k, l cannot be n, $2 \le j = k = l \le n-1$. As

(i) $\alpha x_k^m + \beta y_k^m + z_k^m = \alpha x_{k-1}^{\frac{m}{2}} x_{k+1}^{\frac{m}{2}} + \beta y_{k-1}^{\frac{m}{2}} y_{k+1}^{\frac{m}{2}} + z_{k-1}^{\frac{m}{2}} z_{k+1}^{\frac{m}{2}}$ for m is even,

(ii) $\alpha x_k^m + \beta y_k^m + z_k^m = \alpha x_{k-1}^{\frac{m-1}{2}} x_k x_{k+1}^{\frac{m-1}{2}} + \beta y_{k-1}^{\frac{m-1}{2}} y_k y_{k+1}^{\frac{m-1}{2}} + z_{k-1}^{\frac{m-1}{2}} z_k z_{k+1}^{\frac{m-1}{2}}$ for m is odd,

we have
$$\begin{cases} \alpha + \beta + 1 = 0, \\ \alpha x_k + \beta y_k + z_k = 0, \\ \alpha x_k^2 + \beta y_k^2 + z_k^2 = 0, \\ \alpha x_k^3 + \beta y_k^3 + z_k^3 = 0. \end{cases}$$

Then
$$\begin{cases} \alpha x_k (x_k - z_k) + \beta y_k (y_k - z_k) = 0, \\ \alpha x_k (x_k^2 - z_k^2) + \beta y_k (y_k^2 - z_k^2) = 0. \end{cases}$$

(i) If $x_k = 0, y_k = z_k$, we get $\beta = -1$, $\alpha = 0$, which is a contradiction.

(ii) If $y_k = 0, x_k = z_k$, then similar to (i), $\beta = 0$, which is also a contradiction.

(iii) If $x_k, y_k \ne 0, x_k \ne z_k, y_k \ne z_k$, from $x_k + z_k = y_k + z_k$, we get $x_k = y_k$.

(iv) Assume $x_k = y_k = 0$.

(v) Assume $x_k = y_k = z_k$.

From (iii) to (v), we all get $x_k = y_k$. If we continue checking $x_{k+1}, ..., x_n$, we will find $\mathbf{x} = \mathbf{y}$, or $\alpha = 0$, or $\beta = 0$, all situations cause contradictions.

Up till now, we have proved if there exists a rank-3 basic PSD Hankel tensor $\mathcal{A}$ with dimension no less than 3, $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are mutually distinct Vandermonde vectors. However, by Lemma 2, $r = 3 \leq n$, $\mathcal{A}$ is a strong Hankel tensor, which is a contradiction. □

Therefore, we put forward the following theorem.

Theorem 5.2. *For any non-basic PSD Hankel tensor* $\mathcal{A}$ *with* $\operatorname{rank}(\mathcal{A}) \geq 2$, $n \geq 3$, $\mathcal{A}$ *can be expressed as*

$$\mathcal{A} = \sum_{k=1}^{r} \alpha_k \mathcal{B}_k, \tag{20}$$

where $r \in \mathbb{N}$, $\mathcal{B}_k$ *are basic PSD Hankel tensors with* $\operatorname{rank}(\mathcal{B}_k) \geq 4$ *or* $\operatorname{rank}(\mathcal{B}_k) = 1$.

This theorem can be proved by Theorem 2 and 3 straightforwardly.

6. An example of basic PSD Hankel tensors with rank higher than 2

We shall present an example of basic PSD Hankel tensors in this section with rank > 2. Consider the following example. Let $\mathcal{A}$ be a 4th-order 2-dimensional Hankel tensor generated by $\mathbf{v} = \left(1, 0, -\frac{1}{3}, 0, 1\right)^\top$. For $\mathbf{x} = (x_1, x_2)^\top \in \mathbb{R}^2 \backslash \{\mathbf{0}\}$, the Hankel polynomial satisfies

$$f_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^4 = x_1^4 - 2x_1^2x_2^2 + x_2^4 = (x_1^2 - x_2^2)^2 \geq 0.$$

However, the associated Hankel matrix

$$A = \begin{pmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & -\frac{1}{3} & 0 \\ -\frac{1}{3} & 0 & 1 \end{pmatrix}$$

is apparently not positive semi-definite since there is a negative entry on its diagonal. Therefore, $\mathcal{A}$ is a PSD Hankel tensor, but not a strong Hankel tensor. Furthermore, $\mathcal{A}$ has a following decomposition that

$$\mathcal{A} = \frac{4}{3}\begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\otimes 4} - \frac{1}{6}\begin{pmatrix} 1 \\ 1 \end{pmatrix}^{\otimes 4} - \frac{1}{6}\begin{pmatrix} 1 \\ -1 \end{pmatrix}^{\otimes 4} + \frac{4}{3}\begin{pmatrix} 0 \\ 1 \end{pmatrix}^{\otimes 4},$$

thus $\operatorname{rank}(\mathcal{A}) \leq 4$.

Next we prove that $\mathcal{A}$ is a basic PSD Hankel tensor. Assume $\mathcal{A}$ is not basic, then there exist two PSD Hankel tensors $\mathcal{B}$ and $\mathcal{C}$ such that $\mathcal{A} = \mathcal{B} + \mathcal{C}$ and for any vector $\mathbf{x} \in \mathbb{R}^2$, the Hankel polynomial $f_{\mathcal{A}}(\mathbf{x}) = f_{\mathcal{B}}(\mathbf{x}) + f_{\mathcal{C}}(\mathbf{x}) = (x_1^2 - x_2^2)^2$, and $f_{\mathcal{B}}(\mathbf{x}), f_{\mathcal{C}}(\mathbf{x}) \geq 0$. If $f_{\mathcal{B}}(\mathbf{x})$ does not have the factors $x_1 + x_2$ or $x_1 - x_2$, then take $x_1 = \pm x_2$, and we have $f_{\mathcal{B}}(\mathbf{x}) > 0$, $f_{\mathcal{C}}(\mathbf{x}) = 0 - f_{\mathcal{B}}(\mathbf{x}) < 0$, which is a contradiction. If both $x_1 + x_2$ and $x_1 - x_2$ are the factors of $f_{\mathcal{B}}(\mathbf{x})$, then by D. Hilbert [10], for 2-dimensional homogeneous polynomials, a PSD polynomial is definitely an SOS polynomial, hence $f_{\mathcal{B}}(\mathbf{x}) = \alpha(x_1^2 - x_2^2)^2$, $\alpha > 0$, tensor $\mathcal{B}$ is proportional to $\mathcal{A}$, which is a contradiction.

Therefore, we have found a basic PSD Hankel tensor whose rank equals 4, which implies that basic PSD Hankel tensor with rank > 2 does exist. To verify whether this tensor is exactly rank 4, we use Tensorlab toolbox in Matlab software to decompose this symmetric tensor. After running 1000 times, the minimum error (calculated by the Frobenius norm of tensor $\mathcal{A}$ minus the recombinated tensor) of finding the rank-3 decomposition is about 10^{-3} while that of find the rank-4 decomposition is around 10^{-17}. Therefore, there is a high probability that the rank of this tensor is exactly 4.

Generally, 4th-order 2-dimensional Hankel tensors generated by vectors of the form $\mathbf{v} = \big(a, 0, -b, 0, c\big)^\top$, where $a, c > 0$, $b \geq 0$, and $ac = 9b^2$, are all basic PSD Hankel tensors with rank not larger than 4. This can be proved by similar derivations as above. Thus, a set of low rank basic PSD Hankel tensors has been found.

7. Conclusions and Conjectures

In this paper, we have introduced a new subclass of Hankel tensors called basic PSD Hankel tensors. It is proved that for mth-order n-dimensional positive semi-definite Hankel tensors, there are no rank-2 basic PSD Hankel tensors. Moreover, rank-3 basic PSD Hankel tensors with dimension no less than 3 do not exist, either. Furthermore, a set of low rank basic PSD Hankel tensors are found. These results may be used for further research on PSD non-strong Hankel tensors.

In the previous section, an example is given to show the existence of a basic PSD Hankel tensor with rank ≥ 3. It is thus reasonable to conject the existence of other basic PSD Hankel tensors. The critical truncated Hankel tensor $\mathcal{A}$ in [29] may also be a basic PSD Hankel tensor. $\mathcal{A}$ is a sixth-order three-dimensional PSD truncated Hankel tensor, and the elements of its generating vector $(v_0, 0, \cdots, 0, v_6, 0, \cdots, 0, v_{12})^\top$ satisfy $\sqrt{v_0 v_{12}} = (560 + 70\sqrt{70})v_6$.

For a rank-3 PSD Hankel tensor with $n = 2$, we cannot prove or disprove it is a basic PSD Hankel tensor. Therefore, we put forward a conjecture that 2-dimensional rank-3 basic PSD Hankel tensors do not exist and the rank of tensor $\mathcal{A}$ in the given example in Section 6 is 4.

References

[1] R. Badeau, R. Boyer: *Fast multilinear singular value decomposition for structured tensors*, SIAM J. Matrix Analysis Appl. 30 (2008) 1008–1021.

[2] C. Berg: *The multidimensional moment problem and semigroups, moments in Mathematics*, Proc. of Symposia in Applied Mathematics 37 (1987) 110–124.

[3] R. Boyer, L. De Lathauwer, K. Abed-Meraim: *Higher order tensor-based method for delayed exponential fitting*, IEEE Trans. Signal Processing 55 (2007) 2795–2809.

[4] Y. Chen, L. Qi, Q. Wang: *Computing extreme eigenvalues of large scale Hankel tensors*, J. Scientific Computing 68 (2016) 716–738.

[5] Y. Chen, L. Qi, Q. Wang: *Positive semi-definiteness and sum-of-squares property of fourth order four dimensional Hankel tensors*, J. Comp. Appl. Math. 302 (2016) 356–368.

[6] G. Chesi: *On the gap between positive polynomials and SOS of polynomials*, IEEE Trans. Automatic Control 52 (2007) 1066–1072.

[7] W. Ding, L. Qi, Y. Wei: *Fast Hankel tensor-vector products and application to exponential data fitting*, Numer. Linear Algebra Appl. 22 (2015) 814–832.

[8] W. Ding, L. Qi, Y. Wei: *Inheritance properties and sum-of-squares decomposition of Hankel tensors: Theory and algorithms*, BIT Numerical Mathematics 57 (2017) 169–190.

[9] G. H. Golub, C. F. Van Loan: *Matrix Computations*, 4th ed., Johns Hopkins Studies in the Mathematical Sciences, Johns Hopkins University Press, Baltimore (2013).

[10] D. Hilbert: *Über die Darstellung definiter Formen als Summe von Formenquadraten*, Math. Annalen 32 (1888) 342–350.

[11] S. Hu, G. Li, L. Qi, Y. Song: *Finding the maximum eigenvalue of essentially nonnegative symmetric tensors via sum of squares programming*, J. Optimization Theory Appl. 158 (2013) 717–738.

[12] J. B. Lasserre: *Global optimization with polynomials and the problem of moments*, SIAM J. Optimization 11 (2001) 796–817.

[13] G. Li, L. Qi, Q. Wang: *Positive semi-definiteness of generalized anti-circulant tensors*, Comm. Math. Sciences 14 (2016) 941–952.

[14] G. Li, L. Qi, Y. Xu: *SOS-Hankel tensors: theory and application*, preprint, arXiv:1410.6989 (2014).

[15] J. Luque, J. Thibon, *Hankel hyperdeterminants and Selberg integrals*, J. Physics A: Mathematical and General 36 (2003) 5267–5292.

[16] T. S. Motzkin: *The arithmetic-geometric inequality*, in: *Inequalities*, O. Shisha (ed.), Academic Press, New York (1967) 205–224.

[17] J. Nie, K. Ye: *Hankel tensor decompositions and ranks*, SIAM J. Matrix Anal. Appl. 40(2) (2019) 486–516.

[18] V. Oropeza, M. Sacchi: *Simultaneous seismic data denoising and reconstruction via multichannel singular spectrum analysis*, Geophysics 76 (2011) V25–V32.

[19] J. M. Papy, L. De Lauauwer, S. Van Huffel: *Exponential data fitting using multilinear algebra: The single-channel and multi-channel case*, Numer. Linear Algebra Appl. 12 (2005) 809–826.

[20] L. Qi: *Eigenvalues of a real supersymmetric tensor*, J. Symbolic Computation 40 (2005) 1302–1324.

[21] L. Qi: *Hankel tensors: Associated Hankel matrices and Vandermonde decomposition*, Comm. Math. Sciences 13 (2015) 113–125.

[22] L. Qi: *A note on the multidimensional moment problem*, in: *Contemporary Computational Mathematics – A Celebration of the 80th Birthday of Ian Sloan*, J. Dick, F. Kuo, H. Wozniakowski (eds.), Springer, New York (2018) 1075–1080.

[23] L. Qi, H. Chen, Y. Chen: *Tensor Eigenvalues and Their Applications*, Springer, Cham (2018).

[24] L. Qi, Z. Luo: *Tensor Analysis: Spectral Theory and Special Tensors*, SIAM, Philadelphia (2017).

[25] B. Reznick: *On Hilbert's construction of positive polynomials*, preprint, arXiv:0707.2156 (2007).

[26] B. Sturmfels: *Three open problems on Hankel tensors*, preprint (2017).

[27] R. S. Smith: *Frequency domain subspace identification using nuclear norm minimization and Hankel matrix realizations*, IEEE Trans. Automatic Control 59 (2014) 2886–2896.

[28] S. Trickett, L. Burroughs, A. Milton: *Interpolating using Hankel tensor completion*, in: *SEG Annual Meeting* (2013) 3634–3638.

[29] Q. Wang, G. Li, L. Qi, Y. Xu: *New classes of positive semi-definite Hankel tensors*, Minimax Theory Appl. 2 (2017) 231–248.