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Generalized Nash Equilibrium Problems and Variational Inequalities in Lebesgue Spaces

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We study generalized Nash equilibrium problems (GNEPs) in Lebesgue spaces by means of a family of variational inequalities (VIs) parametrized by an L^∞ vector $r(t)$. The solutions of this family of VIs constitute a subset of the solution set of the GNEP. For each choice of $r(t)$, the VI solutions thus obtained are solutions of the GNEP which can be characterized by a certain relationship among the Karush-Kuhn-Tucker (KKT) multipliers of the players. This result extends a previous one, where only the case in which the parameter r is a constant vector was investigated, and can be considered as a full generalization, to Lebesgue spaces, of a classical property proven by J. B. Rosen [*Existence and uniqueness of equilibrium points for concave n person games*, Econometrica 33 (1965) 520–534] in finite dimensional spaces.

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1. Introduction

In a standard noncooperative game, players choose their strategies in a feasible set which is independent of the rivals' strategies. Nash equilibrium problems (NEP) are games of this kind. On the other hand GNEPs are noncooperative games where the feasible set of each player depends on the strategies chosen by the other players. GNEPs have become very popular in the last decades because of their role in modeling complex economic systems, such as oligopolies, transportation networks or energy distribution systems. In this respect, it is of particular interest the case where the players have to share a common resource. This class of GNEPs has been formulated by Rosen in his seminal paper [13]. More recently, Rosen's

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problems have been investigated in the light of VIs theory, [4]. This approach has been further developed in [11, 12]. The VI approach to GNEPs with shared constraints is very useful from the computational point of view and allows an interesting characterization of the solutions thus obtained.

Indeed, in [4] it is shown how the solutions of the GNEP obtained solving a specific VI have the properties that the KKT multipliers of all players are equal. In [11, 12] the authors associate to a given GNEP with shared constraints a family of VIs parametrized by a positive weights vector $r = (r_1, \ldots, r_m)$ and prove that, for each fixed r, every solution of the corresponding VI is a solution of the GNEP where the KKT multipliers of the player i are given by a common multiplier, divided by the corresponding weight r_i (the case $r = (1, \ldots, 1)$ is the one treated in [4]).

The above mentioned papers are all set in finite dimensional spaces but, recently, some extensions to the infinite dimensional setting have been considered [5, 8, 9]. An infinite dimensional formulation of GNEP has been proposed in [5], where the authors also discuss the delicate issue of the regularity conditions that have to be imposed both on the GNEP and the VI, to derive their corresponding KKT system. Indeed, because the cone of nonnegative functions in many infinite dimensional spaces has empty topological interior, the standard regularity conditions cannot be applied. In this paper we consider a GNEP with shared constraints in the Lebesgue space $L^2([0,T], \mathbb{R}^n)$ and assume that the constraints are satisfied almost everywhere (a.e.) in $[0,T]$. We construct a family of of VIs parametrized by a positive L^∞ vector $r(t)$ whose components are bounded away from zero and whose solutions are also solutions of the given GNEP, such that the KKT multipliers of the players have a particular functional form, thus extending the result of [11, 12, 13] from the space $\mathbb{R}^n$ to the space $L^2([0,T], \mathbb{R}^n)$.

The paper is structured as follows. In Section 2 we briefly provide the formulation of the GNEP within the cadre of a general Banach space; in Section 3 we focus on GNEPs in the Lebesgue space $L^2([0,T], \mathbb{R}^n)$ and construct a family of VIs associated to those GNEPs; in Sections 4 and 5 we recall some recent results of infinite dimensional duality theory which are needed to cope with the problem that the ordering cone of nonnegative L^2 functions has empty topological interior; finally, in Section 6, we prove the theorem which characterizes the VI solutions of GNEPs in terms of the KKT multipliers of the players.

2. The setting of the game

We describe the setting of our Nash game. For notational simplicity, and to make our main result more evident, we deal with two players (the case of N players being easily deduced).

Assume that X_1 and X_2 are two Banach spaces, and denote by $X = X_1 \times X_2$ the product space and by $u = (u^1, u^2)$ the generic element of X, that is u^1 and u^2 are the variables respectively controlled by the first and the second player. Let also $K \subset X$ be a non empty and convex set, J_1 and $J_2 \colon X \to \mathbb{R}$ be two functionals such that $J_1(\cdot, u^2)$ is convex and Gâteaux differentiable for every fixed $u^2 \in X_2$ and $J_2(u^1, \cdot)$ is convex and Gâteaux differentiable for every fixed $u^1 \in X_1$. Any of these functions is called the utility function of the player i or the payoff function or the loss function depending on the particular application in which the GNEP arises.

For every $u = (u^1, u^2) \in X$, the feasible strategy sets of the two players are of the following kind:

$$K_1(u) = \{v^1 \in X_1 : (v^1, u^2) \in K\} \subset X_1$$

and

$$K_2(u) = \{v^2 \in X_2 : (u^1, v^2) \in K\} \subset X_2.$$

Notice that if $u \in K$ then the above sets are non empty ($u^i \in K_i(u)$) and convex. This class of strategy sets, introduced by Rosen in [13], is often referred to as the *jointly convex case* or GNEPs *with coupled constraints* motivated by the fact that the feasible sets are linked through a shared or common constraint.

The goal of each player i, given the strategy of the rival, is to choose a strategy which minimizes the functional J_i on its feasible set. The following definition describes the aim of the game: to find an *equilibrium* point for both players, that is a vector $(\bar{u}^1, \bar{u}^2)$ such that no player can decrease their loss function by changing unilaterally $\bar{u}^i$ to any other feasible point.

Definition 2.1. We say that $\bar{u} = (\bar{u}^1, \bar{u}^2)$ is a *solution of the GNEP* if $\bar{u} \in K$ and the following conditions hold:

$$\begin{cases} J_1(\bar{u}^1, \bar{u}^2) = \min_{u^1 \in K_1(\bar{u})} J_1(u^1, \bar{u}^2), \\ J_2(\bar{u}^1, \bar{u}^2) = \min_{u^2 \in K_2(\bar{u})} J_2(\bar{u}^1, u^2). \end{cases} \tag{1}$$

We recall that if Y is a Banach space, a function $I\colon Y \to \mathbb{R}$ is said to be *Gâteaux differentiable* in $\bar{u} \in Y$ if there exists $\varphi \in Y^\star$ (the topological dual space of Y) such that

$$\lim_{\lambda \to 0^+} \frac{I(\bar{u} + \lambda u) - I(\bar{u})}{\lambda} = \varphi(u) \qquad \forall\, u \in Y.$$

The functional φ is called the *Gâteaux derivative* of I and denoted by $DI(\bar{u})$.

Remark 2.2. By the first order optimality condition applied to (2.1), $\bar{u} = (\bar{u}^1, \bar{u}^2)$ is a solution of GNEP iff $\bar{u} \in K$ and

$$\begin{cases} D_1 J_1(\bar{u}^1, \bar{u}^2)(u^1 - \bar{u}^1) \geq 0 & \forall\, u^1 \in K_1(\bar{u}), \\ D_2 J_2(\bar{u}^1, \bar{u}^2)(u^2 - \bar{u}^2) \geq 0 & \forall\, u^2 \in K_2(\bar{u}) \end{cases} \tag{2}$$

where D_1 and D_2 stand for the Gâteaux derivative of $J_1(\cdot, \bar{u}^2)$ and $J_2(\bar{u}^1, \cdot)$ respectively.

Denote by $\Gamma\colon X \to X^* = X_1^\star \times X_2^\star$, the mapping

$$\Gamma(u^1, u^2) = \left(D_1 J_1(u^1, u^2), D_2 J_2(u^1, u^2)\right). \tag{3}$$

With the above notation, it is clear that (2) is equivalent to

$$\Gamma(\bar{u})^T(u-\bar{u}) \geq 0, \qquad \forall u \in K_1(\bar{u}) \times K_2(\bar{u}), \tag{4}$$

where $\Gamma(\bar{u})^T(u-\bar{u}) := D_1J_1(\bar{u}^1,\bar{u}^2)(u_1-\bar{u}_1) + D_2J_2(\bar{u}^1,\bar{u}^2)(u_2-\bar{u}_2)$.

The formulation (4) is known as *quasi-variational inequality* (QVI), since the convex sets $K_i(\bar{u})$ depend on the solution. Following [4], it is possible to reduce the problem to the VI associated to Γ on the feasible set K (in short, VI(Γ,K)), which means finding a point $\bar{u} = (\bar{u}^1, \bar{u}^2) \in K$ such that

$$\Gamma(\bar{u})^T(u-\bar{u}) \geq 0, \qquad \forall u \in K. \tag{5}$$

Analogously to [3, 4], we have the following

Theorem 2.3. *Every solution of* VI(Γ,K) *is a solution of* GNEP*.*

A solution of the GNEP that is also a solution of VI(Γ,K) is usually referred to as a *variational equilibrium.*

3. Lebesgue space formulation of GNEPs

Let $T > 0$, $n_1, n_2 \in N$ and $n = n_1 + n_2$. Set $X_1 = L^2_{n_1} = L^2([0,T],\mathbb{R}^{n_1})$, $X_2 = L^2_{n_2} = L^2([0,T],\mathbb{R}^{n_2})$. Denote by $x = (x^1, x^2)$ the generic point of $\mathbb{R}^n$ where $x^1 = (x_1, x_2, ..., x_{n_1})$ and $x^2 = (x_{n_1+1}, x_{n_1+2}, ..., x_n)$. The symbol $\cdot$ represents the scalar product in the corresponding Euclidean space, $\mathbb{R}^{n_1}$ or $\mathbb{R}^{n_2}$ and $\| \ \|$ represents the norm in the given space.

Let $\theta_1 \colon [0,T] \times \mathbb{R}^n \to \mathbb{R}$ be a function satisfying the following conditions:

(a_{θ_1}) $\theta_1(\cdot, x)$ is measurable for every $x \in \mathbb{R}^n$ and $\theta_1(t,\cdot)$ is of class $C^1(\mathbb{R}^n)$ for a.e. $t \in [0,T]$;

(b_{θ_1}) $\theta_1(\cdot, 0) \in L^1([0,T])$;

(c_{θ_1}) $\theta_1(t, (\cdot, x^2))$ is convex for a.e. $t \in [0,T]$ and every $x^2 \in \mathbb{R}^{n_2}$;

(d_{θ_1}) $|\nabla_x \theta_1(t,x)| \leq c(1 + \|x\|)$ for a.e. $t \in [0,T]$ and every $x \in \mathbb{R}^n$ ($c > 0$).

Define now the functional $J_1 \colon X \to \mathbb{R}$ by

$$J_1(u^1,u^2) = \int_0^T \theta_1(t, u^1(t), u^2(t))dt.$$

Lemma 3.1. (see [5] for the proof). *Assume that $\theta_1 \colon [0,T] \times \mathbb{R}^n \to \mathbb{R}$ verifies the assumptions (a_{θ_1})–(d_{θ_1}). Then the functional J_1 is well-defined in X, $J_1(\cdot, u^2)$ is convex and Gâteaux differentiable in X_1 for every $u^2 \in X_2$, with Gâteaux derivative for all $v^1 = (v_1, v_2, ..v_{n_1}) \in X_1$ given by*

$$\begin{aligned} D_1J_1(u^1,u^2)\, v^1 &= \int_0^T \nabla_{x^1}\theta_1(t, u^1(t), u^2(t)) \cdot v^1(t)\, dt \\ &= \int_0^T \sum_{r=1}^{n_1} \frac{\partial}{\partial x_r} \theta_1(t, u^1(t), u^2(t)) v_r(t)\, dt \end{aligned} \tag{6}$$

Let $\theta_2\colon [0,T]\times\mathbb{R}^n\to\mathbb{R}$ be a function satisfying the following conditions:

(a_{θ_2}) $\theta_2(\cdot,x)$ is measurable for every $x\in\mathbb{R}^n$ and $\theta_2(t,\cdot)$ is of class $C^1(\mathbb{R}^n)$ for a.e. $t\in[0,T]$;

(b_{θ_2}) $\theta_2(\cdot,0)\in L^1([0,T])$;

(c_{θ_2}) $\theta_2(t,(x^1,\cdot))$ is convex for a.e. $t\in[0,T]$ and every $x^1\in\mathbb{R}^{n_1}$;

(d_{θ_2}) $|\nabla_x\theta_2(t,x)|\le c(1+\|x\|)$ for a.e. $t\in[0,T]$, and every $x\in\mathbb{R}^n$ $(c>0)$.

Let $J_2\colon X\to\mathbb{R}$ be the functional

$$J_2(u^1,u^2)=\int_0^T\theta_2(t,u^1(t),u^2(t))dt.$$

As above, the functional J_2 is well-defined in X, $J_2(u^1,\cdot)$ is convex and Gâteaux differentiable in X_2 for every $u^1\in X_1$, with Gâteaux derivative given by

$$\begin{aligned}D_2J_2(u^1,u^2)v^2&=\int_0^T\nabla_{x^2}\theta_2(t,u^1(t),u^2(t))\cdot v^2(t)\,dt\\&=\int_0^T\sum_{r=n_1+1}^{n}\frac{\partial}{\partial x_r}\theta_2(t,u^1(t),u^2(t))v_r(t)\,dt\end{aligned}\tag{7}$$

for all $v^2=(v_{n_1+1},...v_n)\in X_2$. For the development of our theory we now describe the constraints that define the feasible set:

(1) $G_i\colon L^2_{n_1}\times L^2_{n_2}\to L^2$ are convex and Fréchet differentiable mappings which describe the i-th shared constraint:

$$G_i(u^1(t),u^2(t))\le 0,\ \text{a.e. } t\in[0,T],\ i=1,\dots,m;$$

(2) $H^1_j\colon L^2_{n_1}\to L^2$ are convex and Gâteaux differentiable mappings which represent the j-th constraint of player 1 depending only on his strategy

$$H^1_j(u^1(t))\le 0,\ \text{a.e. } t\in[0,T],\ j=1,\dots,m_1;$$

(3) $H^2_j\colon L^2_{n_2}\to L^2$ are convex and Gâteaux differentiable mappings which represent the j-th constraint of player 2 depending only on his strategy

$$H^2_j(u^2(t))\le 0,\ \text{a.e. } t\in[0,T],\ j=1,\dots,m_2.$$

The set K has thus the specific form:

$$K=\left\{(u^1,u^2)\in L^2_{n_1}\times L^2_{n_2}\ \middle|\ \begin{array}{l}G_i(u^1(t),u^2(t))\le 0,\ i=1,\dots,m,\\ H^1_j(u^1(t))\le 0,\ j=1,\dots,m_1,\\ H^2_j(u^2(t))\le 0,\ j=1,\dots,m_2,\ \text{a.e. } t\in[0,T]\end{array}\right\},\tag{8}$$

while the constraint set of the two players are given by, respectively:

$$K_1(u) = \{v^1 \in L^2_{n_1} : (v^1, u^2) \in K\}$$
$$= \left\{ v^1 \in L^2_{n_1} \;\middle|\; \begin{array}{l} G_i(v^1(t), u^2(t)) \leq 0, \; i = 1, \ldots, m, \\ H^1_j(v^1(t)) \leq 0, \; j = 1, \ldots, m_1, \; \text{a.e. } t \in [0,T] \end{array} \right\}, \tag{9}$$

$$K_2(u) = \{v^2 \in L^2_{n_2} : (u^1, v^2) \in K\}$$
$$= \left\{ v^2 \in L^2_{n_2} \;\middle|\; \begin{array}{l} G_i(u^1(t), v^2(t)) \leq 0, \; i = 1, \ldots, m, \\ H^2_j(v^2(t)) \leq 0, \; j = 1, \ldots, m_2, \; \text{a.e. } t \in [0,T] \end{array} \right\}. \tag{10}$$

Thus, our GNEP consists of finding $(\bar{u}^1, \bar{u}^2) \in K$ such that

$$\begin{cases} \int_0^T \theta_1(t, \bar{u}^1(t), \bar{u}^2(t))dt = \min_{u^1 \in K_1(\bar{u})} \int_0^T \theta_1(t, u^1(t), \bar{u}^2(t))dt \\ \int_0^T \theta_2(t, \bar{u}^1(t), \bar{u}^2(t))dt = \min_{u^2 \in K_2(\bar{u})} \int_0^T \theta_2(t, \bar{u}^1(t), u^2(t))dt. \end{cases} \tag{11}$$

We now introduce a family of VIs whose solutions are also solutions of GNEP (11). To this end, consider two positive numbers $\underline{r}_1, \underline{r}_2 > 0$ and two functions $r_1, r_2 \in L^\infty([0,T])$, such that $0 < \underline{r}_1 < r_1(t)$, $0 < \underline{r}_2 < r_2(t)$, and denote with $r = r(t)$ the vector $(\, r_1(t), r_2(t)\,)$. For each r we define a mapping $\Gamma^r \colon L^2_n \to L^2_n$ as follows:

$$\Gamma^r(u) = (\, \Gamma_1^{r_1}(u), \Gamma_2^{r_2}(u)\,) = \left(r_1(t) \nabla_{x^1} \theta_1(t, u^1(t), u^2(t)), r_2(t) \nabla_{x^2} \theta_2(t, u^1(t), u^2(t)) \right).$$

$VI(\Gamma^r, K)$ is the problem of finding $\bar{u} = (\bar{u}^1, \bar{u}^2)$ such that for all $(v^1, v^2) \in K$:

$$\int_0^T [r_1(t) \nabla_{x^1} \theta_1(t, \bar{u}^1(t), \bar{u}^2(t)) \cdot (v^1(t) - \bar{u}^1(t))$$
$$+ r_2(t) \nabla_{x^2} \theta_2(t, \bar{u}^1(t), \bar{u}^2(t)) \cdot (v^2(t) - \bar{u}^2(t))]dt \geq 0. \tag{12}$$

The relationship between solutions of $VI(\Gamma^r, K)$ (12) and GNEP (11) will be derived by making use of the KKT conditions that can be written for both problems, under suitable regularity assumptions. Thus, in the following section we review some recent results in infinite dimensional duality theory which allow us to derive the KKT multipliers associated with $VI(\Gamma^r, K)$ (12) and GNEP (11). In section 5, we will show that, for a given $r = (r_1(t), r_2(t)\,)$, every solution of $VI(\Gamma^r, K)$ is also a solution of GNEP, with the particular property that the multipliers of the two players are obtained by dividing a common multiplier for the weights $r_1(t)$ and $r_2(t)$, respectively.

4. Duality in infinite dimensional spaces

In the present section, for the reader's convenience, we outline the main features of duality theory for an infinite dimensional cone constrained optimization problem, devoting particular attention to KKT type optimality conditions.

Let us consider the following setting that will be assumed throughout this section.

We denote by $\mathbb{R}^+$ the interval $[0, +\infty)$, and by $\mathbb{R}^{++}$ the open interval $(0, +\infty)$.

Let $\hat{S}$ be a convex subset of a normed space X. Let $(Y, ||\cdot||_Y)$ be a real normed space partially ordered by the closed convex cone C and $(Z, ||\cdot||_Z)$ be a real normed space. Denote by $\langle \cdot, \cdot \rangle$, the duality pairing between Y and Y^*. The dual cone of C is defined by $C^* := \{l \in Y^* : \langle l, y \rangle \geq 0, \ \forall y \in C\}$.

Let $f: \hat{S} \to \mathbb{R}$ be a given objective functional, $g: \hat{S} \to Y$ and $h: \hat{S} \to Z$ given constraint mappings with h linear and assume furthermore that the mapping $(f, g, h): \hat{S} \to \mathbb{R} \times Y \times Z$ is $(\mathbb{R}^+ \times C \times \{0_Z\})$-convex, i.e., $\forall x, y \in \hat{S}$, $\forall t \in [0,1]$ we have

$$t\,(f, g, h)(x) + (1-t)(f, g, h)(y) - (f, g, h)(t\,x + (1-t)y) \in (\mathbb{R}^+ \times C \times \{0_Z\}).$$

We consider the following optimization problem, called *Primal Problem*:

$$\min_{x \in S} f(x), \tag{13}$$

where $S := \{x \in \hat{S} : g(x) \in -C, \ h(x) = 0_Z\}$ is supposed to be nonempty.

The following results are well-known; proofs can be found, e.g., in [2]. The next lemma states that problem (13) is equivalent to the optimization problem

$$\min_{x \in \hat{S}} \sup_{(u,v) \in C^* \times Z^*} L(x, u, v), \tag{14}$$

where the functional $L: \hat{S} \times C^* \times Z^* \to \mathbb{R}$ defined by

$$L(x, u, v) := f(x) + \langle u, g(x) \rangle + \langle v, h(x) \rangle$$

is called the Lagrange functional associated with problem (13).

Lemma 4.1. *$\overline{x} \in S$ is an optimal solution of problem* (13) *iff $\overline{x}$ is an optimal solution of problem* (14) *and the optimal values of both problems are equal.*

We can associate with problem (13) the following optimization problem

$$\max_{(u,v) \in C^* \times Z^*} \inf_{x \in \hat{S}} L(x, u, v), \tag{15}$$

which is called Dual Problem associated with (13) or equivalently (14).

There are some fundamental relationships between the Primal and the Dual Problem. The first one is the weak duality theorem.

Theorem 4.2. *The maximal value of the Dual Problem is bounded from above by the minimal value of the Primal Problem i.e.*

$$\max_{(u,v) \in C^* \times Z^*} \inf_{x \in \hat{S}} L(x, u, v) \leq \min_{x \in \hat{S}} \sup_{(u,v) \in C^* \times Z^*} L(x, u, v). \tag{16}$$

Definition 4.3. A point $(\overline{x}, \overline{u}, \overline{v}) \in \hat{S} \times C^* \times Z^*$ is called a *saddle point of the Lagrange functional* L on $\hat{S} \times C^* \times Z^*$ iff

$$L(\overline{x}, u, v) \leq L(\overline{x}, \overline{u}, \overline{v}) \leq L(x, \overline{u}, \overline{v}) \quad \forall x \in \hat{S}, \ u \in C^*, \ v \in Z^*. \tag{17}$$

A saddle point of the Lagrange functional can be characterized by the following minimax theorem which states that both the Primal and Dual Problem admit an optimal solution and their optimal values are equal.

Theorem 4.4. *A point* $(\overline{x}, \overline{u}, \overline{v}) \in \hat{S} \times C^* \times Z^*$ *is a saddle point of the Lagrange functional* L *iff*

$$\min_{x \in \hat{S}} \sup_{(u,v) \in C^* \times Z^*} L(x,u,v) = \max_{(u,v) \in C^* \times Z^*} \inf_{x \in \hat{S}} L(x,u,v) \tag{18}$$

and $\overline{x}$ *and* $(\overline{u}, \overline{v})$ *are optimal solutions of the Primal and the Dual Problem, resp.*

Let us introduce the set:

$$\mathcal{E} = (f(\bar{x}), 0_Y, 0_Z) - (f, g, h)(\hat{S}) - (\mathbb{R}^+ \times C \times \{0_Z\}),$$

which is called the *extended image* associated with (13).

The optimality of a feasible point $\bar{x}$ can be expressed by means of the set $\mathcal{E}$ as shown in the following result (see e.g., [2, 7]).

Proposition 4.5. $\bar{x} \in S$ *is an optimal solution of problem* (13) *iff*

$$\mathcal{E} \cap (\mathbb{R}^{++} \times C \times \{0_Z\}) = \emptyset. \tag{19}$$

Definition 4.6. The set defined as

$$T_{\hat{S}}(\overline{x}) = \{y \in X |\ y = \lim_{n \to \infty} \lambda_n (x_n - \overline{x}),\ \lambda_n > 0,\ x_n \in \hat{S},\ \forall n \in \mathbb{N},\ \lim_{n \to \infty} x_n = \overline{x}\}$$

is called the *contingent* (or *Bouligand tangent*) *cone* of the set $\hat{S}$ at the point $\overline{x}$.

The next result [2, 6, 7] provides a characterization of the saddle point condition in terms of suitable properties of the contingent cone of the set $\mathcal{E}$.

Theorem 4.7. $(\bar{x}, \overline{u}, \overline{v}) \in (\hat{S} \times C^* \times Z^*)$ *is a saddle point of the Lagrange functional* L *iff* $\bar{x} \in S$ *and*

$$T_{\mathcal{E}}(0, 0_Y, 0_Z) \cap (\mathbb{R}^{++} \times \{0_Y\} \times \{0_Z\}) = \emptyset. \tag{20}$$

The following proposition shows that, under the convexity hypotheses assumed on the given problem, (20) is a sufficient optimality condition for the point $\bar{x} \in S$.

Proposition 4.8. *Let* $\bar{x} \in S$ *and assume that* (20) *is fulfilled. Then,* $\bar{x}$ *is an optimal solution of problem* (13).

Proof. We recall that the assumption (f, g, h) is $(\mathbb{R}^+ \times C \times \{0_Z\})$–convex implies that $\mathcal{E}$ is convex [7]. Therefore, the inclusion $\mathcal{E} \subseteq T_{\mathcal{E}}(0, 0_Y, 0_Z)$ holds, so that (20) yields

$$\mathcal{E} \cap (\mathbb{R}^{++} \times \{0_Y\} \times \{0_Z\}) = \emptyset. \tag{21}$$

Let us prove that (21) implies (19). By contradiction assume that there exists $(\bar{u}, \bar{v}, 0) \in \mathcal{E} \cap (\mathbb{R}^{++} \times C \times \{0_Z\})$. Since

$$\mathcal{E} - (\mathbb{R}^+ \times C \times \{0_Z\}) =$$

$$= (f(\bar{x}), 0_Y, 0_Z) - (f, g, h)(\hat{S}) - [(\mathbb{R}^+ \times C \times \{0_Z\}) - (\mathbb{R}^+ \times C \times \{0_Z\})] = \mathcal{E},$$

then $(\bar{u}, \bar{v}, 0) - (0, \bar{v}, 0) = (\bar{u}, 0, 0) \in \mathcal{E}$. Note that $(\bar{u}, 0, 0) \in \mathbb{R}^{++} \times \{0_Y\} \times \{0_Z\})$ which contradicts (21). By Proposition 4.5 we complete the proof. □

Denote by $$Dg(\bar{x})(v) := \lim_{t\to 0^+} \frac{g(\bar{x}+tv)-g(\bar{x})}{t}$$

the *directional derivative* of g at $\bar{x} \in \hat{S}$ in the direction $v := x - \bar{x}$, where $x \in \hat{S}$.

Theorem 4.9. *Let $\bar{x}\in S$ and assume that f, g have directional derivatives at $\bar{x}$ in every direction $x-\bar{x}$, where $x\in\hat{S}$. Then, the following statements are equivalent:*

(i) (20) *is fulfilled;*

(ii) *There exists $(\overline{u}, \overline{v}) \in (C^* \times Z^*)$ such that $(\bar{x}, \overline{u}, \overline{v})$ is a saddle point of the Lagrange functional L;*

(iii) *There exists $(\overline{u}, \overline{v}) \in (C^* \times Z^*)$ such that*

$$[Df(\bar{x}) + \overline{u} \circ Dg(\bar{x}) + \overline{v} \circ Dh(\bar{x})](x-\bar{x}) \geq 0, \quad \forall x \in \hat{S} \tag{22a}$$

$$\langle \overline{u}, g(\bar{x})\rangle = 0. \tag{22b}$$

Moreover, any of the conditions (i)–(iii) *guarantees that $\bar{x}$ is an optimal solution of problem* (13).

Proof. Taking into account Theorem 4.7, it is enough to show that (ii) and (iii) are equivalent.

Assume that (ii) holds. By the blanket assumptions, $L(\cdot, \bar{u}, \bar{v})$ is a convex functional having directional derivative at $\bar{x}$, in every direction $(x-\bar{x})$, with $x \in \hat{S}$. Taking into account that $\hat{S}$ is a convex set, the second inequality in (17) is equivalent to the condition

$$D_x L(\bar{x}, \bar{u}, \bar{v})(x-\bar{x}) \geq 0, \quad \forall x \in \hat{S},$$

i.e., (22a). By the first inequality in (17) evaluated at $(u, v) = (0, 0)$ we have:

$$0 \leq \langle \bar{u}, g(\bar{x})\rangle + \langle \bar{v}, h(\bar{x})\rangle = \langle \bar{u}, g(\bar{x})\rangle.$$

Recalling that $g(\bar{x}) \in -C$ and $\bar{u} \in C^*$, we have also

$$\langle \bar{u}, g(\bar{x})\rangle \leq 0,$$

and, in turn (22b).

Conversely, assume (iii) holds. We have already observed that (22a) implies the second inequality in (17). Let us prove the first inequality in (17), i.e.,

$$\langle u, g(\bar{x})\rangle \leq \langle \bar{u}, g(\bar{x})\rangle = 0, \quad \forall u \in C^*. \tag{23}$$

which is true, since $g(\bar{x}) \in -C$. This shows that (ii) holds.

We have thus proven that (i), (ii) and (iii) are equivalent. If any of the previous conditions holds, applying Proposition 4.8, we obtain that $\bar{x}$ is an optimal solution of (13). □

Let us observe that (22a) and (22b) are the KKT conditions for the problem (13) at $\bar{x} \in S$.

Remark 4.10. In the case when $\hat{S}$ is an open set and the functions involved are Gâteaux differentiable at $\bar{x}$, then we can replace the inequality in (22a) with an equality.

5. Duality for VI and GNEP

Theorem 4.9 deals with constrained optimization problems. However, it can be reformulated both for VIs and GNEPs as follows. Let us consider first $VI(\Gamma^r, K)$.

By making the position $f^r(u) := \Gamma^r(\bar{u})^T(u-\bar{u}), \forall u \in K$, it is evident that $\bar{u}$ is a solution to $VI(\Gamma^r, K)$ iff $\bar{u}$ is an optimal solution for the problem

$$\min_{u\in K} f^r(u). \tag{24}$$

Let us denote by $C_G := L^2_+([0,T],\mathbb{R}^m) =$

$$= \{\xi = (\xi_1,\dots,\xi_m) : \xi_j \in L^2([0,T]),\ \xi_j(t) \geq 0,\ \text{a. e. } t \in [0,T],\ j = 1,..,m\},$$

$$C_{H^1} = L^2_+([0,T],\mathbb{R}^{m_1}), \quad C_{H^2} = L^2_+([0,T],\mathbb{R}^{m_2}).$$

Setting in (13) $f := f^r$, $\hat{S} := L^2_{n_1} \times L^2_{n_2}$, $C := C_G \times C_{H^1} \times C_{H^2}$, $g := (G, H^1, H^2)$, $h \equiv 0$, the extended image $\mathcal{E}$ associated with (24) is given by

$$\mathcal{E} = -(f^r, G, H^1, H^2)(\hat{S}) - (\mathbb{R}^+ \times C) \subseteq \mathbb{R} \times Y,$$

where $Y := L^2_m \times L^2_{m_1} \times L^2_{m_2}$. By Proposition 4.5 we infer that $\bar{u} \in K$ is a solution to $VI(\Gamma^r, K)$ iff $\mathcal{E} \cap (\mathbb{R}^{++} \times C) = \emptyset$. We will make the following assumption

$$T_{\mathcal{E}}(0, 0_Y) \cap (\mathbb{R}^{++} \times \{0_Y\}) = \emptyset \tag{25}$$

Noticing that the Gâteaux derivative of f^r at $\bar{u}$ is given by $Df^r(\bar{u}) = \Gamma^r(\bar{u})$, by Theorem 4.9, it follows that there exist multipliers:

$$\bar{w} = (\bar{w}_1,\dots,\bar{w}_m) \in C_G,\ \bar{\lambda} = (\bar{\lambda}_1,\dots,\bar{\lambda}_{m_1}) \in C_{H^1},\ \bar{\delta} = (\bar{\delta}_1,\dots,\bar{\delta}_{m_2}) \in C_{H^2}$$

such that

$$\Gamma^r(\bar{u}) + \bar{w} \circ DG(\bar{u}) + \bar{\lambda} \circ D_1H^1(\bar{u}^1) + \bar{\delta} \circ D_2H^2(\bar{u}^2) = 0 \tag{26a}$$

$$\langle \bar{w}, G(\bar{u})\rangle_{L^2_m} + \langle \bar{\lambda}, H^1(\bar{u}^1)\rangle_{L^2_{m_1}} + \langle \bar{\delta}, H^2(\bar{u}^2)\rangle_{L^2_{m_2}} = 0. \tag{26b}$$

We aim now at writing the KKT conditions for the two optimization problems that appear in (2.1). Let us consider the first one

$$J_1(\bar{u}^1, \bar{u}^2) = \min_{u^1\in K_1(\bar{u})} J_1(u^1, \bar{u}^2), \tag{27}$$

The extended image $\mathcal{E}_1 \subseteq \mathbb{R} \times (L^2_m \times L^2_{m_1})$, associated with (27), is given by

$$\mathcal{E}_1 := (J_1(\bar{u}^1,\bar{u}^2), 0_{L^2_m\times L^2_{m_1}}) - (J_1(\cdot,\bar{u}^2), G(\cdot,\bar{u}^2), H^1)(L^2_{n_1}) - (\mathbb{R}^+ \times C_G \times C_{H^1}).$$

By Proposition 4.5 we infer that $\bar{u}^1 \in K_1(\bar{u})$ is an optimal solution to (27) iff

$$\mathcal{E}_1 \cap (\mathbb{R}^{++} \times (C_G \times C_{H^1})) = \emptyset.$$

We will make the following assumption on (27)

$$T_{\mathcal{E}_1}(0, 0_{L^2_m\times L^2_{m_1}}) \cap (\mathbb{R}^{++} \times \{0_{L^2_m\times L^2_{m_1}}\}) = \emptyset. \tag{28}$$

By Theorem 4.9, it follows that there exist multipliers

$$\bar{w} = (\bar{w}_1, \ldots, \bar{w}_m) \in C_G, \quad \bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_{m_1}) \in C_{H^1},$$

such that
$$D_1 J_1(\bar{u}_1, \bar{u}_2) + \bar{w} \circ D_1 G(\bar{u}_1, \bar{u}_2) + \bar{\lambda} \circ D_1 H^1(\bar{u}^1) = 0 \tag{29a}$$
$$\langle \bar{w}, G(\bar{u}_1, \bar{u}_2)\rangle_{L^2_m} + \langle \bar{\lambda}, H^1(\bar{u}^1)\rangle_{L^2_{m_1}} = 0 \tag{29b}$$

Let us consider the second problem in (1):

$$J_2(\bar{u}^1, \bar{u}^2) = \min_{u^2 \in K_2(\bar{u})} J_2(\bar{u}^1, u^2). \tag{30}$$

The extended image $\mathcal{E}_2 \subseteq \mathbb{R} \times (L^2_m \times L^2_{m_2})$, associated with (30), is given by

$$\mathcal{E}_2 := (J_2(\bar{u}^1, \bar{u}^2), 0_{L^2_m \times L^2_{m_2}}) - (J_2(\bar{u}^1, \cdot), G(\bar{u}^1, \cdot), H^2)(L^2_{n_2}) - (\mathbb{R}^+ \times C_G \times C_{H^2}).$$

By Proposition 4.5 we infer that $\bar{u}^2 \in K_2(\bar{u})$ is an optimal solution to (30) iff

$$\mathcal{E}_2 \cap (\mathbb{R}^{++} \times (C_G \times C_{H^2})) = \emptyset.$$

We will make the following assumption on (30)

$$T_{\mathcal{E}_2}(0, 0_{L^2_m \times L^2_{m_2}}) \cap (\mathbb{R}^{++} \times \{0_{L^2_m \times L^2_{m_2}}\}) = \emptyset. \tag{31}$$

By Theorem 4.9, it follows that there exist multipliers

$$\bar{w} = (\bar{w}_1, \ldots, \bar{w}_m) \in C_G, \; \bar{\delta} = (\bar{\delta}_1, \ldots, \bar{\delta}_{m_2}) \in C_{H^2},$$

such that
$$D_2 J_2(\bar{u}_1, \bar{u}_2) + \bar{w} \circ D_2 G(\bar{u}_1, \bar{u}_2) + \bar{\delta} \circ D_2 H^2(\bar{u}^2) = 0 \tag{32a}$$
$$\langle \bar{w}, G(\bar{u}_1, \bar{u}_2)\rangle_{L^2_m} + \langle \bar{\delta}, H^2(\bar{u}^2)\rangle_{L^2_{m_2}} = 0. \tag{32b}$$

6. VI solutions of GNEP

We are now in position to prove the main result of this paper, which is a generalization to infinite dimensional GNEPs of a classical property proven by Rosen in finite dimensional spaces [11, 13].

Theorem 6.1. *The following statements hold:*

(i) *Let $\bar{u} = (\bar{u}^1, \bar{u}^2)$ be a solution of $VI(\Gamma^r, K)$, where $r = (r_1(t), r_2(t))$ and assume that (25) is satisfied. Then, $\bar{u}$ is a solution of GNEP such that the multipliers of the two players, $\gamma^1 \in L^2_{m+m_1}, \gamma^2 \in L^2_{m+m_2}$ can be written as:*

$$\gamma^1(t) = \frac{1}{r_1(t)}(w(t), \lambda(t)), \quad \gamma^2(t) = \frac{1}{r_2(t)}(w(t), \delta(t)), \tag{33}$$

where $(w(t), \lambda(t), \delta(t))$ are the multipliers associated with $VI(\Gamma^r, K)$ at $\bar{u}$.

(ii) *Let* $\bar{u} = (\bar{u}^1, \bar{u}^2)$ *be a solution of GNEP, such that* (28) *and* (31) *are fulfilled. Let* α^1 *and* α^2 *be the multipliers of the two players, associated with the shared constraints, and assume that:*

$$\alpha^1(t) = \frac{w(t)}{r_1(t)}, \quad \alpha^2(t) = \frac{w(t)}{r_2(t)}.$$

Moreover, let λ *be the multiplier of player* 1 *associated with the constraint* H^1 *and let* δ *be the multiplier of player* 2 *associated with the constraint* H^2. *We then get that* $\bar{u}$ *is a solution to* $VI(\Gamma^r, K)$, *with* $r = (r_1(t), r_2(t))$, *and* $(\bar{u}, w(t), r_1(t)\lambda(t), r_2(t)\delta(t))$ *satisfy the KKT conditions for* $VI(\Gamma^r, K)$.

Proof. (i): As observed in Section 5, in order to write the KKT conditions for $VI(\Gamma^r, K)$ at $\bar{u}$, we take the position $f^r(u) := \Gamma^r(\bar{u})^T(u - \bar{u}), \forall u \in K$, and we consider problem (24). We thus obtain that there exist multipliers:

$$\bar{w} = (\bar{w}_1, \dots, \bar{w}_m) \in C_G, \ \bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_{m_1}) \in C_{H^1}, \quad \bar{\delta} = (\bar{\delta}_1, \dots, \bar{\delta}_{m_2}) \in C_{H^2}$$

such that (26a) and (26b) are fulfilled, i.e.,

$$\Gamma^r(\bar{u})^T v + (\bar{w} \circ DG(\bar{u}))(v) + (\bar{\lambda} \circ D_1 H^1(\bar{u}^1))(v^1) + \\ +(\bar{\delta} \circ D_2 H^2(\bar{u}^2))(v^2) = 0, \quad \forall v = (v_1, v_2) \tag{34a}$$

$$\langle \bar{w}, G(\bar{u}) \rangle_{L^2_m} + \langle \bar{\lambda}, H^1(\bar{u}^1) \rangle_{L^2_{m_1}} + \langle \bar{\delta}, H^2(\bar{u}^2) \rangle_{L^2_{m_2}} = 0. \tag{34b}$$

To compute the derivative of the shared constraints $G = (G_1, \dots, G_m)$ note that:

$$DG(\bar{u})(z) = (\ D_1 G_1(\bar{u}) z^1 + D_2 G_1(\bar{u}) z^2, \dots, D_1 G_m(\bar{u}) z^1 + D_2 G_m(\bar{u}) z^2\ ),$$

where $z = (z^1, z^2)$. Now we write (34a) explicitly for all $v = (v_1, v_2)$:

$$\int_0^T \left[r_1(t) \nabla_{x^1} \theta_1(t, \bar{u}^1(t), \bar{u}^2(t)\,) \cdot v^1(t) + r_2(t) \nabla_{x^2} \theta_2(t, \bar{u}^1(t), \bar{u}^2(t)\ ) \cdot v^2(t) \right] dt \\ + \int_0^T \left\{ \sum_{i=1}^m \bar{w}_i(t) [D_1 G_i(\bar{u}^1(t), \bar{u}^2(t)\,) \cdot v^1(t) + D_2 G_i(\bar{u}^1(t), \bar{u}^2(t)\,) \cdot v^2(t)] \right\} dt \\ + \int_0^T \left\{ \sum_{j=1}^{m_1} \bar{\lambda}_j(t) D_1 H_j^1(\bar{u}^1(t)) \cdot v^1(t) + \sum_{j=1}^{m_2} \bar{\delta}_j(t) D_2 H_j^2(\bar{u}^2(t)) \cdot v^2(t) \right\} dt = 0. \tag{35}$$

In (35) we can put $v^2 = 0$ and get

$$\int_0^T \left\{ r_1(t) \nabla_{x^1} \theta_1(t, \bar{u}^1(t), \bar{u}^2(t)\,) + \sum_{i=1}^m \bar{w}_i(t) D_1 G_i(\bar{u}^1(t), \bar{u}^2(t)) + \right. \\ \left. + \sum_{j=1}^{m_1} \bar{\lambda}_j(t) D_1 H_j^1(\bar{u}^1(t)) \right\} \cdot v^1(t) dt = 0, \quad \forall v_1. \tag{36}$$

From the arbitrariness of v^1, we get that the quantity in curly brackets vanishes a. e., and a further division by $r_1(t)$ yields

$$\nabla_{x^1}\theta_1(t,\bar u^1(t),\bar u^2(t)\,)+\sum_{i=1}^{m}\frac{\bar w_i(t)}{r_1(t)}D_1G_i(\bar u^1(t),\bar u^2(t))+ \\ +\sum_{j=1}^{m_1}\frac{\bar\lambda_j(t)}{r_1(t)}D_1H_j^1(\bar u^1(t))=0,\quad \text{a.e. } t\in[0,T]. \tag{37}$$

Multiplying (37) by an arbitrary $v^1\in L^2_{n_1}$ and integrating over $[0,T]$ we get:

$$\int_0^T\nabla_{x^1}\theta_1(t,\bar u^1(t),\bar u^2(t)\,)\cdot v^1(t)dt+\int_0^T\big[\sum_{i=1}^{m}\frac{\bar w_i(t)}{r_1(t)}D_1G_i(\bar u^1(t),\bar u^2(t))+ \\ +\sum_{j=1}^{m_1}\frac{\bar\lambda_j(t)}{r_1(t)}D_1H_j^1(\bar u^1(t))\big]\cdot v^1(t)dt=0. \tag{38}$$

We can now observe that (38) is nothing else than the first equation of the KKT system associated with the problem $\min_{v^1\in K_1(\bar u)}J_1(v^1)$, which is satisfied by $\bar u^1$ and by the multipliers $\bar\alpha_i^1(t)=\frac{\bar w_i(t)}{r_1(t)},\ i=1,...,m$ and $\bar\beta_j^1(t)=\frac{\bar\lambda_j(t)}{r_1(t)},\ j=1,...,m_1$.

To complete the KKT system for player 1 we need to derive their complementarity conditions. To this aim, let us write equation (34b) explicitly:

$$\int_0^T\big\{\sum_{i=1}^{m}\bar w_i(t)G_i(\bar u^1(t),\bar u^2(t))+\sum_{j=1}^{m_1}\bar\lambda_j(t)H_j^1(\bar u^1(t))+ \\ +\sum_{j=1}^{m_2}\bar\delta_j(t)H_j^2(\bar u^2(t))\big\}dt=0. \tag{39}$$

Let us notice that each summand inside the integral is nonpositive a.e. in $[0,T]$, which implies that the integrand vanishes a.e. In particular, we get a.e. in $[0,T]$:

$$\bar w_i(t)G_i(\bar u^1(t),\bar u^2(t))=0,\ i=1,..,m,\ \ \bar\lambda_j(t)H_j^1(\bar u^1(t))=0, j=1,..,m_1,$$

whence, dividing by $r_1(t)$, summing over the respective indexes and integrating again we get:

$$\int_0^T\big[\sum_{i=1}^{m}\frac{\bar w_i(t)}{r_1(t)}G_i(\,\bar u^1(t),\bar u^2(t)\,)+\sum_{j=1}^{m_1}\frac{\bar\lambda_j(t)}{r_1(t)}H_j^1(\bar u^1(t))\big]dt=0. \tag{40}$$

Thus, (38) and (40) constitute the KKT system for the problem

$$\min_{u^1\in K_1(\bar u)}\int_0^T\theta_1(t,u^1(t),\bar u^2(t))dt,$$

which is satisfied by $\bar{u}^1$, $\frac{\bar{w}(t)}{r_1(t)}$, $\frac{\bar{\lambda}(t)}{r_1(t)}$. The sufficiency of the KKT system for this convex problem implies that $\bar{u}^1$ is a solution of the minimum problem of player 1. Analogous computations can be applied to the KKT system for the problem

$$\min_{u^2 \in K_2(\bar{u})} \int_0^T \theta_2(t, \bar{u}^1(t), u^2(t))dt,$$

which turns out that to be satisfied by $\bar{u}^2$, $\frac{\bar{w}(t)}{r_2(t)}$, $\frac{\bar{\delta}(t)}{r_2(t)}$, and (i) is proven.

(ii): Let us write the first equation of the KKT system of player 1 (see (29a)):

$$\int_0^T \nabla_{x^1}\theta_1(t, \bar{u}^1(t), \bar{u}^2(t)) \cdot v^1(t)\, dt + \tag{41}$$
$$+ \int_0^T \left[\sum_{i=1}^m \frac{w_i(t)}{r_1(t)} D_1 G_i(\bar{u}^1(t), \bar{u}^2(t)) + \sum_{j=1}^{m_1} \lambda_j(t) D_1 H_j^1(\bar{u}^1(t)) \right] \cdot v^1(t)\, dt = 0$$

for all v^1. Analogously, for player 2 the first KKT equation reads (see (32a)):

$$\int_0^T \nabla_{x^2}\theta_2(t, \bar{u}^1(t), \bar{u}^2(t)) \cdot v^2(t)\, dt + \tag{42}$$
$$+ \int_0^T \left[\sum_{i=1}^m \frac{w_i(t)}{r_2(t)} D_2 G_i(\bar{u}^1(t), \bar{u}^2(t)) + \sum_{j=1}^{m_2} \delta_j(t) D_2 H_j^2(\bar{u}^2(t)) \right] \cdot v^2(t)\, dt = 0$$

for all v^2. From the arbitrariness of v^1 in (41) we get:

$$\nabla_{x^1}\theta_1(t, \bar{u}^1(t), \bar{u}^2(t)) + \sum_{i=1}^m \frac{w_i(t)}{r_1(t)} D_1 G_i(\bar{u}^1(t), \bar{u}^2(t)) +$$
$$\sum_{j=1}^{m_1} \lambda_j(t) D_1 H_j^1(\bar{u}^1(t)) = 0, \quad \text{a.e. } t \in [0, T], \tag{43}$$

which, multiplying by $r_1(t)$, multiplying by arbitrary $v^1(t)$ and integrating over $[0, T]$ yields for all v^1:

$$\int_0^T r_1(t) \nabla_{x^1}\theta_1(t, \bar{u}^1(t), \bar{u}^2(t)) \cdot v^1(t)\, dt + \tag{44}$$
$$+ \int_0^T \left[\sum_{i=1}^m w_i(t) D_1 G_i(\bar{u}^1(t), \bar{u}^2(t)) + \sum_{j=1}^{m_1} r_1(t) \lambda_j(t) D_1 H_j^1(\bar{u}^1(t)) \right] \cdot v^1(t)\, dt = 0.$$

The same procedure applied to player 2 yields for all v^2:

$$\int_0^T r_2(t) \nabla_{x^2}\theta_2(t, \bar{u}^1(t), \bar{u}^2(t)) \cdot v^2(t)\, dt + \tag{45}$$
$$+ \int_0^T \left[\sum_{i=1}^m w_i(t) D_2 G_i(\bar{u}^1(t), \bar{u}^2(t)) + \sum_{j=1}^{m_2} r_2(t) \delta_j(t) D_2 H_j^2(\bar{u}^2(t)) \right] \cdot v^2(t)\, dt = 0,$$

By summing (44) and (45) we get

$$\begin{aligned}
&\int_0^T [\, r_1(t)\nabla_{x^1}\theta_1(t,\bar{u}^1(t),\bar{u}^2(t)\,)\cdot v^1(t) + r_2(t)\nabla_{x^2}\theta_2(t,\bar{u}^1(t),\bar{u}^2(t))\cdot v^2(t)]dt \\
&+\int_0^T \sum_{i=1}^m w_i(t)\left[D_1G_i(\bar{u}^1(t),\bar{u}^2(t))\cdot v^1(t) + D_2G_i(\bar{u}^1(t),\bar{u}^2(t))\cdot v^2(t)\right] dt \qquad (46)\\
&+\int_0^T [\sum_{j=1}^{m_1} r_1(t)\lambda_j(t)\, D_1H_j^1(\bar{u}^1(t))\cdot v^1(t) + \sum_{j=1}^{m_2} r_2(t)\delta_j(t)D_2H_j^2(\bar{u}^2(t))\cdot v^2(t)]dt = 0,
\end{aligned}$$

which states that the first KKT equation for the problem $VI(\Gamma^r, K)$ is satisfied by $\bar{u}(t)$ and the multipliers $w(t), r_1(t)\lambda(t), r_2(t)\delta(t)$.

The derivation of the second KKT equation goes along the same lines. Indeed the complementarity condition for player 1 reads:

$$\int_0^T \left[\sum_{i=1}^m \frac{w_i(t)}{r_1(t)} G_i(\,\bar{u}^1(t),\bar{u}^2(t)\,) + \sum_{j=1}^{m_1} \lambda_j(t)H_j^1(\bar{u}^1(t))\right] dt = 0, \qquad (47)$$

which implies that the integrand vanishes a.e. and, after multiplication by $r_1(t)$, yields for all $i = 1,\dots,m$, for all $j = 1,\dots,m_1$, and for a.e. t in $[0,T]$:

$$w_i(t)G_i(\,\bar{u}^1(t),\bar{u}^2(t)\,) = 0, \quad r_1(t)\lambda_j(t)H_j^1(\bar{u}^1(t)) = 0. \qquad (48)$$

Analogously, the complementarity condition for player 2 reads:

$$\int_0^T \left[\sum_{i=1}^m \frac{w_i(t)}{r_2(t)} G_i(\,\bar{u}^1(t),\bar{u}^2(t)\,) + \sum_{j=1}^{m_2} \delta_j(t)H_j^2(\bar{u}^2(t))\right] dt = 0, \qquad (49)$$

which yields for all $i = 1,\dots,m$, for all $j = 1,\dots,m_1$, and for a.e. t in $[0,T]$:

$$w_i(t)G_i(\,\bar{u}^1(t),\bar{u}^2(t)\,) = 0, \quad r_2(t)\delta_j(t)H_j^2(\bar{u}^2(t)) = 0. \qquad (50)$$

From equations (48) and (50) we get

$$\begin{aligned}
\int_0^T [\sum_{i=1}^m w_i(t)G_i(\bar{u}^1(t),\bar{u}^2(t)) + \sum_{j=1}^{m_1} r_1(t)\lambda_j(t)H_j^1(\bar{u}^1(t)) + \\
+\sum_{j=1}^{m_2} r_2(t)\delta_j(t)H_j^2(\bar{u}^2(t))\big]dt = 0, \qquad (51)
\end{aligned}$$

which completes the proof. □

From the proof of Theorem 6.1 we obtain the following result that clarifies the relationship between the regularity conditions assumed for $VI(\Gamma^r, K)$ and GNEP.

Corollary 6.2. *Condition* (25) *implies both conditions* (28) *and* (31).

Proof. It is enough to observe that in the proof of (i) of Theorem 6.1 we have shown that (25) implies that KKT conditions hold for both optimization problems that appear in GNEP, namely (27) and (30). By Theorem 4.9 such conditions are equivalent to (28) and (31). □

We end this section with an example illustrating Theorem 6.1.

Example 6.3. Define $J_1, J_2 \colon L^2_2 \to \mathbb{R}$ as

$$J_1(x_1,x_2) = \int_0^T (x_1(t)-1)^2\,dt, \quad J_2(x_1,x_2) = \int_0^T \left(x_2(t)-\frac{1}{2}\right)^2 dt.$$

The problems outlined in Definition 2.1 are given by:

$$\begin{cases} \min \ J_1(\cdot, x_2) \\ x_1 : x_1(t)+x_2(t) \leq 1, \qquad \text{a.e. } t \in [0,T], \end{cases} \tag{52}$$

$$\begin{cases} \min \ J_2(x_1, \cdot) \\ x_2 : x_1(t)+x_2(t) \leq 1, \qquad \text{a.e. } t \in [0,T]. \end{cases} \tag{53}$$

Let $\alpha(t) \in L^2([0,T])$, such that $\frac{1}{2} \leq \alpha(t) \leq 1$, a.e. $t \in [0,T]$, we will show that $(\alpha(t), 1-\alpha(t))$ is a solution to GNEP.

We derive the solutions to GNEP by means of the KKT conditions for $VI(\Gamma^r, K)$ according to Theorem 6.1 (i). We easily compute:

$$\begin{aligned} D_1J_1(x_1,x_2)v^1 &= \int_0^T 2[x_1(t)-1]v^1(t)dt, \quad \forall v^1 \in L^2([0,T]), \\ D_2J_2(x_1,x_2)v^1 &= \int_0^T 2\left[x_2(t)-\frac{1}{2}\right]v^2(t)dt, \quad \forall v^2 \in L^2([0,T]), \\ G_1(x_1,x_2) &= x_1(t)+x_2(t)-1, \quad H^i \equiv 0,\ i=1,2, \\ D_1G_1(x_1,x_2)v^1 &= \int_0^T v^1(t)dt, \quad \forall v^1 \in L^2([0,T]), \\ D_2G_1(x_1,x_2)v^2 &= \int_0^T v^2(t)dt, \quad \forall v^2 \in L^2([0,T]). \end{aligned}$$

Equation (35) reads, for all (v^1, v^2):

$$\int_0^T \{2r_1(t)[x_1(t)-1]v^1(t) + 2r_2(t)\left[x_2(t)-\tfrac{1}{2}\right]v^2(t) + w(t)[v^1(t)+v^2(t)]\}dt = 0.$$

The previous equation leads to:

$$2r_1(t)[x_1(t)-1] + w(t) = 0, \qquad \text{a.e. } t \in [0,T], \tag{54}$$

$$2r_2(t)\left[x_2(t)-\tfrac{1}{2}\right] + w(t) = 0, \qquad \text{a.e. } t \in [0,T]. \tag{55}$$

The complementarity and the feasibility conditions read:

$$w(t)[x_1(t)+x_2(t)-1] = 0 \quad \text{a.e. } t \in [0,T],$$

$$w \in L^2_+([0,T]), \quad x_1(t)+x_2(t) \leq 1, \quad \text{a.e. } t \in [0,T].$$

Taking into account that $r_1(t) > \underline{r}_1$, $r_2(t) > \underline{r}_2$ for a.e. $t \in [0,T]$ and (54) and (55), it follows that there exists $\Gamma \subseteq [0,T]$, with measure $m(\Gamma) > 0$ such that $w(t) > 0$ for all $t \in \Gamma$. Indeed, if not, we would have $w(t) = 0$, a.e. $t \in [0,T]$. This would imply $x_1(t) = 1$, $x_2(t) = \frac{1}{2}$, a.e. $t \in [0,T]$, which contradicts the constraint.

Then, $x_1(t) + x_2(t) = 1$, $\forall t \in \Gamma$. By (54) and (55) it follows that

$$w(t) = 2r_1(t)x_2(t) = 2r_2(t)\left[\frac{1}{2} - x_2(t)\right], \ \forall t \in \Gamma,$$

$$\bar{x}_1(t) = 1 - \frac{r_2(t)}{2[r_1(t) + r_2(t)]}, \quad \bar{x}_2(t) = \frac{r_2(t)}{2[r_1(t) + r_2(t)]}, \quad w(t) = \frac{r_1(t)r_2(t)}{[r_1(t) + r_2(t)]}.$$

Note that, if $t \in \Gamma^c$ then $w(t) \le 0$, and it follows that $m(\Gamma^c) = 0$ yielding that the above solution can be extended to the whole interval $[0,T]$.

Setting $\alpha(t) = 1 - \dfrac{r_2(t)}{2[r_1(t) + r_2(t)]}$, we have $\bar{x}_1(t) = \alpha(t)$, $\bar{x}_2(t) = 1 - \alpha(t)$.

Finally, by (54), we get

$$1 - \alpha(t) = 1 - \bar{x}_1(t) = \frac{w(t)}{2r_1(t)} > 0, \ \forall t \in \Gamma,$$

which yields $\alpha(t) \le 1$ for a.e. $t \in [0,T]$. Similarly, by (55), we get

$$\frac{1}{2} - [1 - \alpha(t)] = \frac{1}{2} - \bar{x}_2(t) = \frac{w(t)}{2r_2(t)} > 0, \ \forall t \in \Gamma,$$

which yields $\alpha(t) \ge \frac{1}{2}$ for a.e. $t \in [0,T]$, as expected.

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References

[1] R. I. Bot, E. R. Csetnek, G. Wanka: *Regularity conditions via quasi-relative interior in convex programming*, SIAM J. Optimization 19(1) (2008) 217–233.

[2] A. Causa, G. Mastroeni, F. Raciti: *On Lagrangian duality in infinite dimension and its applications*, in: *Applications of Nonlinear Anaysis*, T. Rassias (ed.), Springer Optimization and its Applications 134, Springer, Cham (2018) 37–59.

[3] E. Cavazzuti, M. Pappalardo, M. Passacantando: *Nash equilibria, variational inequalities, and dynamical systems*, J. Optim. Theory Appl. 114(3) (2002) 491–506.

[4] F. Facchinei, A. Fischer, V. Piccialli: *On generalized Nash games and variational inequalities*, Oper. Res. Lett. 35 (2007) 159–164.

[5] F. Faraci, F. Raciti: *On generalized Nash equilibrium in infinite dimension: the Lagrange multipliers approach*, Optimization 64(2) (2015) 321–338.

[6] F. Flores-Bazán, G. Mastroeni: *Strong duality in cone constrained nonconvex optimization*, SIAM J. Optimization 23(1) (2013) 153–169.

[7] F. Giannessi: *Constrained Optimization and Image Space Analysis. Volume 1: Separation of Sets and Optimality Conditions*, Mathematical Concepts and Methods in Science and Engineering 49, Springer, New York (2005).

[8] M. Hintermuller, T. Surowiec, A. Kammler: *Generalized Nash equilibrium problems in Banach spaces: Theory, Nikaido-Isoda-based path-following methods, and applications*, SIAM J. Optimization 25(3) (2015) 1826–1856.

[9] B. Jadamba, F. Raciti: *Variational inequality approach to stochastic Nash equilibrium problems with an application to Cournot oligopoly*, J. Optim. Theory Appl. 165 (2015) 1050–1070.

[10] A. Maugeri, F. Raciti: *Remarks on infinite dimensional duality*, J. Global Optimization 46 (2010) 581–588.

[11] K. Nabetani: *Variational Inequality Approaches to Generalized Nash Equilibrium Problems*, Master Thesis, Kyoto University, Kyoto (2008).

[12] K. Nabetani, P. Tseng, M. Fukushima: *Parametrized variational inequality approaches to generalized Nash equilibrium problems with shared constraints*, Comput. Optim. Appl. 48 (2011) 423–452.

[13] J. B. Rosen: *Existence and uniqueness of equilibrium points for concave n person games*, Econometrica 33 (1965) 520–534.