Minimax Theory and its Applications
Volume 5 (2020), No. 1, 65–76

On the Existence of a Saddle Value for Nonconvex and Noncoercive Bifunctions

Felipe Lara
Dep. de Matemáticas, Facultad de Ciencias, Universidad de Tarapacá, Arica, Chile
flarao@uta.cl

Received: July 13, 2019
Accepted: September 30, 2019

We provide necessary and sufficient conditions for ensuring the existence of a saddle value for classes of nonconvex and noncoercive bifunctions. To that end, we use special classes of asymptotic (recession) directions and generalized asymptotic functions introduced and studied previously in the literature. We apply our theoretical results for providing sufficient conditions for zero duality gap for classes of quasiconvex cone constraint mathematical programming problems.

Keywords: Saddle value, asymptotic directions, asymptotic functions, duality, quasiconvexity, noncoercive optimization, nonconvex programming.

2010 Mathematics Subject Classification: 49J35, 90C47, 90C26.

1. Introduction

Let P be a closed, convex and pointed cone in $\mathbb{R}^m$, $H\colon \mathbb{R}^n \to \mathbb{R}^m$ be a vector valued function, and $h_0\colon \mathbb{R}^n \to \mathbb{R}$ be a proper function. Let us consider the cone constraint mathematical programming problem:

$$\text{(P)} \qquad \mu := \inf_{x\in C} h_0(x),$$

where $C := \{x \in \mathbb{R}^n : H(x) \in -P\}$ is assumed to be nonempty.

The (Lagrangian) dual problem associated to (P) is given by

$$\text{(D)} \qquad \nu := \sup_{q\in P^*} \inf_{x\in C} \left(h_0(x) + \langle q, H(x)\rangle\right),$$

where P^* is the positive polar cone of P. We say that problem (P) has a Lagrangian *zero duality gap* if the two optimal values coincides, that is, $\nu = \mu$. Problem (P) is said to have *strong duality* if it has zero duality gap and its dual problem (D) has a solution.

A more general problem related to (P) and (D) is the problem on the existence of a saddle value for bifunctions, that is, given two nonempty sets $A_1 \subseteq \mathbb{R}^n$ and

ISSN 2199-1413 (printed), ISSN 2199-1421 (electronic) / \$ 2.50

$A_2 \subseteq \mathbb{R}^m$, we say that a bifunction $f\colon A_1 \times A_2 \to \mathbb{R} \cup \{+\infty\}$ has a *saddle value* on $A_1 \times A_2$ if

$$\sup_{y \in A_2} \inf_{x \in A_1} f(x,y) = \inf_{x \in A_1} \sup_{y \in A_2} f(x,y). \tag{1}$$

If $A_1 = C$, $A_2 = P^*$ and $f(x,q) = h_0(x) + \langle q, H(x) \rangle$, then the left-hand side in (1) coincides with the dual value ν, while the right-hand side coincides with the primal value μ.

The study of the existence of a saddle value dates back to von Neumann in the context of Game theory for systems of linear inequalities, i.e., when f has linear arguments (see [18]). For convex arguments, we mention the works of Rockafellar (see [11, 12]).

If the bifunction f is neither convex nor concave on its first argument, then the problem of finding sufficient conditions for ensuring the existence of a saddle value becomes harder. An outstanding result on this direction is the famous Sion's theorem for quasiconvex-quasiconcave bifunctions given in [13] when at least one of the sets should be compact. Refined versions of Sion's theorem may be found in [16, 17] by using a coercivity condition on one of the arguments of the bifunction instead of the compactness of the set.

We recall that a proper, lsc and convex function is *coercive* iff there is no nonzero direction of recession, or equivalently, the asymptotic (recession) function is strictly greater than zero for all nonzero vectors (see equation (12) below). In the case of quasiconvex functions, a coercivity assumption is too restrictive, i.e., a refined version of direction of recession need to be used. Following this, the authors in [1, 10, 14] provide finer definitions of direction of recession for nonconvex sets and functions.

In this paper, we use the special classes of directions of recession and the *qx*-asymptotic function (see [3, 5, 7]) for providing sufficient conditions for the existence of a saddle value of a bifunction which is proper, lsc and quasiconvex in one argument. As a consequence, we provide sufficient conditions for zero duality gap for classes of quasiconvex cone constraint mathematical programming problems for which the sum of quasiconvex functions is quasiconvex.

The paper is organized as follows. In Section 2, we set up notation and preliminaries. We review some standard facts on asymptotic analysis and generalized convexity. In Section 3, we provide sufficient conditions for the existence of a saddle value in the nonconvex case. In Section 4, we apply our results for proving zero duality gap for quasiconvex cone constraint mathematical programming problems. Finally, examples of classes of quasiconvex functions are presented.

2. Preliminaries and basic definitions

In this paper, we denote the scalar product between two elements of $\mathbb{R}^n$ by $\langle \cdot, \cdot \rangle$ and the norm by $\|\cdot\|$. For $K \subseteq \mathbb{R}^n$, its closure is denoted by $\operatorname{cl} K$, its boundary by $\operatorname{bd} K$, its topological interior by $\operatorname{int} K$, its relative interior by $\operatorname{ri} K$ and its convex hull by $\operatorname{conv} K$. By K^* we denote the positive polar cone of K.

For a given closed and convex cone $P \subsetneq \mathbb{R}^n$, by the bipolar theorem, we have

$$p \in P \iff \langle q, p\rangle \geq 0,\ \forall\, q \in P^*. \tag{2}$$

and if $\operatorname{int} P \neq \emptyset$,
$$p \in \operatorname{int} P \iff \langle q, p\rangle > 0,\ \forall\, q \in P^* \setminus \{0\}. \tag{3}$$

For a nonempty set $K \subseteq \mathbb{R}^n$, its asymptotic cone is defined by

$$K^\infty := \left\{ u \in \mathbb{R}^n :\ \exists\, t_k \to +\infty,\ \exists\, x_k \in K,\ \frac{x_k}{t_k} \to u \right\}.$$

We adopt the convention that $(\emptyset)^\infty = \emptyset$. If K is closed and convex, then its asymptotic (recession) cone is equal to (see [2, Proposition 2.1.5])

$$K^\infty = \left\{ u \in \mathbb{R}^n :\ x_0 + \lambda u \in K,\ \forall\, \lambda \geq 0 \right\} \text{ for any } x_0 \in K. \tag{4}$$

When K is not necessarily convex or closed, several generalized asymptotic cones have been introduced and studied (see [1, 2, 10, 12, 14]). Recall that, the generalized recession cone of a nonempty set K in $\mathbb{R}^n$ is defined by (see [14] for instance)

$$\text{rec } K := \{u \in \mathbb{R}^n :\ x + tu \in K,\ \forall\, t > 0,\ \forall\, x \in K\}. \tag{5}$$

Clearly, $\operatorname{rec} K \subseteq K^\infty$. The inclusion may be strict as the set $K = \operatorname{epi} f$ with $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sqrt{|x|}$ shows. If K is closed and convex, then $\operatorname{rec} K$ is independent of the choice of $x \in K$, i.e.,

$$\text{rec } K = K^\infty = \{u \in \mathbb{R}^n :\ x_0 + \lambda u \in K,\ \forall\, \lambda > 0\},\quad \text{for any } x_0 \in K. \tag{6}$$

Remark 2.1. If K is convex, then rec K and K^∞ are not equal in general. In fact, consider $K := \{(x_1, x_2) \in \mathbb{R}^2 :\ x_1 > 0, x_2 > 0\} \cup \{(0,0)\}$. Here K is an nonclosed convex set, rec $K = K$ and $K^\infty = \{(x_1, x_2) \in \mathbb{R}^2 :\ x_1, x_2 \geq 0\}$, i.e., $u = (1,0)$ belongs to K^∞ without being in rec K. Moreover, the same example shows that rec K is not always closed.

The basic properties of the generalized recession cone are listed below.

Proposition 2.2. ([8, Proposition 2.1]) *Let K be a nonempty set in $\mathbb{R}^n$. Then the following assertions hold:*

(a) *If K is bounded, then* rec $K = \{0\}$.

(b) rec K *is a convex cone.*

(c) *Let K_1, K_2 be nonempty sets in $\mathbb{R}^n$. Then*

$$\text{rec } K_1 + \text{rec } K_2 \subseteq \text{rec } (K_1 + K_2).$$

(d) *Let $\{K_i\}_{i\in I}$ be a family of nonempty sets in $\mathbb{R}^n$. If $\bigcap_{i\in I} K_i \neq \emptyset$, then*

$$\bigcap_{i\in I} (\text{rec } K_i) \subseteq \text{rec}\left(\bigcap_{i\in I} K_i\right), \tag{7}$$

and equality holds when every K_i is closed and convex.

As we can see in the next remark, equality in Proposition 2.2$(c), (d)$ does not hold in general.

Remark 2.3. (i) The reverse statement in Proposition 2.2(a) does not hold in general. Indeed, for $K := \mathbb{R}\backslash\{0\}$ we have that $\text{rec}\ K = \{0\}$.

(ii) The reverse inclusion in Proposition 2.2(c) does not hold in general. Take $K_1 := [0, +\infty[$ and $K_2 :=] - \infty, 0]\backslash\{-1\}$. Thus, $\text{rec}\ K_1 = [0, +\infty[$ and $\text{rec}\ K_2 = \{0\}$. Then $\text{rec}\ K_1 + \text{rec}\ K_2 = [0, +\infty[$ while $\text{rec}\ (K_1 + K_2) = \mathbb{R}$.

(iii) The reverse inclusion in Proposition 2.2(d) does not hold in general. Indeed, take $h\colon \mathbb{R} \to \mathbb{R}$ with $h(x) = \sqrt{|x|}$. Set $K_1 := \text{epi}\ h$ and

$$K_2 := \{(x, y) \in \mathbb{R}^2 : x \geq 0\}\backslash\{(0, y) \in \mathbb{R}^2 :\ |y| \leq 5\}.$$

Observe that $u = (0, 1) \in \text{rec}\left(\bigcap_{i=1}^{2} K_i\right)$, but $u \notin \text{rec}\ K_2$.

Given any function $h\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the *effective domain* of h is defined by $\text{dom}\, h := \{x \in \mathbb{R}^n :\ h(x) < +\infty\}$. We say that h is a *proper function* if $\text{dom}\, h$ is nonempty. For a function h, we adopt the usual convention $\inf_{\emptyset} h := +\infty$ and $\sup_{\emptyset} h := -\infty$. We denote by $\text{epi}\, h := \{(x, t) \in \mathbb{R}^n \times \mathbb{R} :\ h(x) \leq t\}$ its *epigraph*, and for a given $\lambda \in \mathbb{R}$, by $S_\lambda(h) := \{x \in \mathbb{R}^n :\ h(x) \leq \lambda\}$ we denote its *sublevel set* at the height λ. As usual, $\text{argmin}_{\mathbb{R}^n} h := \{x \in \mathbb{R}^n :\ h(x) \leq h(y),\ \forall\ y \in \mathbb{R}^n\}$.

As usual, a function h with convex domain is said to be:

(a) *convex* if, given any $x, y \in \text{dom}\, h$, then

$$h(\lambda x + (1 - \lambda)y) \leq \lambda h(x) + (1 - \lambda)h(y),\ \forall\ \lambda \in [0, 1],$$

(b) *quasiconvex* if, given any $x, y \in \text{dom}\, h$, then

$$h(\lambda x + (1 - \lambda)y) \leq \max\{h(x), h(y)\},\ \forall\ \lambda \in [0, 1].$$

Clearly, every convex function is quasiconvex since:

$$h \text{ is convex} \iff \text{epi}\, h \text{ is a convex set.}$$
$$h \text{ is quasiconvex} \iff S_\lambda(h) \text{ is a convex set, for all } \lambda \in \mathbb{R}.$$

The usual *subdifferential* of h at $\overline{x} \in \text{dom}\, h$ is defined by:

$$\partial h(\overline{x}) := \{\xi \in \mathbb{R}^n :\ h(y) \geq h(\overline{x}) + \langle \xi, y - \overline{x}\rangle,\ \forall\ y \in \mathbb{R}^n\}. \tag{8}$$

If h is not convex (for instance, quasiconvex), then many different subdifferentials trying to extend the good properties of ∂h have been introduced. Here we recall the following notion: The *Greenberg-Pierskalla subdifferential* (see [6]) of h at $\overline{x} \in \text{dom}\, h$, is defined by

$$\partial^{GP} h(\overline{x}) := \{\xi \in \mathbb{R}^n :\ \langle \xi, y - \overline{x}\rangle \geq 0 \text{ implies } h(y) \geq h(\overline{x})\}. \tag{9}$$

The *asymptotic function* $h^\infty \colon \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ of a proper function h, is the function for which

$$\operatorname{epi} h^\infty := (\operatorname{epi} h)^\infty. \tag{10}$$

When h is lsc and convex, for all $x_0 \in \operatorname{dom} h$ we have

$$h^\infty(u) = \sup_{t>0} \frac{h(x_0 + tu) - h(x_0)}{t} = \lim_{t\to+\infty} \frac{h(x_0 + tu) - h(x_0)}{t}. \tag{11}$$

A function h is called *coercive* if $h(x) \to +\infty$ as $\|x\| \to +\infty$. If $h^\infty(u) > 0$ for all $u \neq 0$, then h is coercive. In addition, if h is proper, lsc and convex, then (see [2, Proposition 3.1.3])

$$h \text{ is coercive} \iff \operatorname{argmin}_{\mathbb{R}^n} h \neq \emptyset \text{ and compact} \iff h^\infty(u) > 0,\ \forall\, u \neq 0. \tag{12}$$

If the function h is quasiconvex, then the usual notion of h^∞ is not good enough as was noted in [1, 10]. For that reason, the authors in [7] introduced the notion of qx-asymptotic function as follows.

The *qx-asymptotic function* $h^{qx} \colon \mathbb{R}^n \to \overline{\mathbb{R}}$ of a proper lsc and quasiconvex function h is defined by

$$h^{qx}(u) := \inf\{\lambda : u \in (S_\lambda(h))^\infty\}, \tag{13}$$

and $h^{qx} \equiv +\infty$ if $h \equiv +\infty$. Furthermore, the following analytic formulas hold (see [7, Proposition 3.1 and Proposition 3.2])

$$h^{qx}(u) = \inf_{x\in\mathbb{R}^n} \sup_{t\geq 0} h(x + tu), \tag{14}$$

$$h^{qx}(u) = \inf_{x\in\mathbb{R}^n} \lim_{t\to+\infty} h(x + tu). \tag{15}$$

The qx-asymptotic function has useful properties and calculus rules (see [7]). Furthermore, for a proper lsc and quasiconvex function h, by Theorem 4.1 of [7] we have

$$\operatorname{argmin}_{\mathbb{R}^n} h \neq \emptyset \text{ and compact} \iff h^{qx}(u) > h^{qx}(0),\ \forall\, u \neq 0. \tag{16}$$

This characterization goes beyond coercivity as the continuous quasiconvex function $h \colon \mathbb{R} \to \mathbb{R}$ given by $h(x) = \min\{1, |x|\}$ shows.

For a further study on asymptotic (recession) directions and functions we refer to [2, 3, 5, 7, 8, 10, 15] and references therein.

3. Existence of a saddle value

Let C and D two nonempty closed and convex sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $f \colon C \times D \to \mathbb{R}$ be a lsc and quasiconvex function on its first argument. Our goal is to provide sufficient conditions for ensuring the following equality:

$$\text{(SV)} \qquad \sup_{y\in D} \inf_{x\in C} f(x, y) = \inf_{x\in C} \sup_{y\in D} f(x, y).$$

The right-hand side (resp. left-hand side) of (SV) is called the *Primal problem* (resp. *Dual problem*) and its value is denoted by V_P (resp. V_D). It is well-known that we always have the following *weak duality* relation:

$$V_D = \sup_{y\in D}\inf_{x\in C} f(x,y) \leq \inf_{x\in C}\sup_{y\in D} f(x,y) = V_P.$$

If $V_P = V_D$, then we say that there exists a *saddle value* for f, or equivalent, (SV) holds. A point $(\overline{x},\overline{y}) \in C \times D$ is said to be a *saddle point* of f if

$$f(\overline{x},y) \leq f(\overline{x},\overline{y}) \leq f(x,\overline{y}),\ \forall\ (x,y) \in C \times D. \tag{17}$$

If the saddle point exists, then $f(\overline{x},\overline{y}) = V_P = V_D$ is the saddle value of f. In general, the existence of a saddle value does not imply the existence of a saddle point.

The following definition, which is motivated by the good properties of the qx-asymptotic function, will be useful in the sequel.

Definition 3.1. Let $f\colon C \times D \to \mathbb{R}$ be a lsc and quasiconvex function on its first argument. Given any $y \in D$, the *qx-asymptotic function* of $f(\cdot,y)$ at the direction $u \in \mathbb{R}^n$ is defined by

$$(f(\cdot,y))^{qx}(u) := \inf_{x\in C}\sup_{t\geq 0} f(x+tu,y), \tag{18}$$

where we put $f(x,y) = +\infty$ for $x \notin C$, $y \in D$.

Note that if $f(\cdot,y)$ is lsc and quasiconvex for all $y \in D$, then the function $x \mapsto \sup_{y\in D} f(x,y)$ is lsc and quasiconvex.

Using [7, Remark 3.3], the existence of a saddle value for f could be studied using qx-asymptotic functions:

Proposition 3.2. *Let $f\colon C \times D \to \mathbb{R}$ be a lsc and quasiconvex function on its first argument. Then*

$$V_P = V_D \iff \left(\sup_{y\in D} f(\cdot,y)\right)^{qx}(0) = \sup_{y\in D}(f(\cdot,y))^{qx}(0). \tag{19}$$

Proof. We simply observe that

$$\sup_{y\in D}(f(\cdot,y))^{qx}(0) = \sup_{y\in D}\left(\inf_{x\in C}\sup_{t\geq 0} f(x,y)\right) = \sup_{y\in D}\inf_{x\in C} f(x,y) = V_D,$$

$$\left(\sup_{y\in D} f(\cdot,y)\right)^{qx}(0) = \inf_{x\in C}\sup_{t\geq 0}\sup_{y\in D} f(x,y) = \inf_{x\in C}\sup_{y\in D} f(x,y) = V_P,$$

and equivalence (19) follows. $\square$

Our first result, which provides a sufficient condition for the existence of a saddle value by using relation (16), is given below. Note that, only for this proposition, an additional assumption on the second argument of f is needed.

Proposition 3.3. *Let $f\colon C \times D \to \mathbb{R}$ be such that $f(\cdot, y)$ and $-f(x, \cdot)$ are lsc and quasiconvex for each $y \in D$ and each $x \in C$, respectively. If*

$$\left(\sup_{y\in D} f(\cdot, y)\right)^{qx}(u) > \left(\sup_{y\in D} f(\cdot, y)\right)^{qx}(0), \ \forall\, u \neq 0, \tag{20}$$

then f possesses a saddle value on $C \times D$.

Proof. Set $g(x) := \sup_{y\in D} f(x, y)$. Then equation (20) implies $g^{qx}(u) > g^{qx}(0)$ for all $u \neq 0$. Then $\operatorname{argmin}_C g$ is nonempty and compact by relation (16). It follows from [17, Theorem 4(*K*)] that f possesses a saddle value on $C \times D$. □

Remark 3.4. Note that condition (20) does not imply the coerciveness of f on its first argument (see the example below equation (16)).

For dealing with more general cases, some special notions of directions or recession need to be used. To that end, we recall that given a proper, lsc and quasiconvex function $h\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, and since $\operatorname{epi} h$ is not necessarily convex, a convenient asymptotic cone in this case is $\operatorname{rec}(\operatorname{epi} h)$. Hence, by equation (5), a vector $u \in \mathbb{R}^n$ is a direction of recession of h (see [7, 8, 14]) if

$$\begin{aligned} u \in \operatorname{rec}(\operatorname{epi} h) &\Longleftrightarrow u \in \{v \in \mathbb{R}^n : \ h(x + tv) \leq h(x), \ \forall\, x \in \operatorname{dom} h, \ \forall\, t \geq 0\} \\ &\Longleftrightarrow \ u \in \{v \in \mathbb{R}^n : \sup_{t\geq 0} h(x + tv) \leq h(x), \ \forall\, x \in \operatorname{dom} h\}. \end{aligned} \tag{21}$$

Therefore, if $u \in \mathbb{R}^n$ is a direction of recession of a proper, lsc and quasiconvex function h, then (by using [7, Remark 3.3]) we have

$$h^{qx}(u) \leq \inf_{x\in\mathbb{R}^n} h(x) = h^{qx}(0). \tag{22}$$

By using this, we adapt the following definitions from [3, 5] for quasiconvex functions. In virtue of Theorem 3.8 (see below), we define the following notions without a properness assumption on h.

Definition 3.5. Let C be a nonempty closed and convex set in $\mathbb{R}^n$, and let $h\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a lsc and quasiconvex function. A nonzero vector $u \in \mathbb{R}^n$ is said to be:

(a) An *ia-direction of recession of* h on C if

$$\lim_{t\to+\infty} h(x + tu) = h^{qx}(0), \ \forall\, x \in C. \tag{23}$$

(b) A *partial ia-direction of recession of* h on C if there exists $\overline{x} \in C$ such that

$$\lim_{t\to+\infty} h(\overline{x} + tu) = h^{qx}(0). \tag{24}$$

Clearly, every ia-direction of recession is a partial ia-direction of recession.

Remark 3.6. Note that for a quasiconvex (in particular, convex) function, the function $t \mapsto h(x+tu), t > 0$, is monotone for large values of t, so the limit always exist and the lim inf used in [3, Definition 2.4] and [5, Definition 4.3] is not needed.

We also adapt Definition 3.2 of [3] for the quasiconvex case.

Definition 3.7. Let C and D be two nonempty closed and convex sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $f\colon C \times D \to \mathbb{R}$ be a lsc and quasiconvex function on its first argument. A direction of recession $u \in C^\infty$ is a *saddle ia-direction of recession of* f on C if there exists $\overline{x} \in C$ such that

$$\lim_{t\to+\infty} \sup_{y\in D} f(\overline{x}+tu, y) = \sup_{y\in D} (f(\cdot, y))^{qx}(0). \tag{25}$$

Our main result, which provides a sufficient condition for the existence of a saddle value for f, is given below.

Theorem 3.8. *Let C and D be two nonempty closed and convex sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $f\colon C\times D \to \mathbb{R}$ be a lsc and quasiconvex function on its first argument. Then a direction of recession $u \in C^\infty$ is a saddle ia-direction of recession of f on C iff it is a partial ia-direction of recession of $x \mapsto \sup_{y\in D} f(\cdot, y)$ on C and* (SV) *holds.*

Proof. ($\Rightarrow$) Let $u \in C^\infty$ be a saddle ia-direction of recession of f on C. Then there exists $\overline{x} \in C$ such that

$$\begin{aligned} \lim_{t\to+\infty} \sup_{y\in D} f(\overline{x}+tu, y) &= \sup_{y\in D} \left(f(\cdot, y)\right)^{qx}(0) = \sup_{y\in D} \inf_{x\in C} f(x,y) \\ &\leq \inf_{x\in C} \sup_{y\in D} f(x,y). \end{aligned} \tag{26}$$

Observe that for any $x \in C$, we have

$$\inf_{x\in C} \sup_{y\in D} f(x,y) \leq \liminf_{t\to+\infty} \sup_{y\in D} f(x+tu,y) = \lim_{t\to+\infty} \sup_{y\in D} f(x+tu,y), \tag{27}$$

where the last equality holds from Remark 3.6. By taking $x = \overline{x} \in C$ in (27), it follows from (26) and (27) that

$$\lim_{t\to+\infty} \sup_{y\in D} f(\overline{x}+tu,y) = \sup_{y\in D} \inf_{x\in C} f(x,y) \leq \inf_{x\in C} \sup_{y\in D} f(x,y) \tag{28}$$

$$\leq \lim_{t\to+\infty} \sup_{y\in D} f(\overline{x}+tu,y). \tag{29}$$

Therefore, $u \in C^\infty$ its a partial ia-direction of recession of $\sup_{y\in D} f(\cdot, y)$ on C. Moreover, from equations (28) and (29), it follows that (SV) holds.

($\Leftarrow$) Set $u \in C^\infty$ a partial ia-direction of recession of $x \mapsto \sup_{y\in D} f(\cdot, y)$ on C. Thus there exists $x_0 \in C$ such that

$$\lim_{t\to+\infty} \left(\sup_{y\in D} f(x_0+tu,y)\right) = \left(\sup_{y\in D} f(\cdot,y)\right)^{qx}(0) = \inf_{x\in C}\sup_{y\in D} f(x,y),$$

and since (SV) holds, we have

$$\begin{aligned}\lim_{t\to+\infty}\left(\sup_{y\in D} f(x_0+tu,y)\right) &= \inf_{x\in C}\sup_{y\in D} f(x,y) = \sup_{y\in D}\inf_{x\in C} f(x,y)\\ &= \sup_{y\in D}(f(\cdot,y))^{qx}(0).\end{aligned}$$

Therefore, $u \in C^{\infty}$ is a saddle ia-direction of recession of f on C. □

Theorem 3.8 applies for classes of nonconvex bifunctions for which both arguments has empty solution sets as we show in the following example.

Example 3.9. Let $f\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be the continuous and real valued bifunction given by $f(x,y) = \frac{x}{1+|x|} - \frac{y}{1+|y|}$. Note that f is quasiconvex (resp. quasiconcave) on its first (resp. second) argument. However, since f is not convex or coercive on its first argument, the results in [3, 5, 15, 16] canot be applied.

Take the recession direction $u = -1$. Observe that, for all $x \in \mathbb{R}$, we have

$$\begin{aligned}\lim_{t\to+\infty}\sup_{y\in\mathbb{R}} f(x-t,y) &= \lim_{t\to+\infty}\left(\frac{x-t}{1+|x-t|} - \inf_{y\in\mathbb{R}}\frac{y}{1+|y|}\right)\\ &= \lim_{t\to+\infty}\left(\frac{x-t}{1+|x-t|}\right)+1 = 0.\end{aligned}$$

On the other hand,

$$\sup_{y\in\mathbb{R}}(f(,y))^{qx}(0) = \sup_{y\in\mathbb{R}}\left(\inf_{x\in\mathbb{R}}\left(\frac{x}{1+|x|}\right) - \frac{y}{1+|y|}\right) = -1 - \inf_{y\in\mathbb{R}}\frac{y}{1+|y|} = 0.$$

Therefore, $u = -1$ is a saddle ia-direction of recession of f on $\mathbb{R}$. It follows from Theorem 3.8 that f has a saddle value on $\mathbb{R}\times\mathbb{R}$.

4. Applications to nonconvex programming

Let $h_i\colon \mathbb{R}^n \to \mathbb{R}$ be lsc and quasiconvex functions for all $i \in \{0,1,\ldots,m\}$, let $H\colon \mathbb{R}^n \to \mathbb{R}^m$ be the vector valued function given by $H := (h_1,\ldots,h_m)$, and let P be a nonempty, closed, convex and pointed cone in $\mathbb{R}^m$.

We consider the cone constraint mathematical programming problem:

$$\inf_{x\in C} h_0(x), \tag{30}$$

where $C := \{x \in \mathbb{R}^n : H(x) \in -P\}$ is assumed to be nonempty.

The Lagrangian of problem (30) is given by $L\colon \mathbb{R}^n \times P^* \to \mathbb{R}$ with

$$L(x;q) := h_0(x) + \langle q, H(x)\rangle = \langle (1,q),(h_0,H)(x)\rangle, \text{ with } q \in P^*. \tag{31}$$

By using relation (2), its follows easily that

$$\sup_{q\in P^*} L(x,q) = L(x,0) = h_0(x), \ \forall\, x \in C. \tag{32}$$

Therefore, if the saddle value exists, then it is equal to the infimum of the objective function h_0.

Next result provides general sufficient conditions for ensuring zero duality gap for problem (30). Note that part (b) is property (a) in [3, Theorem 3.2].

Proposition 4.1. *Let $L\colon \mathbb{R}^n \times P^* \to \mathbb{R}$ be the Lagrangian defined in* (31). *Suppose that L is lsc and quasiconvex on its first argument. Consider the following statements:*

(a) *$u \in C^\infty$ is a saddle ia-direction of recession of L.*

(b) $\inf_{x\in C} h_0(x) \leq \sup_{q\in P^*} \inf_{x\in C} L(x,q)$.

(c) *Zero duality gap holds for problem* (30).

Then
$$(a) \implies (b) \implies (c). \tag{33}$$

Proof. Take $u \in C^\infty$ a saddle ia-direction of recession of L. Then there exists $\overline{x} \in C$ such that

$$\begin{aligned}\inf_{x\in C} h_0(x) &\leq \lim_{t\to+\infty} h_0(\overline{x}+tu) = \lim_{t\to+\infty} \sup_{q\in P^*} L(\overline{x}+tu,q)\\ &= \sup_{q\in P^*}\inf_{x\in C} L(x,q) \leq \inf_{x\in C}\sup_{q\in P^*} L(x,q) = \inf_{x\in C} h_0(x).\end{aligned}$$

Hence, (a)⇒(b)⇒(c). □

Theorem 3.8 is restricted to the quasiconvexity of the Lagrangian L on its first argument. Recall that, in contrast to convex functions, the sum of quasiconvex functions is not necessarily quasiconvex. Indeed, set $K = [\varepsilon, +\infty[$ for $\varepsilon > 0$ small enough, and the quasiconvex functions $f, g\colon K \to \mathbb{R}$ given by $f(x) = -x$ and $g = -\frac{1}{x}$. Clearly, $(f+g)(x) = -x - \frac{1}{x}$ is not quasiconvex on K.

The problem of finding sufficient conditions for ensuring the quasiconvexity of the sum of quasiconvex functions have been studied deeply. Partial results on this direction may be found in [9, 15, 19]. Two examples of classes of quasiconvex functions for which its sum is quasiconvex are given below.

The following class of quasiconvex vector valued functions was introduced in the Definitions 2.2 and 2.3 in [9].

Definition 4.2. Let P be a closed, convex and pointed cone in $\mathbb{R}^m$, and let K be a convex set in $\mathbb{R}^n$. A vector valued function $F\colon K \to \mathbb{R}^m$ is said to be $*$-*quasiconvex* (resp. $*$-*lsc*) on K with respect to the cone P if the function

$$x \mapsto \langle q, F(x)\rangle \text{ is quasiconvex (resp. lsc) on } K \text{ for all } q \in P^*. \tag{34}$$

If $P = \mathbb{R}^m_+$ and h_i is lsc and convex for all $i \in \{0, 1, \ldots, m\}$, then F is $*$-quasiconvex and $*$-lsc on any $K \subseteq \mathbb{R}^n$.

We illustrate the use of the $*$-quasiconvex and $*$-lsc functions below.

Example 4.3. Set $\widehat{P} := [0, +\infty[\times P$ and $\widehat{H} := (h_0, H)$. If $\widehat{H}$ is $*$-lsc and $*$-quasiconvex on C with respect to $\widehat{P}$, and there exists a saddle ia-direction of recession $u \in C^\infty$, then problem (30) has zero duality gap and

$$\sup_{q\in P^*} \inf_{x\in C} L(x;q) = \inf_{x\in C} h_0(x). \tag{35}$$

Indeed, since $\widehat{H}$ is $*$-lsc and $*$-quasiconvex on C with respect to $\widehat{P}$, $\langle \widehat{q}, \widehat{H}(\cdot)\rangle$ is lsc and quasiconvex on C for all $\widehat{q} = (1, q) \in (\widehat{P})^*$. In particular, $L(\cdot; q)$ is lsc and quasiconvex on C for all $q \in P^*$. Then the result follows from Theorem 3.8 and equation (32).

Another class of quasiconvex functions for which its sum is quasiconvex is the class of GP functions introduced in [15, Section 4]:

Definition 4.4. Let $h\colon \mathbb{R}^n \to \mathbb{R}$ be a real-valued, usc and quasiconvex function. The class of *GP-functions* related to h is defined by

$$\Phi_h := \{g\colon \mathbb{R}^n \to \mathbb{R} :\ g \text{ is usc}, \ \partial^{GP} h(x) \subseteq \partial^{GP} g(x),\ \forall\, x \in \mathbb{R}^n\}. \tag{36}$$

By [15, Theorem 4.4], $h \in \Phi_h$ and for all $g \in \Phi_h$, g is quasiconvex. Furthermore, Φ_h is closed under addition and multiplication by positive scalars.

In order to apply Theorem 3.8 for GP-functions, we consider problem (30) with $P = \mathbb{R}^m_+$, that is,

$$\inf_{x\in C_0} h_0(x), \tag{37}$$

where $C_0 := \{x \in \mathbb{R}^n :\ h_i(x) \leq 0,\ \forall\, i \in \{1, \ldots, m\}\}$ is assumed to be nonempty.

Example 4.5. Let $P = \mathbb{R}^m_+$, and let h_i be continuous, real-valued and quasiconvex functions for all $i \in \{0, 1, \ldots, m\}$. If $h_1, h_2, \ldots, h_m \in \Phi_{h_0}$, and there exists $u \in (C_0)^\infty$ a saddle ia-direction of recession of L on C_0, then problem (30) has zero duality gap and

$$\sup_{q\in\mathbb{R}^m_+} \inf_{x\in C_0} L(x;q) = \inf_{x\in C_0} h_0(x). \tag{38}$$

Since $h_0 \in \Phi_{h_0}$, $h_0, h_1, \ldots, h_m \in \Phi_{h_0}$ by [15, Theorem 4.4]. Since $P^* = \mathbb{R}^m_+$, it follows that $L(\cdot, q)$ is continuous and quasiconvex on its first argument for all $q \in \mathbb{R}^m_+$ due to [15, Theorem 4.1]. Then the result follows from Theorem 3.8 and equation (32).

Acknowledgements. This research was partially supported by Conicyt–Chile under the project Fondecyt Iniciación 11180320. The author wishes to thank J. E. Martínez-Legaz and an anonymous referee for their valuable comments and suggestions that helped to improve this paper.

References

[1] Ch. Amara: *Directions de majoration d'une fonction quasiconvexe et applications*, Serdica Math. J. 24 (1998) 289–306.

[2] A. Auslender, M. Teboulle: *Asymptotic Cones and Functions in Optimization and Variational Inequalities*, Springer, New York (2003).

[3] F. Bonenti, J. E. Martínez-Legaz: *On the existence of a saddle value*, J. Optim. Theory Appl. 165 (2015) 785–792.

[4] J. R. Duffin: *Clark's Theorem on linear programs holds for convex programs*, Proc. Natl. Acad. Sci. USA 75 (1978) 1624–1626.

[5] E. Ernst, M. Volle: *Zero duality gap for convex programs: a generalization of the Clark-Duffin theorem*, J. Optim. Theory Appl. 158 (2013) 668–686.

[6] H. J. Greenberg, W. P. Pierskalla: *Quasi-conjugate functions and surrogate duality*, Cah. Cent. Étud. Rech, Opér. 15 (1973) 437–448.

[7] N. Hadjisavvas, F. Lara, J. E. Martínez-Legaz: *A quasiconvex asymptotic function with applications in optimization*, J. Optim. Theory Appl. 180 (2019) 170–186.

[8] A. Iusem, F. Lara: *Quasiconvex optimization problems and asymptotic analysis in Banach spaces*, Optimization, doi: 10.1080/02331934.2019.1612893 (2019).

[9] V. Jeyakumar, W. Oettli, M. Natividad: *A solvability theorem for a class of quasiconvex mappings with applications to optimization*, J. Math. Anal. Appl. 179 (1993) 537–546.

[10] J. P. Penot: *What is quasiconvex analysis?*, Optimization 47 (2000) 35–110.

[11] R. T. Rockafellar: *A general correspondence between dual minimax problems and convex programs*, Pacific J. Math. 25 (1968) 597–611.

[12] R. T. Rockafellar: *Convex Analysis*, Princeton University Press, Princeton (1970).

[13] M. Sion: *On general minimax theorems*, Pacific J. Math. 8 (1958) 171–176.

[14] V. Soltan: *Asymptotic planes and closedness conditions for linear images and vector sums of sets*, J. Convex Analysis 25 (2018) 1183–1196.

[15] S. Suzuki: *Quasiconvexity of sums of quasiconvex functions*, Linear Nonlinear Analysis 3 (2017) 287–295.

[16] H. Tuy: *Minimax theorems revisited*, Acta Math. Vietnamica 29(3) (2004) 217–229.

[17] H. Tuy: *Minimax: existence and stability*, in: *Pareto Optimality, Game Theory and Equilibria*, A. Chinchuluun et al. (eds.), Springer, New York (2008) 3–22.

[18] J. von Neumann: *Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes*, Ergebn. Math. Kolloq. Wien 8 (1937) 73–83.

[19] A. Zaffaroni: *Is every radiant function the sum of quasiconvex functions?*, Math. Meth. Oper. Res. 59 (2004) 221–233.