Minimax Theory and its Applications
Volume 5 (2020), No. 1, 77–101

Mutual Support Equilibrium

Zhicheng Gao
School of Mathematics and Statistics, Carleton University, Ottawa, Ont. K1S5B6, Canada
Zhicheng.Gao@carleton.ca

Moussa Larbani
School of Mathematics and Statistics, Carleton University, Ottawa, Ont. K1S5B6, Canada
Moussa.Larbani@carleton.ca

Received: August 10, 2019
Accepted: December 27, 2019

In recent years, experimental evidence has shown that in normal form games noncooperative behavior is not the only behavior that the players show. Mutual support, altruism, reciprocity and fairness are among behaviors that the players exhibit during the game. These considerable advances in experimental research have not been completely formalized. New concepts of solution and models that reflect other regarding preferences need to be introduced, investigated and applied to real-world game situations. This paper is a contribution to this direction of research. We introduce two new solution concepts for normal form games that reflect altruism and mutual support among players, the altruistic equilibrium and the mutual support equilibrium. We present their properties and provide sufficient conditions for their existence. Further, they are compared to related existing solution concepts. Some research problems are stated in the conclusion.

Keywords: Minimax, Nash equilibrium, altruistic equilibrium, mutual support equilibrium.

2010 Mathematics Subject Classification: 05A15, 05A16.

1. Introduction

The formalization of cooperation in the framework of normal form games goes back to the 50s of the last century. In 1959, Aumann [3] introduced the concept of strong Nash equilibrium (SNE). SNE is based on the notion of ”blocking”. A coalition can block a strategy profile if it can find a deviation that improves the payoff of all its members. An SNE is a strategy profile that cannot be blocked by any coalition. Unfortunately, the set of SNE of most games is empty [6]. Later, Aumann [4] introduced the α-core and the β-core that are also based on the notion of blocking. In contrast with the SNE and the β-core, in 1971 Scarf [30] proved that the α-core is nonempty under common mathematical conditions (compactness and convexity of strategy sets and the continuity and quasiconcavity of the payoff functions). Earlier, in 1957 Berge [5] introduced Strong Berge equilibrium (SBE), that is not well-known in literature. The existence of SBE has been studied in [21]. SBE differs from the three previously mentioned concepts. At SBE any deviation of any coalition containing all the players but one cannot improve the payoff of any of its members.

ISSN 2199-1413 (printed), ISSN 2199-1421 (electronic) / \$ 2.50

On the other hand, in the last two decades, much experimental research provides strong evidence that, in games and social interaction, players do show other social behaviors than competitive behavior (Nash behavior) [15, 23, 28, 33], such as fairness, mutual support, altruism and reciprocity. Such social behavioral aspects have been investigated in psychological games (PG) [8, 15]. In such games, a player's utility is constructed based on three approaches: 1) psychological plausibility, 2) axioms and 3) other-regarding preferences. The latter approach uses payoff transformation of a player's utility to reflect his/her behavior. The most used transformations are those using parameters to reflect the fact that a player considers the utilities of the others when he/she selects his/her strategy. When it comes to applications, psychological games involve difficulties related to complexity and multiple equilibria [15]. The investigation of PG is mainly based on laboratory experiments, their mathematical analysis is at an early stage.

Berge equilibrium (BE), which was introduced by Zhukoveskiy in 1985 [36], is the first equilibrium based on other-regarding preferences that reflects mutual support and altruism among players in a normal form game. Indeed, at BE each player's payoff function is maximized by all the other players. In [11, 17, 37], it is shown that BE reflects the well-known Golden Rule in social interactions.

The common drawbacks of the mentioned solution concepts are the existence problem and the difficulty of their computation. Most games have an empty set of SNE, SBE, and an empty β-core. The α-core and BE are exceptions. BE existence has been intensively studied in the recent years. So far there are no existence results on BE that are based on common mathematical conditions such as continuity and concavity of payoff functions and convexity and compactness of the strategy sets. All BE existence results involve some extremal condition [22]. In addition, when these concepts exist in a finite or infinite game, it is difficult to compute them. Scarf's theorem on nonemptiness of the α-core is not constructive in the sense that no algorithm has been derived from it. The existence and determination problems limit considerably the scope of application of the mentioned concepts to real-world problems.

In the present paper, we introduce two new solution concepts for normal form games that reflect altruism and mutual support, the altruistic equilibrium (AE) and the Mutual Support equilibrium (MSE). These equilibria have the property of being Nash equilibria of a special game derived from the initial game. This makes the problems of their existence and determination more tractable mathematically and computationally than the problems of existence and determination of the existing concepts of cooperative solution of normal form games as Nash equilibrium is well-studied in terms of existence and determination. Moreover, we show that MSE is a generalization of BE.

The paper is organized as follows. In Section 2, we introduce AE and MSE and present their properties. Section 3 is devoted to the problem of existence of these equilibria and their determination. Section 4 compares AE and MSE with related solution concepts for normal form games. Section 5 concludes the paper.

2. Altruistic and mutual support equilibria

In this section we formally introduce the altruistic and mutual support equilibria. For this purpose, we need to recall Nash Equilibrium. Before doing so we introduce some notations.

- N denotes a finite set of n players.
- For each $i \in N$, S_i is the player i's strategy set, which can be infinite.
- $\mathbf{S} = \prod\limits_{i \in N} S_i$ is the set of strategy profiles.
- $u_i : \mathbf{S} \mapsto \mathbb{R}$ is the payoff function for player i, and $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ denotes the vector of payoffs.
- $G = (N, \mathbf{S}, \mathbf{u})$ denotes the game with the set N of players, the set $\mathbf{S}$ of strategy profiles, and the payoff vector $\mathbf{u}$.
- For a subset $K \subseteq N$, denote $-K = N - K$ and define $u_K := \sum_{j \in K} u_j$. When $K = \{i\}, i \in N$, $N - \{i\} = -i$.
- We also write $\mathbf{s} := (s_1, s_2, \ldots, s_n) := (s_i, \mathbf{s}_{-i}) := (\mathbf{s}_K, \mathbf{s}_{-K})$.
- For each $i \in N$, we denote by $u_{-i}(\mathbf{s}) = \sum_{j \in -i} u_j(\mathbf{s}), \mathbf{s} \in \mathbf{S}$, the aggregate payoff function of all the players other than player i.

We recall the following well-known definition [24, 25].

Definition 2.1. A strategy profile $\mathbf{s}^* \in S$ is called *Nash Equilibrium* (NE) of the game G if

$$u_i(s_i, \mathbf{s}^*_{-i},) \leq u_i(\mathbf{s}^*) \text{ for all } s_i \in S_i \text{ and } i \in N. \tag{1}$$

Equation (1) can be reformulated in terms of optimization as follows.

$$\max_{s_i \in S_i} u_i(s_i, s^*_{-i}) = u_i(s^*), \text{ for all}\, i \in N. \tag{2}$$

Definition 2.2. A strategy profile $\mathbf{s}^* \in S$ is called an *Altruistic Equilibrium* (AE) if

$$u_{-i}(s_i, \mathbf{s}^*_{-i}) \leq u_{-i}(\mathbf{s}^*), \text{ for all } s_i \in S_i \text{ and } i \in N. \tag{3}$$

Equation (3) can be reformulated in optimization terms as follows.

$$\max_{s_i \in S_i} u_{-i}(s_i, \mathbf{s}^*_{-i},) = u_{-i}(\mathbf{s}^*), \text{ for all } i \in N. \tag{4}$$

For comparison purposes, we denote by $S^N(G)$ and $S^A(G)$ the sets of NE and AE of the game G, respectively. At an AE, each player maximizes the sum of the payoffs of the other players. Here, each player $i \in N$ behaves in an 'altruistic' way with respect to the other players in the counter coalition $-i$, by maximizing their aggregate payoff u_{-i} and ignoring his/her own payoff, u_i. This shows that AE expresses the well-known Golden Rule in social interactions: Behave with others as you like the others to behave with you. We often drop G from $S^N(G)$ and $S^A(G)$ when it is clear from the context.

Given a game $G = (N, \mathbf{S}, \mathbf{u})$, we can define a new game $G^A = (N, \mathbf{S}, \mathbf{v})$ with the same set of players and the same set of strategies for each player such that the new utility function $v_i(\mathbf{s})$ is given by $v_i(\mathbf{s}) = u_{-i}(\mathbf{s})$. That is, G^A is the game obtained from the altruistic payoff transformation. It is easy to see that an AE of the initial game G becomes an NE of the game G^A and vice versa. Thus $S^A(G) = S^N(G^A)$, and studying AE of G is reduced to studying NE of G^A. Of course the definition of AE makes most sense if we assume the addability of players' payoffs; NE does not require this condition. Addability of players' utility may not be satisfied in some real-world game models. Particularly, when payoffs are of very different scales, the maximization of the sum favours the payoffs on the large scales and almost neglects the ones on the small scales. The obtained solution will be biased towards players with large scale payoffs. In other words, AE is not scaling invariant, which is a drawback. A way to overcome this drawback is to normalize the payoffs before computing AE. An example of normalization of the payoff functions of the game G is as follows

$$U_i(\mathbf{s}) = \frac{u_(\mathbf{s}) - \min_{\mathbf{s}\in\mathbf{S}} u_i(\mathbf{s})}{\max_{\mathbf{s}\in\mathbf{S}} u_i(\mathbf{s}) - \min_{\mathbf{s}\in\mathbf{S}} u_i(\mathbf{s})}, \quad \forall i \in N.$$

The new payoff functions are unit free.

The following proposition gives an estimate of what each player gains in an AE.

Define
$$\beta_i = \min_{\mathbf{s}_{-i}\in\mathbf{S}_{-i}} \max_{s_i\in S_i} u_{-i}\left(s_i, \mathbf{s}_{-i}\right), i = 1, 2, \ldots, n. \tag{5}$$

Thus β_i represents the value that player i can guarantee to all the other players in the counter coalition $-i$, whatever strategy they select. Then we have the following proposition.

Proposition 2.3. *Let* $\beta = \min_{i\in N} \beta_i$ *and* $\mathbf{s}^* \in \mathbf{S}$ *be an AE of a game* G. *Then we have*

$$\frac{1}{n} u_N(\mathbf{s}^*) \geq \frac{\beta}{n-1} \tag{6}$$

where $u_N(\mathbf{s}) = \sum_{i\in N} u_i(\mathbf{s})$.

Proof. By the definition of AE, we have

$$\sum_{i\in N} u_i(\mathbf{s}^*) = u_j(\mathbf{s}^*) + u_{-j}(\mathbf{s}^*) = u_j(\mathbf{s}^*) + \max_{s_j\in S_j} u_{-j}(s_j, \mathbf{s}^*_{-j}) \geq u_j(\mathbf{s}^*) + \beta_j.$$

Summing j over N, we obtain $nu_N(\mathbf{s}^*) \geq u_N(\mathbf{s}^*) + n\beta$, which gives (6). □

This result means that at AE, on average, each player receives a utility larger than the average that each player in the counter coalition receives in the smallest value among the guaranteed values in (5).

Remark 2.4. AE expresses a strong altruistic atmosphere among players. As pointed out above, at AE each player $i \in N$ maximizes the aggregate payoff u_{-i} of all the other players and ignores his/her payoff. This reciprocal altruistic behavior of the players expresses the mutual support aspect of AE.

Example 2.5. Consider the following three-player game, where each player has two strategies A and B; the first player selects the rows, the second player selects the columns and the third player selects the matrices. We assume $a, b \geq 2$.

	A	B
A	$(0,0,0)$	$(-1,-1,-1)$
B	$(1,-1,-1)$	$(-1,-1,-1)$

	A	B
A	$(-1,-1,-1)$	$(-1,-1,-1)$
B	$(a,-1,b)$	$(0,0,0)$

It is easy to verify that in this game, $S^N = \{BBB\}$. The corresponding game G^A is as follows

	A	B
A	$(0,0,0)$	$(-2,-2,-2)$
B	$(-2,0,0)$	$(-2,-2,-2)$

	A	B
A	$(-2,-2,-2)$	$(-2,-2,-2)$
B	$(b-1,a+b,a-1)$	$(0,0,0)$

It is also easy to see that $S^A = \{AAA, BAB\}$. This example shows that it can occur that $S^A \cap S^N = \emptyset$. In other words, no AE is an NE and no NE is an AE and the sets S^A and S^N are nonempty. ☐

As at AE players ignore maximizing their individual payoffs, one may expect that in a given game, some AEs may not be individually rational. In other words, there may be some AE, say $\mathbf{s}^*$, where some players get a payoff lower than their guaranteed payoff, that is, for at least one player $i \in N$

$$u_i(\mathbf{s}^*) < \alpha_i = \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, \mathbf{s}_{-i}),$$

where α_i is the guaranteed payoff (security level) of player i, whatever the other players do. The following example illustrates this fact.

Example 2.6. Consider the following three-player game where Rose, Colin, and Larry each have two pure strategies A and B. The left table corresponds to Larry A, and the right table corresponds to Larry B. In both tables, Rose is the row player and Colin is the column player.

	A	B
A	(3,3,3)	(2,4,2)
B	(4,2,2)	(1,1,5)

	A	B
A	(2,2,4)	(5,1,1)
B	(1,5,1)	(0,0,0)

The corresponding G^A game is as follows

	A	B
A	(6,6,6)	(6,4,6)
B	(4,6,6)	(6,6,2)

	A	B
A	(6,6,4)	(2,6,6)
B	(6,2,6)	(0,0,0)

It is not difficult to see that $S^A = \{AAA, ABB, BAB, BBA\}$. And the security levels of the three players are $\alpha_1 = \alpha_2 = \alpha_3 = 2$. Notice that at the AE ABB, the payoff of Larry satisfies $u_3(ABB) = 1 < \alpha_3 = 2$. This means that ABB is not individually rational. Similarly, the AEs BAB and BBA are not individually rational. However, AAA is an individually rational AE. Thus, when AE is not individually rational, all the players or some of them need to be willing to support the other players or be altruistic to a point where they are willing to give up some of the gains that they can guarantee for the sake of the others. This may happen in social interactions where the players have a common goal or objective. For instance, in war situation or when a group of companies produce the same product face a new entrant that threatens their market shares. Non individually rational AEs may be rejected if all players are willing to support each other, but care about their own interests at the same time. □

To overcome this drawback of AE and increase its attractiveness, we include the individual rationality in the definition of AE, which produces the following equilibrium.

Definition 2.7. A strategy profile $\mathbf{s}^* \in S$ is called a *Mutual Support Equilibrium* (MSE) of the game G if it satisfies the following two properties.

(i) Mutual support among players:

$$u_{-i}(s_i, \mathbf{s}^*_{-i}) \leq u_{-i}(\mathbf{s}^*), \text{ for all } s_i \in S_i \text{ and } i \in N. \tag{7}$$

(ii) Individual rationality:

$$\alpha_i = max_{s_i \in S_i} min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) \leq u_i(\mathbf{s}^*),\ i \in N.$$

The MSE reflects mutual support among players as each player maximizes the payoff of all the other players via the maximization of the sum of their payoffs, and at the same time it guarantees for each player a payoff that is at least equal to his/her security level. The condition (i) of Definition 2.7 means also that the MSE $\mathbf{s}^*$ is an NE of the game G^A. Considering again Example 2.6, we see that the strategy profile AAA is an MSE, but ABB, BAB and BBA are not MSEs as they satisfy the first condition of Definition 2.7 and do not satisfy its second condition. For comparison purposes, we denote by $S^M(G)$ the set of MSE of the game G. We may drop G, when there is no risk of confusion about the considered game. Next, we provide an example of a game with infinite strategy sets.

Example 2.8. Consider the following game G, where $u_1(\mathbf{s}) = s_1 + s_2 + s_3$, $u_2(\mathbf{s}) = s_1 + s_2 - s_3$, $u_3(\mathbf{s}) = s_1 - s_2 + s_3$, with $S_i = [0, 1]$, $i = 1, 2, 3$.

The payoff functions in the game G^A are $u_{-1}(\mathbf{s}) = 2s_1$, $u_{-2}(\mathbf{s}) = 2s_1 + 2s_3$, $u_{-3}(\mathbf{s}) = 2s_1 + 2s_2$.

It is easy to verify that the strategy profile $\mathbf{s} = \{(1, 1, 1)\}$ is an AE. In fact, any strategy profile of the form $\mathbf{s} = (1, s_2, s_3)$ with $s_2, s_3 \in [0, 1]$ is an AE. It is also easy to verify that $\mathbf{s} = (1, 1, 1)$ is also an NE of the game G. Therefore, $\mathbf{s} = (1, 1, 1)$ is also individually rational. Thus, $\mathbf{s} = (1, 1, 1)$ is an MSE of the game. □

To illustrate the fact that MSE and AE have an application potential in real-world interactions, we show that the following cooperation practice borrowed from Sugden [33] is actually an MSE. Among British road-users, there is a well-established custom by which, at busy times, drivers on main roads intermittently allow drivers from minor roads to enter the flow of traffic ahead of them. Because most British road junctions are regulated by fixed 'give way' signs rather than by traffic lights or roundabouts, the road network would increase in size if there were not at least a sizeable minority of drivers willing to behave in this way. But there is no legal requirement for this behaviour, and no firm rules about when a main-road vehicle should give way to a minor-road one. Considering the practice as a whole, and comparing it with the default situation in which main-road drivers always insist on their legal rights to priority, it seems reasonable to suppose that it works to the general benefit of all drivers. Since most people use both major and minor roads, and since each act of giving way induces only incremental costs and benefits, it seems equally reasonable to judge that the distribution of benefits is reasonably fair. And each driver's experience of the roads tells him/her that the practice is generally followed. One may reasonably say, if an individual is sufficiently motivated by mutual support, this combination of circumstances will induce him/her to participate in the practice. When he/she gives way to another driver, he/she need not think of this as a gratuitous act of kindness. He/she can think of himself as playing a fair part in a practice that works to everyone's benefit, including his own.

Let us now show that this real-world cooperation practice is actually an MSE. We formulate the situation as a game, where the set of players is the set of drivers, each driver i has the strategy set $S_i = \{0, 1\}$, 1 means observing the practice and 0 means not observing the practice; the payoff function of the player i is

$$u_i(\mathbf{s}) = (\textstyle\sum_{j \neq i} s_j)/(n-1).$$

A strategy profile $\mathbf{s} \in \mathbf{S}$ represents the status of observance of the cooperation practice among players and the value $u_i(\mathbf{s})$ represents the probability that driver i will be given way to the main road if he/she is a minor-road driver when the status of cooperation is $\mathbf{s}$. Let us show that the strategy profile $\mathbf{s}^* = (1, 1, ..., 1)$, where all the drivers observe the practice, as it happen actually, is an MSE. We have $u_{-i}(\mathbf{s}) = \sum_{j \neq i} u_j(\mathbf{s})$, $i = 1, 2, ..., n$ and

$$u_{-i}(\mathbf{s}^*) = \sum_{j \neq i} u_j(\mathbf{s}^*) = (n-1)(n-1)/(n-1) = n-1.$$

Clearly, for all $\mathbf{s} \in \mathbf{S}$ and $i \in N$, we have

$$u_{-i}(s_i, \mathbf{s}^*_{-i}) = (n-1)(n-2)/(n-1) = (n-2) \leq u_{-i}(\mathbf{s}^*) = \sum_{j \neq i} u_j(\mathbf{s}^*) = n-1,$$

if $s_i = 0$ and $u_{-i}(s_i, \mathbf{s}^*_{-i}) = (n-1) = u_{-i}(\mathbf{s}^*)$, if $s_i = 1$, which means that $u_{-i}(s_i, \mathbf{s}^*_{-i}) \leq u_{-i}(\mathbf{s}^*)$. Therefore, $\mathbf{s}^*$ is an NE of the game G^A. It remains to show that $\mathbf{s}^*$ satisfies the individual rationality condition with respect to the game G. It can be easily shown that for all $i \in N$, $\alpha_i = 0 \leq u_i(\mathbf{s}^*) = n-1$.

Other real-world situations where MSE and AE can be adequate are the situations described by the famous prisoner dilemma game (PDG). PDG has numerous real-world applications in business and economics for it provides a framework for understanding and analysing situations involving cooperation and competition for strategic decision-making. In the introduction, we have mentioned that in the last two decades, experimental research has shown that in game phenomena, players do not only show competitive (Nash behavior) but cooperative behavior as well. Specifically, meta analysis of PDG experimental results show that, on average, in 50 % of games, the players choose the cooperative strategy profile (see [7], [23] and [28]). Next, we show that MSE captures this very cooperative strategy profile in PDG. The following example [31, P74] is the general two-player prisoner dilemma game.

Example 2.9. Each player has two pure strategies C (cooperate) and D (defect), and the payoffs satisfy $T > R > U > S$.

	C	D
C	(R, R)	(S, T)
D	(T, S)	(U, U)

The domination argument gives $S^N = \{DD\}$ and $S^A = S^M = \{CC\}$. The Nash equilibrium DD is Pareto inferior to the mutual support equilibrium CC, which is the cooperative profile of the game. □

Further, consider the following three-player dilemma game [31, P139].

Example 2.10. In this three-player game, Rose, Colin, and Larry each have two pure strategies C and D. The left table corresponds to Larry C, and the right table corresponds to Larry D. In both tables, Rose is the row player and Colin is the column player.

	C	D
C	$(1, 1, 1)$	$(0, 3, 0)$
D	$(3, 0, 0)$	$(2, 2, -2)$

	C	D
C	$(0, 0, 3)$	$(-2, 2, 2)$
D	$(2, -2, 2)$	$(-1, -1, -1)$

The corresponding G^A game is

	C	D
C	$(2, 2, 2)$	$(3, 0, 3)$
D	$(0, 3, 3)$	$(0, 0, 4)$

	C	D
C	$(3, 3, 0)$	$(4, 0, 0)$
D	$(0, 4, 0)$	$(-2, -2, -2)$

The domination argument gives $S^N = \{DDD\}$, $S^A = S^M = \{CCC\}$. Again the mutual support equilibrium CCC is the cooperative profile of this game. □

The question arises: under which conditions does MSE exist? We address this problem in the next section.

3. Existence of AE and MSE

As an AE is an NE of a transformed game G^A, the problems of its existence and computation have the same level of difficulty as NE's. However, the problem of existence of MSE has an additional difficulty, the individual rationality condition with respect to the initial game G. In this section, we provide sufficient existence conditions for both AE and MSE. The following theorem is a straightforward application of Nash's existence theorem [13, 16, 12] to the game G^A.

Theorem 3.1. *Assume that the following conditions hold.*

(i) *All strategy sets* S_i, $i \in N$ *are compact and convex in a Hausdorff topological vector space.*

(ii) *The function* $s_i \mapsto u_{-i}(s_i, \mathbf{s}_{-i})$ *is quasi-concave for each* $i \in N$.

(iii) *The function* $\mathbf{s} \mapsto u_{-i}(\mathbf{s})$ *is lower semi-continuous for each* $i \in N$.

Then the game G *has an AE.*

Next, we turn to the problem of existence of MSE, which is not straightforward because of the individual rationality condition. First we consider the set of individually rational strategy profiles

$$A = \{\mathbf{s} \mid \alpha_i = \max_{s_i \in S_i} \min_{s_{-i} \in \mathbf{S}_{-i}} u_i(s_i, s_{-i}) \leq u_i(\mathbf{s}),\ \mathbf{s} \in \mathbf{S},\ i = 1, 2, ..., n\}. \tag{8}$$

Note that the set A is nonempty when the strategy sets are compact and the payoff functions are continuous.

Theorem 3.2. *Assume that the following conditions hold.*

(i) *All strategy sets* $S_i, i \in N$ *are compact and convex in a Hausdorff topological vector space* X.

(ii) *The functions* $s_i \mapsto u_i(s_i, \mathbf{s}_{-i})$ *and* $s_i \mapsto u_{-i}(s_i, \mathbf{s}_{-i})$ *are quasiconcave for each* $i \in N$.

(iii) *The function* $\mathbf{s} \mapsto u_i(\mathbf{s})$ *is continuous for each* $i \in N$.

(iv) $\forall(\mathbf{s}, \mathbf{t}) \in A \times \mathbf{S}$, $\exists \mathbf{z} \in A$ *such that* $\sum_{i \in N} u_{-i}(t_i, \mathbf{s}_{-i}) \leq \sum_{i \in N} u_{-i}(z_i, \mathbf{s}_{-i})$. *That is,* $\forall \mathbf{s} \in A$, $\max_{\mathbf{t} \in \mathbf{S}} \sum_{i \in N} u_{-i}(t_i, \mathbf{s}_{-i}) = \max_{\mathbf{z} \in A} \sum_{i \in N} u_{-i}(z_i, \mathbf{s}_{-i})$.

Then the game G *has an MSE.*

A game interpretation of condition (iv) of Theorem 3.2 can be given by considering the zero-sum two-person game $< \{I, II\}, \phi(\mathbf{t}, \mathbf{s}), \mathbf{S} \times A >$, where $\phi(\mathbf{t}, \mathbf{s}) = \sum_{i \in N} u_{-i}(t_i, \mathbf{s}_{-i})$, player I is the set (coalition) N of all players of the game G, he/she plays $\mathbf{t}$; player II is an artificial player, he/she plays $\mathbf{s}$. Then, condition (iv) means that given a strategy profile $\mathbf{s} \in A$ of the artificial player, there is a "collective" best response strategy profile $\mathbf{z} \in \mathbf{S}$ of all the players N, which is individually rational for each of them, that is, $\mathbf{z} \in A$. This observation makes sense as non-individually rational strategies would not be selected as best responses by a rational player.

Proof. First, we introduce the real-valued functions

$$T(\mathbf{s}, \mathbf{t}) = \sum_{i \in N} \{u_{-i}(t_i, \mathbf{s}_{-i}) - u_{-i}(\mathbf{s})\},\ (\mathbf{s}, \mathbf{t}) \in A \times \mathbf{S} \tag{9}$$

and

$$H(\mathbf{s}, \mathbf{z}) = \sum_{i \in N} \{u_{-i}(z_i, \mathbf{s}_{-i}) - u_{-i}(\mathbf{s})\},\ (\mathbf{s}, \mathbf{z}) \in A \times A. \tag{10}$$

Note that the functions $T(\mathbf{s},\mathbf{t})$ and $H(\mathbf{s},\mathbf{z})$ are similar but differ in their domains. From condition (iv) of Theorem 3.2,

$$\forall(\mathbf{s},\mathbf{t}) \in A \times \mathbf{S},\ \ \exists\mathbf{z} \in A,\ \ \textit{such that}\ \ T(\mathbf{s},\mathbf{t}) \le H(\mathbf{s},\mathbf{z}). \tag{11}$$

Let us now prove that the function $H(\mathbf{s}, z)$ satisfies all the conditions of Fan's minimax inequality [14].

(a) The set A is nonempty. Let $i \in N$ and $\bar{s}_i \in S_i$ be such that

$$\max_{s_i \in S_i} \min_{s_{-i} \in \mathbf{S}_{-i}} u_i(s_i, \mathbf{s}_{-i}) = \min_{s_{-i} \in \mathbf{S}_{-i}} u_i(\bar{s}_i, \mathbf{s}_{-i}).$$

Then the strategy profile $\bar{\mathbf{s}} = (\bar{s_1}, \bar{s_2}, ..., \bar{s_n})$ belongs to A.

(b) The set A is compact and convex. For compactness we first note that each set $A_i = \{\mathbf{s} \,|\, u_i(\mathbf{s}) \ge \alpha_i, \mathbf{s} \in \mathbf{S}\}$ is closed since the set $[\alpha_i, \infty)$ is closed and $u_i(\mathbf{s})$ is continuous. Thus $A = \cap_{i \in N} A_i$ is a closed subset of $\mathbf{S}$. We also note that $\mathbf{S} = \prod_{i \in N} \mathbf{S}_i$ is compact since each $\mathbf{S}_i$ is compact. Let I be an index set and $\mathcal{F} = \{V_\alpha : \alpha \in I\}$ be an arbitrary open cover for A. Since $\mathcal{F}$ together with $\mathbf{S} \setminus A$ is an open cover for $\mathbf{S}$, $\mathbf{S}$ can be covered by a finite number of sets, say, $V_1, ..., V_k$ from $\mathcal{F}$ together with possibly $\mathbf{S} \setminus A$. Since $A \subset \mathbf{S}$, $V_1, ..., V_k$ cover A, and it follows that A is compact.

Next, we prove that A is convex. Let $\mathbf{s}, \mathbf{s}' \in A$ and $\lambda \in [0, 1]$, we need to prove that $\lambda\mathbf{s} + (1-\lambda)\mathbf{s}' \in A$. As $u_i, i = 1, ..., n$ are quasiconcave, we deduce that $\alpha_i \le \min\{u_i(\mathbf{s}), u_i(\mathbf{s}')\} \le u_i(\lambda\mathbf{s} + (1-\lambda)\mathbf{s}')$ for all $i = 1, ..., n$, which means that A is convex.

(c) By construction, $H(\mathbf{s},\mathbf{s}) = 0,\ \forall\mathbf{s} \in A$.

(d) From condition (ii) of Theorem 3.2, we deduce that the function $H(\mathbf{s},\mathbf{z})$ is quasi-concave in $\mathbf{z}$.

(e) Condition (iii) of Theorem 3.2 implies that the function $H(\mathbf{s},\mathbf{z})$ is lower semicontinuous in $\mathbf{s}$.

Thus, all the condition of Fan maxmin inequality are satisfied, then there exists $\bar{\mathbf{s}} \in A$ such that $H(\bar{\mathbf{s}},\mathbf{z}) \le 0,\ \forall\mathbf{z} \in A$. Using condition (iv) and the inequality (11), we obtain $\forall\mathbf{t} \in \mathbf{S}$, $\exists\mathbf{z} \in A$, $T(\bar{\mathbf{s}}, t) \le H(\bar{\mathbf{s}},\mathbf{z}) \le 0$. Then, $\forall\mathbf{t} \in \mathbf{S}$, $T(\bar{\mathbf{s}},\mathbf{t}) \le 0$.

Hence
$$T(\bar{\mathbf{s}},\mathbf{t}) = \sum_{i \in N}\{u_{-i}(t_i, \bar{\mathbf{s}}_{-i}) - u_{-i}(\bar{\mathbf{s}})\} \le 0,\ \ \forall\mathbf{t} \in \mathbf{S}. \tag{12}$$

Next, we prove that $\bar{\mathbf{s}}$ is an MSE. Let $j \in N$ be any player. Let $t_j \in S_j$ be arbitrary, and for each player $i \ne j$ set $t_i = \bar{\mathbf{s}}_i$ in (12). Then all the terms in the sum of the inequality (12) vanish, but the jth term, that is, (12) becomes $u_{-j}(t_j, \bar{\mathbf{s}}_{-j}) - u_{-j}(\bar{\mathbf{s}}) \le 0,\ \forall t_j \in S_j$. This means $u_{-j}(t_j, \bar{\mathbf{s}}_{-j}) \le u_{-j}(\bar{\mathbf{s}})$. Since the player j is chosen arbitrarily, we have $u_{-j}(t_j, \bar{\mathbf{s}}_{-j}) \le u_{-j}(\bar{\mathbf{s}}),\ \forall t_j \in S_j,\ \forall j \in N$. This means that $\bar{\mathbf{s}}$ satisfies condition (i) of Definition 2.7. As $\bar{\mathbf{s}} \in A$, it also satisfies condition (ii) of Definition 2.7, thus, it is an MSE. □

Example 3.3. Assume that the game G is a three-person game where $N = \{1, 2, 3\}$, $S_1 = S_2 = S_3 = [0, 1]$. The payoff functions are

$$u_1(\mathbf{s}) = s_1 + s_2 + 2s_3, \quad u_2(\mathbf{s}) = -s_1 + s_2 - s_3, \quad u_3(\mathbf{s}) = s_1 - s_2 + s_3.$$

As the payoff functions of players are linear and [0,1] is compact and convex, the conditions (i)–(iii) of Theorem 3.2 are satisfied. Let us verify that condition (iv) is satisfied. The security levels of the players 1, 2 and 3 are $\alpha_1 = 1, \alpha_2 = -1$, and $\alpha_3 = 0$, respectively. Since

$$u_{-1}(t_1, \mathbf{s}_{-1}) = 0, \quad u_{-2}(t_2, \mathbf{s}_{-2}) = 2s_1 + 3s_3, \quad u_{-3}(t_3, \mathbf{s}_{-3}) = 2s_2 + t_3.$$

we have
$$\phi(\mathbf{t}, \mathbf{s}) = \sum_{i \in N} u_{-i}(t_i, \mathbf{s}_{-i}) = 2s_1 + 2s_2 + 3s_3 + t_3.$$

Further, $\forall \mathbf{s} \in \mathbf{S}$, $\max_{\mathbf{t} \in \mathbf{S}} \phi(\mathbf{t}, \mathbf{s}) = 2s_1 + 2s_2 + 3s_3 + 1$, reached at $\mathbf{t} = (t_1, t_2, 1)$, $\forall\, t_1, t_2 \in [0, 1]$. Consider the particular solution $\mathbf{z}^0 = (1, 1, 1)$ of this optimization problem. Hence, we have $\forall \mathbf{s} \in \mathbf{S}$, $\max_{\mathbf{t} \in \mathbf{S}} \phi(\mathbf{t}, \mathbf{s}) \leq \phi(\mathbf{t}, \mathbf{z}^0)$, with $\mathbf{z}^0 \in A$. This means that condition (iv) of Theorem 3.2 is satisfied. Thus, the game G has an MSE. Indeed, $\mathbf{z}^0 = (1, 1, 1)$ is an MSE. First, it is individually rational as it satisfies

$$u_1(\mathbf{z}^0) = 4 > \alpha_1 = 1, \quad u_2(\mathbf{z}^0) = -1 \geq \alpha_2 = -1, \quad u_3(\mathbf{z}^0) = 1 > \alpha_3 = 0.$$

Next, it is an NE of the game G^A as it satisfies

$$\begin{aligned} &u_{-1}(t_1, 1, 1) = 0 \leq u_{-1}(1, 1, 1) = 0, \ \forall t_1 \in [0.1], \\ &u_{-2}(1, t_2, 1) = 5 \leq u_{-2}(1, 1, 1) = 5, \ \forall t_2 \in [0.1], \\ &u_{-3}(1, 1, t_3) = 2 + t_3 \leq u_{-3}(1, 1, 1) = 3, \ \forall t_3 \in [0.1]. \end{aligned}$$

Further, one can easily verify that the strategy profile $\mathbf{z}^1 = (1/2, 1/2, 1)$ is another MSE of the game. Indeed, $u_1(\mathbf{z}^1) = 3 > \alpha_1 = 1$, $u_2(\mathbf{z}^1) = -1 \geq \alpha_2 = -1$, $u_3(\mathbf{z}^1) = 1 > \alpha_3 = 0$ and

$$\begin{aligned} u_{-1}(t_1, 1/2, 1) = 0 &\leq u_{-1}(1/2, 1/2, 1) = 0, \ \forall t_1 \in [0.1], \\ u_{-2}(1/2, t_2, 1) = 4 &\leq u_{-2}(1/2, 1/2, 1) = 4, \ \forall t_2 \in [0.1], \\ u_{-3}(1/2, 1/2, t_3) = 1 + t_3 &\leq u_{-3}(1, 1, 1) = 2, \ \forall t_3 \in [0.1]. \end{aligned}$$

As an NE automatically satisfies the individual rationality condition, another approach to the existence of MSE is to find sufficient conditions for the existence of an AE that is also an NE. This approach has two interesting features. The first is that it provides an equilibrium that is self-enforcing (being an NE) and expressing mutual support among players (being an MSE). Such solutions can be very suitable for social conflicts and global problems such as global warming, pollution, etc. In such problems, non self-enforcing cooperative solutions do not work as the players do not care about long term effects of their behaviour and likewise, in non-cooperative solutions, the players ignore the general interest and damage the public goods such as air and water. Thus, we arrive at the following definition.

Definition 3.4. A strategy profile that is AE and NE of the game G is called *altruistic Nash equilibrium* (ANE).

Let S^{AN} be the set of ANE of the game G. From the above discussion, an ANE is an MSE, that is, $S^{AN} \subseteq S^M$. In general $S^{AN} \neq S^M$, as NE of the game G^A need not be NE of the initial game G as the payoff functions of the two games may be considerably different because of the aggregation process. As illustrated in Example 2.6, where the profile (A, A, A) is a MSE but not a NE of the initial game G. Our next theorem establishes the existence of an ANE.

Theorem 3.5. *Assume that the following conditions hold.*

(i) *All strategy sets S_i, $i \in N$ are compact and convex in a Hausdorff topological vector space.*

(ii) *The functions $s_i \mapsto u_{-i}(s_i, \mathbf{s}_{-i})$ and $s_i \mapsto u_i(s_i, \mathbf{s}_{-i})$ are concave for each $i \in N$.*

(iii) *The function $\mathbf{s} \mapsto u_i(\mathbf{s})$ is continuous for each $i \in N$.*

(iv) *For each $\mathbf{s} \in \mathbf{S}$ and for each $i \in N$, there exists $z_i \in S_i$ such that $\max_{t_i \in S_i} u_i(t_i, \mathbf{s}_{-i}) = u_i(z_i, \mathbf{s}_{-i})$ and $\max_{t_i \in S_i} u_{-i}(t_i, \mathbf{s}_{-i}) = u_{-i}(z_i, \mathbf{s}_{-i})$.*

Then $S^A \cap S^N \neq \emptyset$, which means the game G has an ANE.

Conditions (i)–(iii) are standard conditions for the existence of Nash equilibrium. Condition (iv) has a game interpretation. It means: For a given strategy profile $\mathbf{s} \in \mathbf{S}$, each player has a common best reply strategy to $\mathbf{s}$ in both games G and G^A.

Proof. We use the Fan minimax inequality [14]. We first introduce the function $\phi(\mathbf{s}, (\mathbf{t}, \mathbf{d})) : \mathbf{S} \times (\mathbf{S} \times \mathbf{S}) \to \mathbf{R}$ defined by

$$\phi(\mathbf{s}, (\mathbf{t}, \mathbf{d})) = \sum_{i=1}^{n} \{u_{-i}(t_i, \mathbf{s}_{-i}) - u_{-i}(\mathbf{s})\} + \sum_{i=1}^{n} \{u_i(d_i, \mathbf{s}_{-i}) - u_i(\mathbf{s})\}. \tag{13}$$

Next, we establish the relationship between the function $\phi(\mathbf{s}, (\mathbf{t}, \mathbf{d}))$ and MSE of the game G. Assume that there exists $\mathbf{s}^* \in \mathbf{S}$ such that

$$\phi(\mathbf{s}^*, (\mathbf{t}, \mathbf{d}) \leq 0, \text{for all} \, (\mathbf{t}, \mathbf{d}) \in \mathbf{S} \times \mathbf{S}. \tag{14}$$

Then $\mathbf{s}^* \in S^A \cap S^N$. Indeed, consider any player $j \in N$. Assume $\mathbf{d} = \mathbf{s}$, $t_i = s_i$ for all $i \in -j$ and leave t_j free in $\mathbf{S}_j$ in the equation (14), then (14) becomes $u_{-j}(t_j, \mathbf{s}^*_{-j}) - u_{-j}(\mathbf{s}^*) \leq 0$, for all $t_j \in \mathbf{S}_j$, that is, $u_{-j}(t_j, \mathbf{s}^*_{-j}) \leq u_{-j}(\mathbf{s}^*)$, for all $t_j \in \mathbf{S}_j$. As j is arbitrarily chosen, $\mathbf{s}^*$ is an NE of the game G^A. Similarly, we now prove that $\mathbf{s}^* \in \mathbf{S}$ is an NE of the game G. Indeed, let $j \in N$ be any player in the game G. Setting $\mathbf{t} = \mathbf{s}$, $d_i = s_i$ for all $i \in -j$ and leaving d_j free in (14), then (14) becomes $u_j(d_j, \mathbf{s}^*_{-j}) - u_j(\mathbf{s}^*) \leq 0$, for all $d_j \in \mathbf{S}_j$, that is, $u_j(d_j, \mathbf{s}^*_{-j}) \leq u_j(\mathbf{s}^*)$, for all $d_j \in \mathbf{S}_j$. As j is arbitrarily chosen, $\mathbf{s}^*$ is an NE of the game G. Now we prove that there exists at least one strategy profile $\mathbf{s}^* \in \mathbf{S}$ that satisfies the inequality in (14). Using condition (iv) of Theorem 3.5, we obtain that for all $\mathbf{s}, \mathbf{t}$ and $\mathbf{d}$ in $\mathbf{S}$, and $i \in N$, there exists $\mathbf{v} \in \mathbf{S}$ such that

$$u_{-i}(t_i, \mathbf{s}_{-i}) \leq u_{-i}(v_i, \mathbf{s}_{-i}), \text{ and } u_i(d_i, \mathbf{s}_{-i}) \leq u_i(v_i, \mathbf{s}_{-i}).$$

Thus for all $\mathbf{s}, \mathbf{t}$ and $\mathbf{d}$ in $\mathbf{S}$, there exists $\mathbf{v} \in \mathbf{S}$ such that

$$\begin{aligned} \phi(\mathbf{s},(\mathbf{t},\mathbf{d})) &= \sum_{i=1}^{n} \left(u_{-i}(t_i, \mathbf{s}_{-i}) - u_{-i}(\mathbf{s})\right) + \sum_{i=1}^{n} \left(u_i(d_i, \mathbf{s}_{-i}) - u_i(\mathbf{s})\right) \qquad (15) \\ &\le \sum_{i=1}^{n} \left(u_{-i}(v_i, \mathbf{s}_{-i}) - u_{-i}(\mathbf{s})\right) + \sum_{i=1}^{n} \left(u_i(v_i, \mathbf{s}_{-i}) - u_i(\mathbf{s})\right). \end{aligned}$$

Next, consider the function $\psi(.) : \mathbf{S} \times \mathbf{S} \to \mathbf{R}$, defined by

$$\psi(\mathbf{s},\mathbf{v}) = \sum_{i=1}^{n} \left(u_{-i}(v_i, \mathbf{s}_{-i}) - u_{-i}(\mathbf{s})\right) + \sum_{i=1}^{n} \left(u_i(v_i, \mathbf{s}_{-i}) - u_i(\mathbf{s})\right). \qquad (16)$$

Then from (15) and (16), we deduce that for all $\mathbf{s}, \mathbf{t}$ and $\mathbf{d}$ in $\mathbf{S}$, there exists $\mathbf{v} \in \mathbf{S}$ such that

$$\phi(\mathbf{s},(\mathbf{t},\mathbf{d})) \le \psi(\mathbf{s},\mathbf{v}) \qquad (17)$$

It is easy to verify that, thanks to conditions (ii) and (iii) of Theorem 3.5, the function $(\mathbf{s},\mathbf{v}) \mapsto \psi(\mathbf{s},\mathbf{v})$ is continuous, and the function $\mathbf{v} \mapsto \psi(\mathbf{s},\mathbf{v})$ is concave.

Then $\psi(\mathbf{s},\mathbf{v})$ satisfies all the conditions of the Fan Minimax Inequality [14], which implies that there exists $\mathbf{s}^* \in \mathbf{S}$ such that

$$\psi(\mathbf{s}^*,\mathbf{v}) \le \max_{\mathbf{z}\in\mathbf{S}} \psi(\mathbf{z},\mathbf{z}), \text{ for all } \mathbf{v} \in \mathbf{S} \qquad (18)$$

According to (16), we have $\psi(\mathbf{z},\mathbf{z}) = 0$ for all $\mathbf{z} \in \mathbf{S}$. Then (18) implies

$$\psi(\mathbf{s}^*,\mathbf{v}) \le 0, \text{ for all } \mathbf{v} \in \mathbf{S} \qquad (19)$$

From (17) and (19), we deduce that $\phi(\mathbf{s}^*,(\mathbf{t},\mathbf{d})) \le \psi(\mathbf{s}^*,\mathbf{v}) \le 0$ for all $\mathbf{t}$ and $\mathbf{d}$ in $\mathbf{S}$. Thus, from (14) and the discussion that follows it, $\mathbf{s}^* \in S^A \cap S^N$. □

Example 3.6. Recall the game G of Example 2.8, where we showed that $S^A = \{(1, s_2, s_3) : s_2, s_3 \in [0, 1]\}$ and $S^N = \{(1, 1, 1)\}$. Hence $S^N \cap S^A = \{(1, 1, 1)\}$. It is also easy to see that the players' security levels are $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = 0$. This example shows that $S^N \cap S^A$ is a proper subset of S^M. □

Example 3.7. Recall the game G in Example 2.5, where we showed that in pure strategies $S^A = \{AAA, BAB\}$ and $S^N = \{BBB\}$. It is also easy to see that the players' security levels are $\alpha_1 = \alpha_2 = \alpha_3 = -1$. Hence $S^N \cap S^A = \emptyset$ and $S^M = S^A = \{AAA, BAB\}$. □

Example 3.8. We borrow Example 3.1 of [9], where the game G is a 3-player game with $S_i = [-1, 1]$,

$$u_1(s) = s_1 + s_2 - s_3^2, \quad u_2(s) = s_2 - s_3^2, \quad u_3(s) = s_1 + s_2^2(s_3 + 1).$$

Hence $u_{-1}(s) = s_1 + s_2 + s_2^2(s_3 + 1) - s_3^2$, $u_{-2}(s) = 2s_1 + s_2 + s_2^2(s_3 + 1) - s_3^2$, $u_{-3}(s) = s_1 + 2s_2 - 2s_3^2$. It is easy to see that $\alpha_1 = \alpha_3 = -1$, $\alpha_2 = 0$, $S^N = \{(1, 1, 1)\}$, and $S^A = \{(1, 1, 0)\}$. As $(1, 1, 0)$ is individually rational, $S^A = S^M$ and $S^A \cap S^N = \emptyset$. We note the typo $\alpha_1 = 0$ instead of -1 in [9, page 11]. □

From Theorems 3.1, 3.2 and 3.5, one can deduce the following procedures for computing AE and MSE.

Procedure 1. Computation of AE. Assume that the conditions of Theorem 3.1 are satisfied by the game G. Use any of the numerous NE computation methods available in the literature to compute NE of the game G^A introduced in Section 2.

Procedure 2. Computation of ANE. Assume the game G satisfies the conditions of Theorem 3.5.

Step 1. Compute Nash equilibria of the game G.

Step 2. Compute Nash equilibria of the game G^A, introduced in Section 2.

Step 3. Find common Nash equilibria to the two games, G and G^A.

Procedure 3. Computation of MSE. Assume the game G satisfies the conditions of Theorem 3.2.

Step 1. Compute the security levels $\alpha_i, i = 1, ..., n$ of all the players with respect to the initial game G to determine the set A introduced in (8).

Step 2. Compute $\min_{\mathbf{s}\in A} \max_{\mathbf{t}\in \mathbf{S}} \sum_{i\in N}\{u_{-i}(t_i, \mathbf{s}_{-i}) - u_{-i}(\mathbf{s})\}$.
Let $(\mathbf{s}^*, \mathbf{z}^*) \in A \times \mathbf{S}$ be a solution of this optimization problem, then the strategy profile $\mathbf{s}^*$, is an MSE. Indeed, from (12), we deduce that

$$\min_{\mathbf{s}\in A} \max_{\mathbf{t}\in A} \sum_{i\in N}\{u_{-i}(t_i, \mathbf{s}_{-i}) - u_{-1}(\mathbf{s})\} \leq 0$$

then $\max_{\mathbf{t}\in \mathbf{S}} \sum_{i\in N}\{u_{-i}(t_i, \mathbf{s}^*_{-i}) - u_{-1}(\mathbf{s}^*)\} \leq 0$. Then, as in the proof of Theorem 3.2, we deduce that $\mathbf{s}^*$ is an MSE.

4. Related concepts

In this section, we compare AE, MSE and ANE to the most known cooperative solution concepts of normal form games that have similar properties. The comparison will be based on the interaction between players, existence and computational aspects. We first compare our concepts to Berge equilibrium and related concepts that reflect mutual support and altruism, second we compare them to the bargaining solution and finally the traditional cooperative concepts as the Aumann's strong equilibrium, the α-core and the β-core.

4.1. Berge equilibrium and related concepts

The closest concepts of equilibrium to AE, MSE and ANE are Berge equilibrium [36], Vaisman Berge equilibrium [34] and Berge Nash equilibrium [1], respectively. Recently, a generalization of Berge equilibrium, the Unilateral Support equilibrium has been introduced [29]. Further, we will compare AE to the most general equilibrium that involves other regarding preferences introduced by Berge [5], we will call it general Berge eqquilibrium (GBE) to avoid confusion with the above mentioned Berge equilibria. Finally, we will compare MSE and ANE with the mutual advantage practice and mutual beneficial practice concepts of Sugden ([32] and [33]). We start the comparison with the first three mentioned closely related concepts to our concepts and end it by a comparison to the last four mentioned concepts.

Definition 4.1. A strategy profile $\mathbf{s}^* \in S$ is called *Berge equilibrium* (BE) if

$$u_i(\mathbf{s}_i^*, s_{-i}) \leq u_i(\mathbf{s}^*), \text{ for all } s_{-i} \in S_{-i} \text{ and } i \in N. \tag{20}$$

In terms of optimization, BE can be expressed as follows

$$\max_{s_{-i} \in S_{-i}} u_i(\mathbf{s}_i^*, s_{-i}) = u_i(\mathbf{s}^*), \text{ for all } i \in N. \tag{21}$$

At BE each player's payoff function is maximized by the group of all the other players. This is an expression of mutual support. In [17, 37, 38] it has been shown that BE reflects the well-known Golden Rule in social interaction "Behave with the others as you like the others to behave with you". It is important to note that for two-player games, BE is just a Nash equilibrium of the game obtained from the initial game by interchanging the payoff functions of the two players, namely, the game where $N = \{1, 2\}$, the strategy set and payoff function of Player 1 are $\mathbf{S}_1$ and $u_2(\mathbf{s})$, respectively; and the strategy set and payoff function of Player 2 are $\mathbf{S}_2$ and $u_1(\mathbf{s})$, respectively. Further, Van Dam [35] formalized the Golden Rule in a symmetric normal form game, which is a special case of Definition 4.1.

In [36], Vaisman has proved that BE may not satisfy the individual rationality condition, therefore, he introduced the Vaisman Berge equilibrium by adding the individual rationality condition to BE as follows.

Definition 4.2. A strategy profile $\mathbf{s}^* \in S$ is called *Vaisman Berge equilibrium* (VBE) if it satisfies the following two conditions:

(i) It is a BE, that is,

$$u_i(\mathbf{s}_i^*, s_{-i}) \leq u_i(\mathbf{s}^*), \forall s_{-i} \in S_{-i} \text{ and } i \in N. \tag{22}$$

(ii) It is individually rational, that is,
$\alpha_i = \max_{s_i \in S_i} \min_{\mathbf{s}_{-i} \in \mathbf{S}_{-i}} u_i(s_i, \mathbf{s}_{-i}) \leq u_i(\mathbf{s}^*), \forall i \in N$.

Definition 4.3. [1] A Berge equilibrium that is also Nash equilibrium is called Berge-Nash Equilibrium (BNE).

For the purpose of comparison we denote by $S^B(G)$, $S^{VB}(G)$ and $S^{BN}(G)$ the sets of BE, VBE and BNE of the game G, respectively. When there is no risk of confusion, we will often drop G from $S^B(G)$, $S^{VB}(G)$ and $S^{BN}(G)$. From Definitions 2.2 and 2.7, it is easy to see that for any two-player game, BE is equivalent to AE, and hence $S^B(G)$ can be obtained as a set of NE by interchanging the payoff functions of the players. The following proposition shows the relationships between the elements of each of the three pairs: AE and BE, MSE and VBE, and ANE and BNE.

Proposition 4.4. *A BE is an AE, a VBE is an MSE and a BNE is an ANE. That is,* $S^B(G) \subseteq S^A(G)$, $S^{VB}(G) \subseteq S^M(G)$ *and* $S^{BN}(G) \subseteq S^{AN}(G)$.

Proof. We prove that a BE is an AE. The second and third parts of the proposition can be proved similarly. Let $\mathbf{s}^* \in S$ be a BE of the game G. Consider a player $i \in N$ and let $j \in -i$.

According to the definition of BE, we have $u_j(s_j^*, s_i, \mathbf{s}^*_{-\{i,j\}}) \leq u_j(\mathbf{s}^*)$, for all $s_i \in S_i$, where $(s_j^*, s_i, \mathbf{s}^*_{-\{i,j\}})$ is another way of writing the profile $(s_i, \mathbf{s}^*_{-i})$, where s_j^* is highlighted as the strategy selected by the player j.

Therefore, $u_j(s_i, \mathbf{s}^*_{-i}) \leq u_j(\mathbf{s}^*)$, for all $s_i \in S_i$ and for all $j \in -i$. Thus, $\sum_{j \in -i} u_j(s_i, \mathbf{s}^*_{-i}) \leq \sum_{j \in -i} u_j(\mathbf{s}^*)$, for all $s_i \in S_i$. This means that $\mathbf{s}^*$ is an AE of the game G. □

Remark 4.5. Proposition 4.4 shows that AE, MSE and ANE are generalizations of BE, VBE and BNE, respectively. Therefore, from the point of view of players' interaction, the former enjoy all the properties of the latter, including altruism, mutual support and the Golden Rule. The only difference is that at AE, MSE and ANE each player $i \in N$ maximizes only one function, namely, the aggregate payoff u_{-i} of the other players, instead of maximizing the payoff function of each of the other players individually. It is important to note that, in general, this generalization is strict in the sense that $S^B(G) \neq S^A(G)$, $S^{VB}(G) \neq S^M(G)$ and $S^{BN}(G) \neq S^{AN}(G)$. The following example shows that $S^B(G) \neq S^A(G)$ and $S^{VB}(G) \neq S^M(G)$. □

Example 4.6. Recall the game G in Example 2.5, where we had

$$S^A = \{AAA, BAB\}, \quad S^N = \{BBB\}.$$

It is easy to see that $S^B = \{AAA\}$. Clearly, $S^B \subset S^A$ and $S^B \neq S^A$. This example shows that it can occur that $S^B \neq S^A \neq S^N$ and all of the three sets are nonempty.

Example 4.7. Let G be the game in Example 2.8. We showed that $\mathbf{s} = (1,1,1)$ is both AE and NE, and hence it is MSE and ANE. On the other hand, in [26], it was shown that this game has no BE. And consequently, no BVE and no BNE.

This example shows that BE, BVE and BNE may not exist in the basic and simple class of games with linear payoff functions and compact intervals as strategy sets, while AE exists in such games as they are Nash equilibria of games with linear payoff functions obtained from the initial game with the same strategy sets.

From the point of view of existence, as AE, MSE and ANE are generalizations of BE, VBE and BNE, respectively, any existence theorem of the latter is an existence theorem of the former. Moreover, the problem of existence of AE, MSE and ANE is more tractable than the one of existence of BE, VBE and BNE. Indeed, AE is an NE, hence all Nash equilibrium existence results available in literature can be adapted to provide existence conditions of AE in almost all types of games with the most general conditions. The existence of NE is guaranteed by common (standard) conditions such as compactness and convexity of strategy sets and quasiconcavity and continuity of payoff functions (or weaker versions of these conditions). In contrast, almost all existing in literature works on existence conditions of BE involve an extremal condition in addition to the mentioned common conditions [22]. As a result, BE, BVE and BNE may not exist even in games with linear payoff functions and compact intervals as strategy sets.

Example 4.7 is an illustration. The difficulty of establishing the existence of BE has been highlighted in [20] and [26]. Comparing existence theorems of MSE and BVE, the condition (iv) of Theorem 3.2 is easier to satisfy and check than the g-diagonal transfer quasiconcavity condition of Theorem 3.1 in [26].

As for the computational aspect, AE seems to be easier to compute than BE. Indeed, according to (4) and (21), at an AE, say $\mathbf{s}^*$, each player $i \in N$ selects a strategy that maximizes only one function, namely, $u_{-i}(s_i, \mathbf{s}^*_{-i})$ with respect to s_i, while at BE, each player $i \in N$ selects a strategy that simultaneously maximizes $n - 1$ functions, namely, $u_j(\mathbf{s}^*_j, s_i, \mathbf{s}^*_{-\{i,j\}})$ with respect to s_i for all $j \in -i$. Moreover, as AE is an NE of a transformed game, AE can be computed using any of the existing methods of finding Nash equilibria of different types of games.

Furthermore, as mentioned above, from Definitions 2.2 and 2.4, in two-player games, BE is equivalent to AE and $S^B(G)$ can be obtained as NEs of the game obtained by interchanging the payoff functions of the players. However, Pottier and Nessah [27] showed that the problem of existence of BE cannot be reduced to the problem of existence of Nash equilibrium of a game that is obtained from the initial game by interchanging the players' payoff functions when the number of players $n \geq 3$.

However, there are two results that relate BE to NE. Let $\mathcal{D}_n$ denote the set of permutations of N that have no fixed element, that is

$$\mathcal{D}_n = \{\sigma | \sigma \text{ permutation of } N \text{ such that } \sigma(i) \neq i,\ \forall i \in N\}.$$

Such permutations are called *derangements* and it is well known that

$$|\mathcal{D}_n| = n! \sum_{p=0}^{n} \frac{(-1)^p}{p!}.$$

Let G_σ be the game obtained from the initial game G such that player i's new utility function is $u_{\sigma(i)}$. The following proposition, first presented by Krim [19] and appeared later in [8], shows that a BE of the game G is an NE of the game G_σ for every $\sigma \in \mathcal{D}_n$.

Proposition 4.8. *We have* $S^B(G) \subseteq \bigcap_{\sigma \in \mathcal{D}_n} S^N(G_\sigma)$.

Since $|\mathcal{D}_n| \sim n!/e$ is a very fast growing function in n, the above proposition does not give an efficient algorithm for BE existence and computation when N is large. A more useful characterization of BE was given by Krim [19] and later appeared in [8], which relates $S^B(G)$ to the common NEs of a collection of n two-player games G_i, $i \in N$ obtained from the initial game. Before stating this result, we need to introduce the collection of two-player games G_i, $i \in N$. In the game G_i, S_i and $\mathbf{S}_{-i}$ are respectively the strategy sets of players 1 and 2, and u_{-i} and u_i are respectively the payoff functions of players 1 and 2.

Proposition 4.9. *We have* $S^B(G) = \bigcap_{i \in N} S^N(G_i)$.

Thus, BE is an NE of n games of type $G_i, i \in N$, while AE is an NE of a single transformed game. Propositions 4.8 and 4.9 hint that, in general, it is more difficult to deal with BE than AE from computational perspective. Any method that computes NE can be used to compute AE, while BE cannot be computed as NE of a unique game obtained by permutations of players' payoff functions when $n \geq 3$. Similar comments may be made when MSE and VBE are compared in terms of computation. For more details on the existence and computation of BE see [2, 20, 22, 26].

Thus, we have seen that AE, MSE and ANE are generalizations of BE, VBE and BNE, respectively, and enjoy the same properties as expressing altruism and mutual support and the Golden Rule. Moreover, AE and MSE are more tractable from the point of view of existence and computation. However, BE has one advantage over AE. AE is not applicable when the utility functions of players represent values that are not addable.

To address the problem of coordination that is necessary for players in $-i$ to support the player i in BE, Schouten et al. [29] introduced the Unilateral Support equilibrium, which is based on the notion of Unilateral Support strategy profile that is defined as follows. Given a permutation σ of N, then $\mathbf{s}^* \in S$ is called Unilateral Support strategy profile with respect to σ if

$$u_{\sigma(i)}(\mathbf{s}_i, \mathbf{s}^*_{-i}) \leq u_{\sigma(i)}(\mathbf{s}^*), \ \forall \mathbf{s}_i \in S_i, \, \forall i \in N.$$

That is, $\mathbf{s}^* \in S$ is a Unilateral Support strategy profile with respect to σ if $\mathbf{s}^*$ is a Nash equilibrium of the game G_σ.

Definition 4.10. A strategy profile $\mathbf{s}^* \in S$ is a *Unilateral Support Equilibrium* (USE) of the game G, if it is a Unilateral Support strategy profile for all permutations $\sigma \in \mathcal{D}_n$ (the set $\mathcal{D}_n$ is defined as above).

An equivalent definition of USE [29] is that $\mathbf{s}^* \in S$ is a USE of the game G if

$$u_i(\mathbf{s}_j, \mathbf{s}^*_{-j}) \leq u_i(\mathbf{s}^*), \, \forall \mathbf{s}_j \in S_j, \, \forall i \in N, \, \forall j \in -i. \tag{23}$$

Let us now compare AE with USE. In fact, AE is a generalization of USE. Indeed, let $\mathbf{s}^* \in S$ be a USE, then for a fixed player $j \in N$ in (23), we get

$$u_i(\mathbf{s}_j, \mathbf{s}^*_{-j}) \leq u_i(\mathbf{s}^*), \ \forall \mathbf{s}_j \in S_j, \, \forall i \in -j. \tag{24}$$

Then summing up in (24) with respect to $i \in -j$, we get

$$u_{-j}(\mathbf{s}_j, \mathbf{s}^*_{-j}) \leq u_{-j}(\mathbf{s}^*), \, \forall \mathbf{s}_j \in S_j,$$

which means that $\mathbf{s}^*$ is also an AE.

Shouten et al. [29] provide the following example of a game that does not have a USE.

Example 4.11. [29] Consider the following three-player game G.

	A	B
A	(0,1,1)	(0,0,0)
B	(0,0,0)	(1,0,0)

	A	B
A	(1,0,0)	(1,0,0)
B	(0,0,0)	(0,1,1)

To show that this game has AEs, we construct the corresponding G^A game:

	A	B
A	(2,1,1)	(0,0,0)
B	(0,0,0)	(0,1,1)

	A	B
A	(0,1,1)	(0,1,1)
B	(0,0,0)	(2,1,1)

It is not difficult to see that (A, A, A) and (B, B, B) are NEs of the game G^A, therefore, they are (by definition) AEs of the game G. Thus, this example shows that AE is a strict generalization of USE. $\square$

Further, in [29], a relation between USE and NE, that is similar to the one between BE and NE (see Proposition 4.8), is established as follows.

$$S^{US}(G) = \bigcap_{\sigma \in \mathcal{D}_n} S^N(G_\sigma) \tag{25}$$

where $S^{US}(G)$ denotes the set of USE of the game G. From (25) and Proposition 4.8, we deduce that a BE is a USE. Conversely, Crettez [10] has established necessary and sufficient conditions for a USE to be a BE. Comparing USE to AE, as the former is a common NE to $|\mathcal{D}_n|$ games of the form G_σ, from computational and existence points of view, like BE, its existence establishment and computation will be more difficult than NE. Consequently, USE will be more difficult with respect to these two aspects than AE, as the latter is an NE of one game only, namely, G^A.

Let us now compare AE to the general Berge equilibrium. We first give its definition.

Definition 4.12. A strategy profile $\mathbf{s}^* \in \mathbf{S}$ is a *general Berge equilibrium* (GBE) of the game G if

$$u_{r_m}(\mathbf{s}_{T_m}, \mathbf{s}^*_{-T_m}) \leq u_{r_m}(\mathbf{s}^*), \text{ for all } m \in M, r_m \in R_m \text{ and } \mathbf{s} \in \mathbf{S}, \tag{26}$$

for all partitions $R = \{R_m\}_{m \in M}$ of N and for all sets $T = \{T_m\}_{m \in M}$ of subsets of N, where M is a finite set.

By definition, for specific choices of R and T, BE, USE and NE are also GBE [2]. The main difference between GBE and AE and MSE is that GBE does not involve aggregation of payoffs of players, while AE and MSE are defined by aggregation of payoffs. Moreover, the sets $-i, i = 1, 2, ...n$ do not form a partition of N to be considered as the set R of the Definition 4.12 to show that AE is a GBE. Thus, generally, AE is not a GBE. Abalo and Kostreva [2] provide some sufficient conditions for the existence of GBE. In [20], it has been pointed out that Abalo and Kostreva conditions were actually not sufficient for the existence of GBE. Moreover, as the definition of GBE involves any partition R and any set T, one can expect that the existence of GBE is difficult to establish. Moreover, as GBE is a BE, all the above comments made on BE existence and computation are valid for GBE. Therefore, AE and MSE are more tractable from the point of view of these two aspects.

Another concept that is related to MSE is the Sugden's concept of Mutual Advantage profile (MAP)[33] that reflect the motivation to participate in a fair cooperation. MAP is defined as follows.

Definition 4.13. Assume that in the game G there is a strategy profile $\mathbf{s}^0 \in \mathbf{S}$ that is a non-cooperative default. A strategy profile $\mathbf{s}^* \in \mathbf{S}$ is called *Mutual Advantage Profile* (MAP) if it satisfies the following conditions:

(i) Mutual Gain: If $\mathbf{s}^*$ is played rather than $\mathbf{s}^0$, every individual benefits.

(ii) Fairness: The benefits gained by playing $\mathbf{s}^*$ rather than $\mathbf{s}^0$ are distributed fairly between all the players.

(iii) Assurance: Each player i has reason to expect that every other player j will play his strategy s_j^* of the joint action $\mathbf{s}^*$. ☐

MSE and ANE share the principle of mutual gain in condition (i) of MAP but without the non-cooperative default profile $\mathbf{s}^0$. Condition (iii) of MAP is implicit in MSE. Indeed, for the MSE to materialize, it is necessary that each player is motivated to play his/her MSE strategy. ANE does not need this assumption as it is an NE, that is, a self-enforcing equilibrium. MSE and ANE do not share the condition (ii) of MAP. Moreover, MAP is not a completely defined concept, as Sugden mentions, it is part of an initial thought about a theory of motivation for mutual advantage that has yet to be built.

Indeed, in condition (ii), he did not specify the way that what is gained by playing MAP instead of the non-cooperative default profile $\mathbf{s}^0$ is distributed between players. Also the non-cooperative default profile in condition (i) may be given explicitly as a numerical value or defined informally by some behavioral and/or psychological conditions. In the latter case, it may require additional computational efforts to find it in complex games. In terms of existence and computation, one can reasonably expect that MAP would be more difficult than MSE or ANE because of the condition (ii) and the requirement of determining the non-cooperative default profile when it is defined informally (in non-numerical terms).

Further, Sugden [32] has introduced another other-regarding concept of solution as follows.

Definition 4.14. A strategy profile $\mathbf{s}^* \in \mathbf{S}$ is called *Mutually Beneficial Practice Profile* (MBP) if it satisfies the following conditions:

(i) $\alpha_i < u_i(\mathbf{s}^*)$, for all players $i \in N$, and

(ii) for all nonempty groups of players K $(K \neq N)$, there is a player $k \in K$ such that: $\min_{\mathbf{s}_{-K} \in \prod_{j \in -K} \mathbf{S}_j} u_k(\mathbf{s}_K^*, \mathbf{s}_{-K}) < u_k(\mathbf{s}^*)$.

Condition (i) of Definition 4.14 means that at an MBP, $\mathbf{s}^*$, the payoff of each player $i \in N$ must be higher than his/her maximin payoff, α_i. We may call it a "strict" individual rationality condition.

Condition (ii) means that for any arbitrary group of players there must exist at least one member of this group for whom the gain with the practice is strictly higher than his/her (group-based) minimum payoff.

MSE and ANE share the individual rationality condition (i) of MBP with the slight difference that it is not strict in MSE and ANE. Further, there is a conceptual difference with respect to condition (ii). Indeed, MSE is a NE of the game G^A, that is, each player's strategy is his/her best response when the other players play MSE, while in MBP players' strategies are not defined based on the concept of individual best best response, they are group-based. Moreover, condition (ii) may not be satisfied for all players in a given group. Crettez [11] proposes two sufficient conditions under which a (strict) BE is an MBP. Strict BE is derived from BE Definition 4.1 when inequalities are strict. To the best of our knowledge there are no works on the existence and computation of MBP. Regrading these two aspects, we may say that MBP would be easier than MSE, however, MBP may exhibit the problem of multiplicity as its conditions are not very restrictive. In such a case, the players may need to negotiate to agree on additional criteria to select an MBP.

4.2. The bargaining solution

Since AE has a cooperative flavour, it is worth comparing it to the Nash Bargaining solution (BS). Generally speaking, $BS \notin S^A$ (see Example 4.15), but it can occur that BS is also an AE (see Example 4.16).

Example 4.15. Consider the 2-player game with the following payoff table.

	A	B
A	(20,20)	(21, 11)
B	(11, 21)	(10, 10)

Here we have $S^N = \{AA\}$, $S^B = S^A = \{AB, BA\}$. The security levels are $\alpha_1 = \alpha_2 = 10$. We also note that the Nash bargaining solution with status quo point (10,10) is AA with payoffs (20,20) which is not in S^A.

Example 4.16. Consider the 3-player game:

$$u_1(s) = s_1 + s_2 + s_3, \quad u_2(s) = s_1 - s_2 + s_3, \quad u_3(s) = s_1 + s_2 - s_3,$$

with $S_i = [0, 1]$, $i = 1, 2, 3$.

We will consider the status quo as the security levels of players.

$$\alpha_i = \max_{s_i \in S_i} \min_{\mathbf{s}_{-i} \in \mathbf{S}_{-i}} u_i(s_i, \mathbf{s}_{-i}), \ i = 1, 2, 3.$$

It is easy to see that $\alpha_1 = 1$, $\alpha_2 = 0$, and $\alpha_3 = 0$. The bargaining solution can be computed as a solution of the following optimization problem:

$$\max \prod_{i \in N} (u_i(s) - \alpha_i) = \max\{s_1^3 - s_2^3 - s_3^3 + s_1^2 s_2 + s_2^2(s_1 + s_3) + s_3^2 s_2 + s_1 s_2 s_3\}$$

$$\text{subject to } \ s_i \in S_i = [0, 1], \ \ i = 1, 2, 3.$$

The bargaining solution is $BS = (1,1,1)$. On the other hand, the payoffs in the game G^A are

$$u_{-1}(\mathbf{s}) = 2s_1, \quad u_{-2}(\mathbf{s}) = 2s_1 + 2s_2, \quad u_{-3}(\mathbf{s}) = 2s_1 + 2s_3.$$

Clearly, $\mathbf{s} = (1,1,1)$ is also an AE of the initial game. Thus, the bargaining solution is an AE as well. ☐

We should point out that the bargaining solution is usually not an AE because the former is always Pareto optimal while the latter is not necessarily Pareto optimal, as shown by the following example. The other difficulty is that AE and BS are of different natures, the former is based on a high level of cooperation (mutual support and altruism), while the latter involves a weak level of cooperation, if the bargaining fails, the players return back to the status quo, which shows that the players pursue their individual interests first and are not ready to give up some of their payoff to others. The equality of BS and AE may be true for some classes of games that need to be found. The same comment can be made when comparing AE with the Kalai Smorodinsky bargaining solution [18] as the latter is also Pareto optimal by definition and involves a status quo point.

Example 4.17. Consider the 2-player game with the following payoff table.

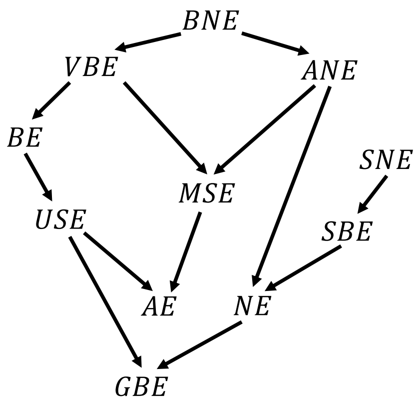
	A	B
A	$(2,2)$	$(3,0)$
B	$(0,3)$	$(1,1)$

Here we have $S^A = \{BB\}$ which is not Pareto optimal. The security levels are $\alpha_1 = \alpha_2 = 2$, and the Nash bargaining solution with status quo point (2,2) is AA. We also note here $S^M = \emptyset$. ☐

4.3. Other cooperative solution concepts

Many other cooperative solutions for games in normal form have been introduced in literature. We compare AE to the most prominent of them, namely, the SNE, the α-core and the β-core that have all been introduced by Aumann. The major difference between AE and MSE and these three concepts is in the way cooperation is expressed. In the latter, a player cares about the others as long as it is beneficial to him/her, in the former each player cares about the others first. In other words, the latter represents a very weak level of cooperation: a player participates in a coalition if he/she can increase his/her payoff, while the former represent a very high level of cooperation that is expressed by mutual support and altruism. Furthermore, it is well known that SNE does not exist in most games [6]. The α-core and the β-core are difficult to compute, while any method computing NE can be used to compute MSE and AE.

The following diagram illustrates the relation between various equilibrium concepts mentioned in this paper, where $A \to B$ means A implies B. The concepts that do not appear in this diagram do not have implication relationship with the others, such as MAP and MBP of Sugden, the α-core, β-core and BS.



5. Conclusion

The concepts of AE, MSE and ANE that express altruism and mutual support in normal form games have been introduced. These equilibria come from "other-regarding preferences". AE represents pure altruism and MSE represents mutual support. The properties of AE, MSE and ANE are presented and illustrated and existence theorems are proved for them. A detailed comparison with similar concepts is given. The comparison shows that AE, MSE and ANE are generalizations of BE, VBE and BNE respectively, and the former are more tractable from computational perspective than the latter. Similar conclusions are drawn when AE and MSE are compared to SNE, SBE, α-core and β-core. However, these results are valid only when AE and MSE apply: the payoffs of players are addable.

In the future we would like to extend this study to different types of games such as differential games, extensive-form games, and games involving uncertainty. It will also be very interesting to obtain sufficient conditions, which are weaker than those given in Theorems 3.2 and 3.5, for the existence of MSE and ANE. Finally, finding sufficient conditions for an AE to be a BS is also an interesting challenging research problem.

Acknowledgement. We thank Andrew MacFie for his early participation in this work. We are also grateful to the two anonymous reviewers for their constructive comments that helped considerably improve the original version of the paper.

References

[1] K. Abalo, M. Kostreva: *Some existence theorems of Nash and Berge equilibria*, Appl. Math. Letters 17 (2004) 569–573.

[2] K. Abalo, M. Kostreva: *Berge equilibrium: some recent results from fixed-point theorems*, Appl. Math. Computation 169 (2005) 624–638.

[3] R. J. Aumann: *Acceptable points in general cooperative n-person games*, in: *Contributions to the Theory of Games IV*, A. W. Tucker and R. D. Luce (eds.), Annals of Mathematics Study 40, Princeton University Press, Princeton (1959) 287–324.

[4] R. J. Aumann: *The core of a cooperative game without side payments*, Trans. Amer. Math. Soc. 98 (1961) 539–552.

[5] C. Berge: *Théorie Générale des Jeux à n Personnes*, Gautier Villars, Paris (1957).

[6] B. D. Bernstein, D. B. Peleg, M. D. Whinston: *Coalition-proof Nash equilibria. I: Concepts*, J. Econ. Theory 42(1) (1987) 1–12.

[7] A. M. Colman: *Game Theory and its Applications in the Social and Biological Sciences*, 2nd ed., Butterworth-Heinemann, Oxford (1995).

[8] A. M. Colman, T. W. Körner, O. Musy, T. Tazdaït: *Mutual support in games: some properties of Berge equilibria*, J. Math. Psychology 55(2) (2011) 166–175.

[9] P. Courtois, R. Nessah, T. Tazdaït: *Existence and computation of Berge equilibrium and of two refinements*, J. Math. Economics 72(C) (2017) 7–15.

[10] B. Crettez: *Unilateral Support Equilibrium, Berge Equilibrium, and Team Problems Solutions*, J. Quant. Economics 17(4) (2019) 727–739.

[11] B. Crettez: *On Sugden's "mutually beneficial practice" and Berge equilibrium*, Int. Revue Economics 64 (2017) 357–366.

[12] G. Debreu: *A social equilibrium existence theorem*, Proc. Nat. Acad. Sci. 38 (1952) 886–893.

[13] K. Fan: *Fixed-point and minimax theorems in locally convex topological linear spaces*, Proc. Nat. Acad. Sci. 38 (1952) 121–126.

[14] K. Fan: *A minimax inequality and applications*, in: Inequalities III, O. Shisha (ed.), Academic Press, New York (1972) 103–113.

[15] E. Fehr, K.-M. Schmidt: *The economics of fairness, reciprocity and altruism-experimental evidence and new theories*, in: *Handbook of the Economics of Giving, Altruism and Reciprocity*, vol. 1 (2006) 615–691.

[16] I. L. Glicksberg: *A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points*, Proc. Amer. Math. Soc. 3 (1952) 170–174.

[17] A. A. Guseinov, V. I. Zhukovskiy, K. N. Kudryavtsev: *Mathematical Foundations of the Moral Golden Rule* (in Russian), Lenand, Moscow (2016).

[18] E. Kalai, M. Smorodinsky: *Other solutions to Nash's bargaining problem*, Econometrica 43(3) (1975) 513–518.

[19] F. Krim: *An In-Depth Study of Berge Equilibrium*, Master's thesis, University of Mouloud Mammeri, Tizi-Ouzou, Algeria (2001).

[20] M. Larbani, R. Nessah: *A note on the existence of Berge and Berge-Nash equilibria*, Math. Social Sciences 55(2) (2008) 258–271.

[21] M. Larbani, R. Nessah: *Sur l'équilibre fort selon Berge*, RAIRO Oper. Research 35 (2001) 439–451.

[22] M. Larbani, V. I. Zhukovskiy: *Berge equilibrium in normal form static games: a literature review*, Izv. Inst. Math. Inform. Udmurskogo Gosudarstvennogo Universiteta 49 (2017) 80–110.

[23] J. O. Ledyard: *Public goods: a survey of experimental research*, in: *Handbook of Experimental Economics*, J. H. Kagel and A. E. Roth (eds.), Princeton University Press, Princeton (1995) 111–194.

[24] J.-F. Nash: *Equilibrium points in n-person games*, Proc. Nat. Acad. Sci. USA 36(1) (1950) 48–49.

[25] J. F. Nash: *Non-cooperative games*, Ann. Math. 54 (1951) 286–295.

[26] R. Nessah, M. Larbani, T. Tazdaït: *A note on Berge equilibrium*, Appl. Math. Letters 20(8) (2007) 926–932.

[27] A. Pottier, R. Nessah: *Berge-Vaisman and Nash equilibria: transformation of games*, Int. Game Theory Review 16(4) (2014) 1450009.

[28] D. Sally: *Conversation and cooperation in social dilemmas: a meta-analysis of experiments from 1958 to 1992*, Rationality and Society 7 (1995) 58–92.

[29] J. Schouten, P. E. M. Borm, R. L. P. Hendrickx: *Unilateral Support Equilibria*, Center for Economic Research, Discussion Paper Vol. 2018-011, Tilburg (2018).

[30] H. E. Scarf: *On the existence of a cooperative solution for a general n-person game*, J. Econ. Theory 32 (1971) 169–181.

[31] P. D. Straffin: *Game Theory and Strategy*, Mathematical Association of America, Washington (1993).

[32] R. Sugden: *Team reasoning and intentional cooperation for mutual benefit*, J. Social Ontoly 1(1) (2014) 143–166.

[33] R. Sugden: *Mutual advantage, conventions and team reasoning*, Int. Review Economics 58 (2011) 9–20.

[34] K. S. Vaisman: *Berge Equilibrium*, Doctoral Thesis, Saint-Petersburg State University, St. Petersburg (1995).

[35] E. Van Dam: *Rationality and the Golden Rule*, Working Paper, Tilburg University (2014).

[36] V. I. Zhukovskiy: *Some problems of non-antagonistic differential games*, in: *Matematicheskie Metody v Issledovaniy Operatsiy*, P. Kenderov (ed.), Bulgarian Academy of Sciences, Sofia (1985) 103–195.

[37] V. I. Zhukovskiy, K. N. Kudryavtsev: *Mathematical foundations of the golden rule. I: Static case*, Automation and Remote Control 78(10) (2017) 1920–1940.

[38] V. I. Zhukovskiy, A. L. Topchishvili: *Mathematical model of golden rule in the form differential positional game of many persons*, Model Assisted Statistics and Applications 22(3) (2016) 203–229.