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A Gårding Inequality Based Unified Approach to Various Classes of Semi-Coercive Variational Inequalities Applied to Non-Monotone Contact Problems with a Nested Max-Min Superpotential

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We present a unified existence and approximation theory for various classes of variational inequalities (VIs) in reflexive Banach spaces. The focus is on semi-coercive problems. Here we abandon projections, which are limited to a Hilbert space setting, instead we adopt semicoercivity of the elliptic linear operator in form of a Gårding inequality. Also we extend the smoothing procedure from [43] to provide smoothing approximations of nested max-min functions. Then we couple this regularization technique with the finite element method to solve numerically semi-coercive hemivariational inequalities (HVIs) involving a nested max-min superpotential and apply our approximation theory for pseudomonotone VIs to these HVIs. As a model example we consider a unilateral semi-coercive contact problem with non-monotone friction on the contact boundary.

Keywords: Hemivariational inequality, pseudomonotonicity, semicoercivity, Gårding inequality, max-min superpotential, smoothing approximation, finite element discretization, non-monotone contact.

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1. Introduction

The purpose of this paper is threefold. Firstly we develop an existence and approximation theory for a general class of linear semi-coercive variational inequalities (VIs, for short) that encompasses VIs of the first kind and of the second kind following the terminology of [20]. Thus we extend the approximation re-

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sult of [26] that was limited to semi-coercive VIs of the first kind. This broader approximation theory applies to the finite element approximation of unilateral Tresca friction contact problems in linear elasticity [28] in the challenging case when in addition to unilateral friction boundary conditions only Neumann but no Dirichlet boundary conditions are present, so the elastic body is not clamped, but only submitted to forces and is nevertheless fixed by the unilateral contact with a rigid foundation. We emphasize that unilateral friction contact problems in linear elasticity can also numerically be solved by boundary element methods. These approximation methods have the advantage that they need the discretization of the boundary only. However, their analysis relies on more involved boundary integral operator theory what is outside of the scope of the present paper, see [30] for more details. Thus using an additional analysis, the approximation theory of the present paper can also applied to the resulting boundary VIs that live on the boundary of the domain of the given elliptic unilateral friction boundary value problem.

Secondly we present an existence theory and approximation theory for a class of nonlinear pseudomonotone VIs that encompass hemivariational inequalities (HVI for short). This class involves linear semi-coercive operators that satisfy a Gårding inequality similar to [26] what replaces the abstract compactness condition $(A1^{\mathrm{S}})$ of [45]. Thus the setting of semi-coerciveness is more versatile - as it is detailed below - than in [45]. More importantly, we do not only recover the result of [45] on weak convergence of the discrete approximations, but can also establish strong convergence results.

Thirdly we extend the smoothing procedure from [43] to provide smoothing approximations of nested max-min functions. Then we couple this regularization technique with the finite element method to solve numerically semi-coercive HVIs involving a nested max-min superpotential and apply our approximation theory for the considered abstract class of pseudomonotone VIs to these HVIs. As a model example we consider a unilateral semi-coercive contact problem with nonmonotone friction on the contact boundary.

Here we substantially deepen the exposition of semicoercivity as given in [2, 22]. There semicoercivity of a linear operator is introduced by the use of the orthogonal projection onto its null space (of its symmetric part) what limits the range of this setting to Hilbert spaces. Here we abandon projections, instead appeal to linear operator theory, in particular to compact operators and Fredholm operators, and adopt semicoercivity of a linear operator in form of a Gårding inequality. Such a Gårding inequality is automatically satisfied by uniformly strongly elliptic operators, see e.g. [17, Proposition 2, Chapter VII],[55, VI,8], [54, 19]. Moreover a Gårding inequality in appropriate function spaces is a key ingredient of the analysis of boundary integral operators and more abstract pseudo-differential operators [33]. Therefore the numerical analysis of semi-coercive unilateral problems via the boundary integral approach [30] is based upon such a Gårding inequality.

This paper is organized as follows. In the subsequent Section 2 we gather some elements of linear operator theory in general Banach spaces thus setting the stage of semi-coercive operators that satisfy a Gårding inequality. Then Section 3 provides the approximation theory in reflexive Banach spaces for a general class of linear semi-coercive VIs. Section 4 is devoted to existence and approximation theory for a class of nonlinear pseudomonotone VIs involving linear semi-coercive operators. As a consequence, an existence result for linear semi-coercive VIs is given, too. Section 5 brings the application to non-monotone contact unilateral problems modelled by a semi-coercive hemivariational inequality and a nested max-min function along with a numerical example. The paper concludes in Section 6 with an outlook to further directions of research.

2. Preliminaries from linear operator theory

Throughout this section, X and Y are general Banach spaces; let $\mathcal{L}(X;Y)$ denote the Banach space of continuous linear operators $A\colon X \to Y$. Then $\mathcal{N}(A) \subset X, \mathcal{R}(A) \subset Y$ denote the null space (kernel), range (image) of A, respectively. With the dual spaces X^*, Y^* of X, Y, respectively, we have the adjoint operator $A' \in \mathcal{L}(Y^*; X^*)$. For M a subspace of X, M° denotes the polar set (annihilator), the set of all continuous linear functionals on X that vanish on M.

We start with the following slight extension of the Fredholm alternative (see e.g. [12, Théorème VI.6], [13, Theorem VI.6]).

Proposition 2.1. *Let $A = B - T\colon X \to Y$ with $B \in \mathcal{L}(X;Y)$ bijective and $T \in \mathcal{L}(X;Y)$ compact. Then*

(i) *$\mathcal{N}(A)$ is of finite dimension,*

(ii) *$\mathcal{R}(A)$ is closed, more precisely $\mathcal{R}(A) = (\mathcal{N}(A'))^\circ$,*

(iii) *$\mathcal{R}(A) = Y$ iff $\mathcal{N}(A') = \{0\}$,*

(iv) *$dim\ \mathcal{N}(A) = dim\ \mathcal{N}(A') = codim\ \mathcal{R}(A) := dim\ Y/(\mathcal{R}(A))$.*

Proof. Let $\tilde{A} := B^{-1}A = B^{-1}(B - T) = I_{X\to X} - B^{-1}T$.

Then $\mathcal{N}(\tilde{A}) = \mathcal{N}(A)$, since $x \in \mathcal{N}(A)$ iff $Bx = Tx$ iff $x = B^{-1}Tx$ iff $x \in \mathcal{N}(\tilde{A})$.

Further $\mathcal{R}(\tilde{A}) \cong \mathcal{R}(A)$, since $y \in \mathcal{R}(A)$ iff $y = Bx - Tx$ for some $x \in X$ iff $B^{-1}y = x - B^{-1}Tx$ for some $x \in X$ iff $B^{-1}y \in \mathcal{R}(\tilde{A})$. Hence dim $\mathcal{R}(\tilde{A}) =$ dim $\mathcal{R}(A)$.

Moreover, $X/\mathcal{R}(\tilde{A}) \cong Y/\mathcal{R}(A)$, since $[y] \in Y/\mathcal{R}(A)$ iff $[y] = y + \mathcal{R}(A)$ for some $y \in Y$ iff $[x] := B^{-1}[y] = B^{-1}y + B^{-1}\mathcal{R}(A) = x + \mathcal{R}(\tilde{A})$ for some $x = B^{-1}y \in X$ iff $[x] = B^{-1}[y] \in X/\mathcal{R}(\tilde{A})$. Hence dim $X/\mathcal{R}(\tilde{A}) =$ dim $Y/\mathcal{R}(A)$.

Since $B^{-1}T$ is compact, items (i)–(iii) follow from [12, Théorème VI.6]. Finally item (iv) is immediate from [49, Part I, Theorem 4.25]. □

Now $A \in \mathcal{L}(X;Y)$ is called a *Fredholm operator* (see e.g. [53, 54, 55, 57]) or a Φ-operator with [39] if

$$\dim \mathcal{N}(A) < \infty, \mathcal{R}(A) \text{ is closed}, \text{codim } \mathcal{R}(A) \equiv \dim Y/(\mathcal{R}(A)) < \infty.$$

Then the *(Fredholm) index* of A is defined by

$$\text{Ind } A := \dim \mathcal{N}(A) - \text{codim } \mathcal{R}(A)\,.$$

Thus we have

Corollary 2.2. *Assume* $A = B - T \colon X \to Y$ *with* $B \in \mathcal{L}(X;Y)$ *bijective and* $T \in \mathcal{L}(X;Y)$ *compact. Then* A *is a Fredholm operator with index zero.*

Remark 2.3. In the proof of Proposition 2.1 we used the left regularization

$$\tilde{A}_X = B^{-1}A = B^{-1}(B-T) = I_{X\to X} - \tilde{T}_X$$

with $\tilde{T}_X = B^{-1}T$ compact. Similarly there is the right regularization

$$\tilde{A}_Y = AB^{-1} = (B-T)B^{-1} = I_{Y\to Y} - \tilde{T}_Y$$

with $\tilde{T}_Y = TB^{-1}$ compact. Thus A admits a left regularization and a right regularization with compact operators, what is equivalent that A is a Fredholm operator by a Theorem of Atkinson, see e.g. [53, Prop. 7.1].

Now we let $Y = X^*$ and obtain

Corollary 2.4. *Let* $A \in \mathcal{L}(X;X^*)$ *satisfy a Gårding inequality, i.e., there holds*

$$\text{(G)} \qquad \langle Ax, x\rangle + \langle Tx, x\rangle \geq \alpha \|x\|^2 \quad (\forall x \in X)$$

for some compact $T \in \mathcal{L}(X;X^*)$ *and* $\alpha > 0$. *Then* A *is a Fredholm operator with index zero.*

Proof. Let $M := A + T \in \mathcal{L}(X;X^*)$. Then M is positive definite or – following the terminology of [57] – uniformly monotone, hence bijective [57, Theorem 5.2]. Thus $A = M - T$ satisfies the assumptions of Corollary 2.2. □

In the subsequent sections we often use that a continuous linear operator $A \in \mathcal{L}(X;X^*)$ correponds to the continuous bilinear form $a : X \times V \to \mathbb{R}$ via $a(x,y) = \langle Ax, y\rangle_{X^*\times X}$. Next we consider monotone operators A (written $A \geq 0$) or corresponding positive semidefinite bilinear forms a, i.e.,

$$\langle Ax, x\rangle = a(x,x) \geq 0 \quad (\forall x \in X) \tag{1}$$

and collect some useful facts in the following lemma.

Lemma 2.5. *Let* $A \in \mathcal{L}(X;X^*), A \geq 0$. *Then*

(i) $\tilde{a}(x) := \langle Ax, x\rangle$ *is convex;*

(ii) $\mathcal{N}_A := \{y \in X : \langle Ay, y\rangle = 0\}$ *is a closed subspace of* X.

(iii) *Let* $K \subset X, F \colon K \to \mathbb{R}$ *be convex.*
Then the following assertions are equivalent:

$$\begin{aligned} &\hat{x} \in K, \quad a(\hat{x}, x - \hat{x}) + f(x) - f(\hat{x}) \geq 0, \forall x \in K \\ &\hat{x} \in K, \quad a(x, \hat{x} - x) + f(\hat{x}) - f(x) \leq 0, \forall x \in K\,. \end{aligned} \tag{2}$$

Proof. (1) In consequence of

$$t(1-t)[a(y,y) - a(z,y) - a(y,z) + a(z,z)] = t(1-t)a(y-z, y-z) \geq 0$$

we obtain for any $t \in (0,1)$

$$\hat{a}(ty + (1-t)z) \leq ta(y,y) + (1-t)a(z,z)\,.$$

(2) follows from (1) and $A \geq 0$.

(3) This is the well-known Minty lemma, see e.g. [5, Lemma 3.5], [25, Behauptung 1.7], [36, Lemma 1.5, Chap. III]. □

To clarify the notion of semicoercivity of the present paper (following [26, 51]) in comparison to [2, 22, 37] we conclude this section with the following remarks.

Remark 2.6. Consider the special case of a real Hilbert space H with inner product $[\cdot,\cdot]$ and norm $\|\cdot\|$.

(1) Then a linear monotone operator $A \in \mathcal{L}(H;H)$, i.e. $[Ax,x] \geq 0$ $(\forall x \in H)$, satisfies a Gårding inequality, if and only if (see [22, Proposition 1.3]):

(i) $Y := \{y \in H : [Ay,y] = 0\} = \mathcal{N}(A+A')$ is of finite dimension;

(ii) there exists $\alpha > 0$ such that $[Ax,x] \geq \alpha \|(I - \mathrm{proj}_Y)x\|^2, \quad \forall x \in H\,.$

Semicoercivity in the sense of (i) along with (ii) goes back to Fichera [19] and Stampacchia [52]; it is used in [2] and more recently in [4].

(2) Further we write $a(x,x) := [Ax,x]; a(x,x) = a_{\mathrm{symm}}(x,x)$, where

$$a_{\mathrm{symm}}(x,y) := 1/2\,[a(x,y) + a(y,x)]$$

is a symmetric bilinear form and satisfies a Cauchy-Schwarz inequality. Hence

$$|x| := a(x,x)^{1/2}$$

is a continuous seminorm and we have $\mathcal{N}_A = \{y \in H : |y| = 0\}\,.$

(3) Let, as considered in [37], H be given as product of Hilbert spaces G and F, $H = G \times F$, F of finite dimension, and for some $\alpha > 0$,

$$\langle Ax, x\rangle \geq \alpha \|y\|_G^2 \quad (\forall x = (y,z) \in H)\,,$$

where $\|\cdot\|_G$ denotes the nom on G. Let

$$\tilde{G} = \{(y, 0_F) : y \in G\}, \quad \tilde{F} = \{(0_G, z) : z \in F\}\,.$$

Then $H = \tilde{G} \oplus \tilde{F}$ and

$$[Ax,x] \geq \alpha \|\mathrm{proj}_{\tilde{G}} x\|^2 = \alpha\{\|x\|^2 - \|\mathrm{proj}_{\tilde{F}} x\|^2\}$$

and the monotone linear operator A satisfies a Gårding inequality with the compact linear map $x \to \mathrm{proj}_{\tilde{F}} x$.

3. Approximation of linear semi-coercive variational inequalities

In this section we deal with approximation of a canonical class of semi-coercive variational inequalities (VIs for short) where the sum of a bilinear form and a convex functional and further a linear functional as right hand side occur and where the constraints are implicitly defined by a convex set. Thus following the terminology of [20] we include VIs of the first kind and of the second kind as well.

3.1. The continuous linear VI problem

More precisely, let V be a real reflexive Banach space; let $K \subset V$ be closed convex. Let $A \in \mathcal{L}(V, V')$ be a continuous linear operator that gives rise to the continuous bilinear form $a\colon V \times V \to \mathbb{R}$ via $a(v, w) = \langle Av, w\rangle_{V' \times V}$. Further let $f\colon V \to \mathbb{R}$ be convex and continuous and $g \in V'$ be fixed. With this data we consider the variational inequality: Find $\hat{u} \in V$ that satisfies

$$\text{(VI)} \qquad u \in K,\ a(\hat{u}, v - \hat{u}) + f(v) - f(\hat{u}) \geq \langle g, v - \hat{u}\rangle,\ \forall v \in K\,. \tag{3}$$

We assume that A is semi-coercive in the following sense: A is monotone (positive semidefinite), i.e.,

$$\langle Av, v\rangle \geq 0 \quad (\forall v \in V) \tag{4}$$

and A satisfies a Gårding inequality, that is, there holds

$$\text{(G)} \qquad \langle Av, v\rangle + \langle Tv, v\rangle \geq c_g \|v\|^2 \quad (\forall v \in V) \tag{5}$$

for some compact $T \in \mathcal{L}(V; V')$ and $c_g > 0$.

Since A is monotone, the Minty lemma applies, and a solution $\hat{u} \in K$ of (VI) is characterized by

$$\hat{u} \in K,\, a(v, \hat{u} - v) + f(\hat{u}) - f(v) \leq \langle g, \hat{u} - v\rangle, \forall v \in K\,. \tag{6}$$

By the strong separation theorem, for the convex lower semicontinuous (lsc, for short) function f, there exist constants $f_0 \in \mathbb{R}, f_1 > 0$ such that

$$f(v) \geq f_0 - f_1\|v\|,\ \forall v \in V\,. \tag{7}$$

Since we do not presuppose boundedness, we employ recession analysis (see e.g. [48]). The recession cone (or asymptotic cone) to the convex closed set K is given for some fixed $x_0 \in K$ by

$$K^\infty = \{z \in V : x_0 + tz \in K\, (\forall t > 0)\}\,.$$

Note that by the closedness of K, this definition is independent of the chosen $x_0 \in K$. Further the recession function to the convex function f on V is given by

$$f^\infty(x) := \lim_{t \to \infty} \frac{f(v + tx) - f(v)}{t}\,, \forall v \in V\,.$$

By the convexity of f, with $\frac{s}{t} \in (0, 1)$,

$$f(v + sy) = f[\tfrac{s}{t}(v + ty) + (1 - \tfrac{s}{t})v] \leq \tfrac{s}{t} f(v + ty) + (1 - \tfrac{s}{t}) f(v)\,,$$

hence in the limit $t \to \infty$, for any $s > 0$,

$$f(v + sy) - f(v) \leq s f^\infty(y)\,,\ \forall v \in V\,. \tag{8}$$

3.2. The discretized linear VI problem

We adapt the approximation scheme of Glowinski [20, Chapter 1] to our variational problem (3) and suppose that we are given a parameter h converging to 0 and a family $\{V_h\}_{h>0}$ of closed finite dimensional subspaces of V. In addition we have a family $\{K_h\}_{h>0}$ of closed convex nonempty sets of V_h , not necessarily contained in K , and a family $\{f_h\}_{h>0}$ of convex continuous functions defined on V_h such that the following two hypotheses (H1) and (H2) are satisfied (denoting weak convergence by $\rightharpoonup$ in contrast to strong convergence denoted by $\to$):

(H1) If for some sequence $\{h_j\}_{j\in\mathbb{N}}$ with $h_j \to 0$, $v_{h_j} \in K_{h_j}$ $(j \in \mathbb{N})$ and $v_{h_j} \rightharpoonup v \in V$ $(j \to \infty)$, then $v \in K$ and

$$\liminf_{j\to\infty} f_{h_j}(v_{h_j}) \geq f(v) .$$

(H2) There exist a subset $\tilde{K} \subset V$ such that cl $\tilde{K} = K$ and mappings $r_h : \tilde{K} \to V_h$ with the property that, for each $v \in \tilde{K}$, $r_h v \to v$ for $h \to 0$, $\lim_{h\to 0} f_h(r_h v) = f(v)$, and $r_h v \in K_h$ for all $h \leq h_0(v)$ for some $h_0(v) > 0$.

Without any loss of generality we can assume that $0 \in \tilde{K}$ and $0 \in K_h$ for all $h > 0$. Thus we approximate the variational problem (3) by the following finite dimensional variational inequality: Find $u_h \in V$ that satisfies

$$u_h \in K_h,\ a(u_h, v_h - u_h) + f_h(v_h) - f_h(u_h) \geq \langle g, v_h - u_h\rangle,\ \forall v_h \in K_h . \quad (9)$$

When simply $\tilde{K} = K$, then (H1) and (H2) express that K_h converges to K and f_h to f with respect to Mosco (set) convergence, see [6], what can be subsumed in epigraphical analysis [7].

Note that we only changed the generally nonlinear functional f to f_h. In most computations, however, it will be necessary to replace also the biliner form a and the linear form g by some approximations a_h and g_h, obtained from a numerical integration rule which is used in the finite element discretization. Since this is covered in the finite element analysis of linear elliptic boundary value problems and variational equalities by the well-known Strang's lemmas, we do not discuss this aspect here.

3.3. The approximation result

Theorem 3.1. *Let* $A \in \mathcal{L}(X; X^*)$ *be semi-coercive, i.e., let* A *satisfy* (4) *and* (5). *Morever let* f, f_h *and* K, K_h *satisfy* (H1) *and* (H2). *Suppose the solution* $\hat{u}$ *of (3) is unique, then* $\lim_{h\to 0} \|u_h - \hat{u}\| = 0$ *holds.*

Proof. We split the proof into five parts. We first show a priori estimates and boundedness of $\{u_h\}$, before we can establish norm convergence.

Step 1: A priori estimates for $\{u_h\}$.

Fix $w_0 \in \tilde{K}$. Let $w_h := r_h w_0 \in K_h$ for $0 < h < h_0(w_0)$. Then $w_h \to w_0$ and with u_h, a solution of (9), we have using (7) and $0 \in K_h$

$$a(u_h, u_h) \leq c_0 + c_1\|u_h\| + g(u_h) \leq c_0 + c_2\|u_h\| . \quad (10)$$

Here and in the following $c_0, c_1, c_2, \dots$ are generic positive constants. Moreover, by Minty's lemma,

$$a(w_h, u_h) + f_h(u_h) - g(u_h) \le a(w_h, w_h) + f_h(w_h) - g(w_h)\,. \tag{11}$$

Step 2: $\{u_h\}$ is bounded in norm.

Here we use recession analysis similar to existence theory of semi-coercive variational inequalities, see e.g. [3, 14], and modify a contradiction argument, which in the existence theory of semi-coercive variational inequalities goes back to Fichera [19] and Stampacchia [52].

We assume that there exists a subsequence $\{u_\ell\} := \{u_{h_\ell}\}$ such that $\|u_\ell\| \to +\infty$ $(\ell \to \infty)$. With $y_\ell := \|u_\ell\|^{-1} u_\ell$ in the reflexive Banach space V, we can extract a subsequence, again denoted by $\{y_\ell\}$, that converges weakly to some $y \in V$. In virtue of (10), we get

$$0 \le a(y_\ell, y_\ell) \le c_0\, \|u_\ell\|^{-2} + c_1\, \|u_\ell\|^{-1}\,,$$

hence $a(y_\ell, y_\ell) \to 0$.

We claim that $y \in K^\infty$. Fix some $w_0 \in \tilde{K} \subset K$ and arbitrary $t > 0$. Then by (H2), $w_\ell := r_{h_\ell} w_0 \in K_\ell := K_{h_\ell}; w_\ell \to w_0$ $(\ell \to \infty)$. Since $\|u_\ell\| \to \infty$, for ℓ large enough,

$$v_\ell := \big(1 - \frac{t}{\|u_\ell\|}\big) w_\ell + \frac{t}{\|u_\ell\|} u_\ell \in K_\ell\,,$$

hence $v_\ell \rightharpoonup w_0 + ty \in K$ by (H1), what proves the claim. Next from (11)

$$f_\ell(u_\ell) := f_{h_\ell}(u_{h_\ell}) \le a(w_\ell, w_\ell - u_\ell) + f_\ell(w_\ell) + \langle g, u_\ell - w_\ell\rangle\,.$$

Hence by division by $\|u_\ell\|$ we obtain the estimate

$$\limsup_{\ell\to\infty} \frac{f_\ell(u_\ell)}{\|u_\ell\|} \le -a(w_0, y) + g(y)\,. \tag{12}$$

Since $\|u_\ell\| \to \infty$, again for ℓ large enough,

$$f_\ell(v_\ell) \le (1 - \frac{t}{\|u_\ell\|}) f_\ell(w_\ell) + \frac{t}{\|u_\ell\|} f_\ell(u_\ell)\,,$$

hence by (H1), (12) implies in the limit $\ell \to \infty$, for any $w_0 \in \tilde{K}$,

$$\begin{aligned} f(w_0 + ty) &\le \liminf_{\ell\to\infty} f_\ell(v_\ell) \le \lim_{\ell\to\infty} (1 - \frac{t}{\|u_\ell\|}) f_\ell(w_\ell) + t \limsup_{\ell\to\infty} \frac{f_\ell(u_\ell)}{\|u_\ell\|} \\ &\le f(w_0) - a(w_0, ty) + g(ty)\,. \end{aligned}$$

By the continuity of a and f this extends to the closure of $\tilde{K}$, hence for all $v \in K$, $t > 0$, we get

$$a(v, ty) + f(v + ty) - f(v) \le g(ty)\,.$$

Divide by $t > 0$, take the limit $t \to \infty$ and obtain for all $v \in K$,

$$f^{\infty}(y) \leq g(y) - a(v, y) \tag{13}$$

Then (8) and (13) together yield for all $s > 0$,

$$f(\hat{u} + sy) - f(\hat{u}) \leq s f^{\infty}(y) \leq g(sy) - a(v, sy) \tag{14}$$

We have $\hat{u} + sy \in K$ for all $s > 0$, since $\hat{u} \in K$ and $y \in K^{\infty}$. Further adding (14) to (6) and using Minty's lemma show that $\hat{u} + sy$ solves (VI) for any $s > 0$. Therefore by uniqueness, we arrive at $y_\ell \rightharpoonup y = 0$.

Now we use the Gårding inequality (5) and obtain

$$a(y_\ell, y_\ell) + \langle T y_\ell, y_\ell \rangle \geq c_g \|y_\ell\|^2 \quad (\forall v \in V)\,.$$

Since $T \in \mathcal{L}(V; V')$ is compact, for some subsequence, again denoted by $\{y_\ell\}$, we have that $T y_\ell$ converges strongly to $Ty = T(0) = 0$. With $a(y_\ell, y_\ell) \to 0$, strong convergence $y_\ell \to 0$ follows. However, $\|y_\ell\| = 1$. A contradiction is reached and the norm boundedness of $\{u_h\}$ is proved.

Step 3: $\{u_h\}$ converges weakly to the solution $\hat{u}$ of (VI).

By uniqueness, it is enough to show that any weak limit point u^* of $\{u_h\}$ solves (VI). So let $u_n := u_{h_n} \rightharpoonup u^*$. Then from the Minty inequality resulting from (9), for any $w_n \in K_n := K_{h_n}$,

$$a(w_n, u_n) + f_{h_n}(u_n) - g(u_n) \leq a(w_n, w_n) + f_{h_n}(w_n) - g(w_n)\,.$$

Fix some arbitrary $w_0 \in \tilde{K}$. Then by (H2), $w_n := r_{h_n} w_0 \in K_n$; $w_n \to w_0$ $(n \to \infty)$. Hence in the limit by (H1), (H2),

$$\begin{aligned} a(w_0, u^*) + f(u^*) - g(u^*) &\leq \liminf_{n \to \infty} a(w_n, u_n) + f_{h_n}(u_n) - g(u_n) \\ \leq \lim_{n \to \infty} a(w_n, w_n) + f_{h_n}(w_n) - g(w_n) &= a(w_0, w_0) + f(w_0) - g(w_0)\,. \end{aligned}$$

By the continuity of a, f, g this extends to the closure of $\tilde{K}$, hence for all $v \in K$,

$$a(v, u^*) + f(u^*) - g(u^*) \leq a(v, v) + f(v) - g(v)\,.$$

From Minty's lemma we conclude that u^* solves (VI).

Step 4: $a(u_h - \hat{u}, u_h - \hat{u}) \to 0\,(h \to 0)$ for the solution $\hat{u}$ of (VI).

Here we modify an argument due to Glowinski [20, Proof of Theorem 5.2, Chapter I]. Again fix some $w_0 \in \tilde{K}$. Then by (H2), $w_h := r_h w_0 \in K_h$; $w_h \to w_0$ $(h \to 0)$. Hence by boundedness of $\{u_h\}$, (9) implies

$$\begin{aligned} a(u_h - \hat{u}, u_h - \hat{u}) &= a(u_h, w_h - \hat{u}) - a(u_h, w_h - u_h) - a(\hat{u}, u_h - \hat{u}) \\ &\leq c_3 \|w_h - \hat{u}\| + f_h(w_h) - f_h(u_h) - g(w_h - u_h) - a(\hat{u}, u_h - \hat{u})\,, \end{aligned}$$

in the limit by (H1), (H2), for any $w_0 \in \tilde{K}$,

$$0 \le \limsup_{h \to 0} a(u_h - \hat{u}, u_h - \hat{u}) \le c_3 \|w_0 - \hat{u}\| + f(w_0) - f(\hat{u}) - g(w_0 - \hat{u}) \,,$$

what extends by continuity of f, g to cl $\tilde{K} = K$. Thus we we can choose $w_0 = \hat{u} \in K$ and obtain the claimed convergence assertion.

Step 5: u_h converges strongly to $\hat{u}$ for $h \to 0$.

Assume there exists a sequence $\{u_n\}$ such that u_n solves (9) for $h = h_n \to 0$ and $\|u_n - \hat{u}\| \ge \delta > 0$ By Step 2, $\|u_n\|$ is bounded. So we can extract a subsequence, again denoted by $\{u_n\}$ such that by Step 3, u_n converges weakly to $\hat{u}$, the solution of (VI).

Finally use the Gårding inequality (5) again and obtain, since $T \in \mathcal{L}(V; V')$ is compact, for some subsequence, again denoted by $\{u_n\}$, that Tu_n converges strongly to $T(\hat{u})$. With $a(u_h - \hat{u}, u_h - \hat{u}) \to 0$ by Step 4, $\|u_n - \hat{u}\| \to 0$ follows. A contradiction is reached and the norm convergence of $\{u_h\}$ is proved. □

To conclude this section, we give some remarks and further references to the literature. Generally the solution set in semi-coercive problems with a positive definite linear operator is not unique, but a closed convex set. In this situation of non-uniqueness Adly and Goeleven [2] coupled Galerkin discretization with Tychonov regularization and established norm convergence of the entire approximation process to the unique solution of minimal norm. On the other hand, if for a solution u of the semi-coercive VI (3) of first kind with $f = 0$ there holds

$$\mathcal{N}_A \cap \mathcal{N}(Au - g) \cap (K - u) = \{0\} \,,$$

then u is unique, see Spann [51, Lemma 2.2].

4. Nonlinear pseudomonotone semi-coercive variational inequalities – existence and approximation

This section is devoted to a class of nonlinear pseudomonotone VIs that – as we shall see – encompass hemivariational inequalites (HVIs). This class involves linear semi-coercive operators that satisfy a Gårding inequality similar to [26]; thus we can dispense with the abstract condition $(A1^{\mathrm{S}})$ of [45]. First we give an existence result that is tailored to the application to HVIs and to non-monotone contact problems in the subsequent sections. Moreover from this existence result we derive an existence result for semi-coercive linear VIs. Then we present a general approximation result.

4.1. The setting and existence results

Let V be a reflexive Banach space and V^* its dual. We denote by $\langle \cdot, \cdot \rangle$ the duality pairing between V and V^*, and by $\| \cdot \|$ and $\| \cdot \|_*$ the norm and the dual norm on V and V^*, respectively.

Let $A \in \mathcal{L}(V; V^*)$ a semi-coercive operator in the form of a Gårding inequality, that is, there holds

$$\text{(G)} \qquad \langle Av, v\rangle + \langle Cv, v\rangle \geq \alpha \|v\|^2 \quad (\forall v \in V) \tag{15}$$

for some compact $C \in \mathcal{L}(V; V^*)$ and $\alpha > 0$.

Further, let $g \in V^*$ be a continuous linear form, $K \subseteq V$ a nonvoid closed convex set such that $0 \in K$, what can always be arranged by a simple translation. Finally let a bifunction $\varphi\colon K \times K \to \mathbb{R}$ be given that satisfies:

(A^φ) (i) $\varphi(u, u) = 0$ for any $u \in K$;

(ii) $\varphi(u, \cdot)$ is continuous and convex for any $u \in K$;

(iii) $\varphi(\cdot, v)$ is upper semicontinuous for any $v \in K$ on the intersection of K with any finite dimensional subspace of V;

(iv) $\varphi(\cdot\,,\,\cdot)$ is pseudomonotone in the sense that $u_n \rightharpoonup u$ (weak convergence) and $\liminf_{n\to\infty} \varphi(u_n, u) \geq 0$ imply $\limsup_{n\to\infty} \varphi(u_n, v) \leq \varphi(u, v)$ for all $v \in K$;

(v) there exist $c_0 \geq 0, c_\varphi > 0$ such that

$$\varphi(v, 0) \leq c_0 + c_\varphi \|v\| \quad \forall v \in V\,. \tag{16}$$

As we shall see in the subsequent section, all the assumptions (A^φ) are satisfied by the nonlinear non-monotone bifunction in the class of hemivariational inequalities considered.

In the semi-coercive unbounded situation we have to impose an appropriate geometric condition on the right hand side g that involves the "directions of escape" given by $K^\infty \cap Y$, where again $Y := \{y \in V : \langle Ay, y\rangle = 0\}$ and where for later use we set $|y| := \sqrt{\langle Ay, y\rangle}$.

In fact, we can relax the condition ($\mathrm{A2}^s$)(iii) of [45] and now require:

(A^s) Either $Y \cap K$ is bounded or

$$\langle g, y\rangle < -c_\varphi \text{ for all } y \in K^\infty \cap Y \text{ with } \|y\| = 1\,.$$

In this setting we consider the semi-coercive nonlinear problem VI (A, φ, g, K): Find $u \in K$ such that

$$\langle Au, v - u\rangle + \varphi(u, v) \geq \langle g, v - u\rangle \quad \forall v \in K \tag{17}$$

and have the following existence result.

Theorem 4.1. *Let $A\colon V \to V^*$ be a semi-coercive operator in the sense of the Gårding inequality* (15) *and let $\varphi\colon V \times V \to \mathbb{R}$ satisfy* (A^φ). *Under the assumption* (A^s)*, the problem* VI (A, φ, g, K) (17) *has a solution.*

Proof. We modify the proof of Theorem 2.1 of [45] and rely upon a contradiction argument that goes back to Hess [32] and Stampacchia [52]. In view of [25, Theorem 3.9] and [32] it is sufficient to show the existence of a constant $R > 0$ such that

$$-\langle Av, v\rangle + \varphi(v, 0) + \langle g, v\rangle < 0 \quad \forall v \in K \text{ with } \|v\| = R.$$

Assume the contrary, i.e., there exists a sequence $\{v_n\} \subset K$ such that $\|v_n\| \to \infty$ and

$$-\langle Av_n, v_n\rangle + \varphi(v_n, 0) + \langle g, v_n\rangle \geq 0.$$

By (15) and (16) it follows that

$$\begin{aligned} \alpha\|v_n\|^2 - \langle Cv_n, v_n\rangle \leq \langle Av_n, v_n\rangle &\leq \langle g, v_n\rangle + \varphi(v_n, 0) \\ &\leq \|g\|_*\|v_n\| + c_0 + c_\varphi\|v_n\|. \end{aligned} \tag{18}$$

Set $y_n = \frac{v_n}{\|v_n\|}$. Since $0 \in K$ and K is convex, it follows for n large enough that $y_n \in K$ as well. Moreover, $\|y_n\| = 1$ and consequently, we can extract a subsequence that converges weakly to some $y \in K$, since K is weakly closed.

Claim I: $y \neq 0$.

Assume the contrary. Then divide (18) by $\|v_n\|^2$ and obtain for some subsequence denoted again by $\{y_n\}$,

$$\alpha = \lim_{n\to\infty}[\alpha - \langle Cy_n, y_n\rangle] \leq \lim_{n\to\infty}[\frac{c_0}{\|v_n\|^2} + \frac{\|g\|_* + c_\varphi}{\|v_n\|}] = 0\,,$$

hence a contradiction. This proves Claim I.

Claim II: $|y_n| = \sqrt{\langle Ay_n, y_n\rangle} \to 0$.

Assume the contary, $|y_n| \geq \beta$ for some $\beta > 0$ and for some subsequence denoted again by $\{y_n\}$. Then divide (18) by $\|v_n\|$ and obtain for n large enough,

$$\|v_n\|\,|y_n|^2 \leq \|g\|_* + 2c_\varphi\,,$$

hence

$$\beta^2 \leq |y_n|^2 \leq \frac{\|g\|_* + 2c_\varphi}{\|v_n\|}\,.$$

Since $\|v_n\| \to \infty$, a contradiction is reached. This proves Claim II.

Now use (15) again and estimate

$$\alpha\|y_n - y\|^2 - \langle C(y_n - y), y_n - y\rangle \leq \langle Ay_n, y_n\rangle - \langle Ay, y_n\rangle - \langle Ay_n, y\rangle + \langle Ay, y\rangle\,,$$

hence for some subsequence again denoted by $\{y_n\}$,

$$0 \leq \alpha \limsup_{n\to\infty} \|y_n - y\|^2 \leq -\langle Ay, y\rangle \leq 0\,.$$

Thus $\langle Ay, y\rangle = |y|^2 = 0$ and the (sub)sequence y_n converges strongly to y in V. In conclusion, $y \in Y \cap K$ and $\|y\| = 1$.

Claim III: λy belongs to K for any $\lambda > 0$.

Indeed, because of $\|v_n\| \to \infty$, there exists n_0 such that $\|v_n\| > \lambda$ for all $n \geq n_0$. By convexity and $0 \in K$, $\lambda y_n \in K$ for all $n \geq n_0$, and by the closedness of K, $\lambda y \in K$. Since $0 \in K$, $y \in K^\infty \cap Y$ follows with $y \neq 0$. Hence, if $Y \cap K$ is bounded, this leads immediately to a contradiction. Otherwise, we obtain from (18) that

$$0 \leq \langle Av_n, v_n \rangle \leq \langle g, v_n \rangle + c_0 + c_\varphi \|v_n\|. \tag{19}$$

Divide (19) by $\|v_n\|$ to arrive at

$$0 \leq \langle g, y_n \rangle + \frac{c_0}{\|v_n\|} + c_\varphi \,,$$

what gives in the limit, $0 \leq \langle g, y \rangle + c_\varphi$. This contradicts ($A^s$), since $y \in K^\infty \cap Y$ with $\|y\| = 1$. Thus the existence theorem is proved. □

We come back to the continuous linear VI problem: Find $\hat{u} \in V$ that satisfies

$$\text{(VI)} \qquad \hat{u} \in K,\ a(\hat{u}, v - u) + f(v) - f(\hat{u}) \geq \langle g, v - \hat{u} \rangle,\ \forall v \in K\,. \tag{20}$$

Recall that by the strong separation theorem, for the convex lsc function f, there exist constants $f_0 \in \mathbb{R}, f_1 > 0$ such that

$$f(v) \geq f_0 - f_1 \|v\|,\ \forall v \in V\,. \tag{21}$$

Corollary 4.2. *Let $A\colon V \to V^*$ be a semi-coercive operator in the sense of the Gårding inequality* (15)*. Suppose the convex function f is continuous and*

(A^f) *either $Y \cap K$ is bounded or $\langle g, y \rangle < -f_1$ for all $y \in K^\infty \cap Y$ with $\|y\| = 1$.*

Then problem (20) *has a solution.*

Proof. We only have to show that the assumptions of the existence theorem 4.1 are satisfied for $\varphi(u, v) := f(v) - f(u)$. With f convex and lsc, hence weakly lsc, the conditions of (A^φ) hold obviously. (A^s) is (A^f) with $c_\varphi = f_1$. □

This existence result can be compared to the main existence result [8, Theorem 3.4] of Baiocchi, Buttazzo, Gastaldi, and Tomarelli. Whereas the latter result needs abstract compactness and compatibility conditions, our condition (A^f) involves the data f and g directly.

4.2. An approximation theory for semi-coercive pseudo-monotone variational inequalities

Let T be a directed set with $0 \in \operatorname{cl} T$. We adapt the approximation scheme of Glowinski [20] and suppose that $\{V_t\}_{t \in T}$ is a family of closed subspaces of V. In addition we have a family $\{K_t\}_{t \in T}$ of closed convex nonempty subsets of V_t, not necessarily contained in K such that the following two hypotheses are satisfied:

(HS1) If $\{v_{t'}\}_{t' \in T', t' \to 0}$ weakly converges to v in V, $v_{t'} \in K_{t'}$ ($t' \in T'$) for a subnet $\{K_{t'}\}_{t' \in T'}$ of the net $\{K_t\}_{t \in T}$, then $v \in K$.

(HS2) There exist a subset $\tilde{K} \subset V$ such that $\operatorname{cl} \tilde{K} = K$ and mappings $r_t\colon \tilde{K} \to V_t$ with the property that, for each $w \in \tilde{K}, r_t w \to w$ for $t \in T, t \to 0$ and $r_t w \in K_t$ for all $t \leq t_0(w)$ for some $t_0(w) \in T$.

Since $0 \in K$, by a translation argument, we can simply assume that $0 \in K_t$ for all $t \in T$. Further, we replace φ by some approximation φ_t satisfying

(HS3) φ_t is pseudo-monotone for any $t \in T$;

(HS4) for any nets $\{u_t\}$ and $\{v_t\}$ such that $u_t \in K_t$, $v_t \in K_t$, $u_t \rightharpoonup u$, and $v_t \to v$ in V it follows that

$$\limsup_{t \in T} \ \varphi_t(u_t, v_t) \leq \varphi(u, v) \, .$$

(HS5) $\varphi_t(u_t, 0) \leq c_0 + c_\varphi \|u_t\| \quad \forall u_t \in V_t \, .$

Altogether $\mathrm{VI}(A, \varphi, g, K)$ is approximated by the problem $\mathrm{VI}(A, \varphi_t, g, K_t)$:
Find $u_t \in K_t$ such that

$$\langle Au_t, v_t - u_t \rangle + \varphi_t(u_t, v_t) \geq \langle g, v_t - u_t \rangle \quad \forall v_t \in K_t. \tag{22}$$

In some computations it will be necessary to replace also A and g by some approximations A_t and g_t, defined for example by a numerical integration procedure which is used in the finite element discretization. Since the main difficulty comes from the treatment of the general nonlinear functional φ, we focus here on its approximation φ_t.

Theorem 4.3. *Let $A \colon V \to V^*$ be a linear monotone operator that satisfies the Gårding inequality* (15). *Under assumption* (A^s) *and hypotheses* (HS1)–(HS5), *the family $\{u_t\}$ of solutions to* $\mathrm{VI}(A, \varphi_t, g, K_t)$ *is uniformly bounded in V and has weak limit points. Any weak limit point of $\{u_t\}$ solves the problem $VI(A, \varphi, g, K)$. Moreover, any weak limit point of $\{u_t\}$ is in fact a strong limit point.*
Furthermore, if the solution $\hat{u}$ of the problem $\mathrm{VI}(A, \varphi, g, K)$ *is unique, then the equality* $\lim_{t \in T} \|u_t - \hat{u}\| = 0$ *holds.*

Proof. We split the proof into four parts. We first show uniform boundedness of $\{u_t\}$, before we can establish the convergences first with respect to the weak topology, then with respect to the functional $\tilde{a}$ and finally with respect to the norm on V.

Step 1: Existence and uniform boundedness of $\{u_t\}$.

Theorem 4.1 immediately implies the existence of a solution u_t to $\mathrm{VI}(A, \varphi_t, g, K_t)$. To show the uniform boundedness of the family $\{u_t\}$ we modify the contradiction argument in the proof of Theorem 4.1 as follows. Assume $\{u_t\}$ is not bounded by norm. Then for any $n \in \mathbb{N}$ there exist $t_n \in T, t_n \to 0$ such that $u_n := u_{t_n}$ satisfy $\|u_n\| \to \infty$. From (22) with $v_n := 0 \in K_{t_n}$ we get

$$-\langle Au_n, u_n \rangle + \varphi_{t_n}(u_n, 0) + \langle g, u_n \rangle \geq 0.$$

By (15) and (HS5), it follows that

$$\begin{aligned} \alpha \|u_n\|^2 - \langle Cu_n, u_n \rangle \leq \langle Au_n, u_n \rangle &\leq \langle g, u_n \rangle + \varphi_{t_n}(u_n, 0) \\ &\leq \|g\|_* \|u_n\| + c_0 + c_\varphi \|u_n\| \, . \end{aligned} \tag{23}$$

Now use (23) instead of (18) and complete the contradiction argument verbatim as in the proof of Theorem 4.1. Consequently $\{u_t\}$ is uniformly bounded in the reflexive Banach space V and possesses weak limit points.

Step 2: Any weak limit point of $\{u_{t\in T}\}$ is a solution of $\mathrm{VI}(A, \varphi_t, g, K_t)$.

Let us extract a subnet, again denoted by $\{u_t\}$, such that $u_t \rightharpoonup u$ for $t \to 0$. In view of (HS1), u belongs to K. Now, take an arbitrary $w \in \tilde{K}$. By (HS2), there exist $w_t := r_t w \in K_t$ such that $w_t \to w$ in V.

Observe, see Lemma 2.5(i), that $\tilde{a}(u) = \langle Au, u\rangle$ is convex and continuous, hence weakly lower semicontinuous. Thus by (HS3), (HS4) and $\mathrm{VI}(A, \varphi_t, g, K_t)$ we get that for $w \in \tilde{K}$

$$\begin{aligned}\langle Au, w-u\rangle + \varphi(u,w) &\geq \limsup_{t\in T}\langle Au_t, w_t - u_t\rangle + \limsup_{t\in T}\varphi_t(u_t, w_t)\\ &\geq \liminf_{t\in T}\{\langle Au_t, w_t - u_t\rangle + \varphi_t(u_t, w_t)\}\\ &\geq \liminf_{t\in T}\langle g, w_t - u_t\rangle = \langle g, w-u\rangle\,.\end{aligned}$$

By continity of $\varphi(u,\cdot)$ this extends to cl $\tilde{K} = K$:

$$\langle Av, u-v\rangle + \varphi(u,v) \geq \langle g, u-v\rangle\,\forall v \in K\,,$$

and conclude that u is a solution to $VI(A, \varphi, g, K)$.

Step 3: Convergence with respect to the functional $\tilde{a}$.

We show that for any subnet $\{u_{t'}\}$ weakly convergent to some $u \in V$, we have $\lim_{t'\in T'} \tilde{a}(u_{t'} - u) = 0$. To this end we modify an argument due to Glowinski [20, Proof of Theorem 5.2 Chapter I]. For any fixed $w \in \tilde{K}$ let $w_t := r_t z \in K_t$ and $w_t \to w$ for $t \to 0$. Then we can write and estimate using (22) and the boundedness of $\{u_{t'}\}$

$$\begin{aligned}\langle A(u_{t'} - u), u_{t'} - u\rangle &= \langle Au_{t'}, w_{t'} - u\rangle - \langle Au_{t'}, w_{t'} - u_{t'}\rangle - \langle Au, u_{t'} - u\rangle\\ &\leq c_A\|w_{t'} - u\| + \langle g, u_{t'} - w_{t'}\rangle + \varphi_{t'}(u_{t'}, w_{t'}) - \langle Au, u_{t'} - u\rangle\,.\end{aligned}$$

This gives in the limit for any $w \in \tilde{K}$,

$$0 \leq \liminf_{t'\in T', t'\to 0} \tilde{a}(u_{t'} - u) \leq c_A\|w - u\| + \langle g, u - w\rangle + \varphi(u,w)\,.$$

This extends to K by continuity. Finally choose $w = u$ and obtain the conclusion.

Step 4: Convergence with respect to the norm $\|\cdot\|$.

Let $\{u_{t'}\}_{t'\in T'}$ a subnet that is weakly convergent to some $u \in V$. According to Step 3 $\tilde{a}(u_{t'} - u) \to 0$ holds. Now we show that $\lim_{t'\in T'}\|u_{t'} - u\| = 0$. Assume the contrary. Then there exist $\delta > 0$ and a further subnet $\{u_s\}_{s\in S}$ with $S \subset T$ such that $\|u_s - u\| \geq \delta$ for all $s \in S$. By compactness of C there exists a further subnet $\{u_r\}_{r\in R}$ with $R \subset S$ such that Cu_r strongly converges to Cu. By (15),

$$0 \leq \alpha\|u_r - u\|^2 \leq \tilde{a}(u_r - u) + \langle C(u_r - u), u_r - u\rangle \to 0\,,$$

$\|u_r - u\| \to 0$ follows, and a contradiction is reached. The proof is complete. □

5. Application to non-monotone contact modelled by a hemi-variational inequality and a nested max-min superpotential

5.1. Unilateral contact with non-monotone friction

In this section, we consider the unilateral contact of an elastic body with a rigid foundation under given forces subject to a non-monotone friction law on the contact boundary. We deal with a similar problem as in [45], but now the semi-coercive HVI modelling this problem involves a nested max-min superpotential. Here we apply the approximation theory of Section 4.2 and thus, we can obtain not only weak convergence but also norm convergence results.

Let $\Omega = (0,1)^2 \subset \mathbb{R}^2$ represent a linear elastic body. The boundary $\Gamma := \partial\Omega$ is decomposed into four disjoint open parts Γ_1, Γ_2, Γ_3 and Γ_c such that we have $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3 \cup \overline{\Gamma}_c$. Γ_1 and Γ_2 are Neumann parts with boundary loads given below. On Γ_c we have unilateral and non-monotone contact condition that are specified later. We emphasize that in our benchmark example no Dirichlet boundary condition is prescribed. In particular, on the part Γ_3 of the boundary we assume that the horizontal displacement u_1 is zero, but the vertical displacement u_2 is free; see Figure 5.2. Thus we model the deformation of a linear elastic block under given loads, where the block can move in a rail that is affected by rust leading to frictional contact.

The body is loaded with horizontal forces $\mathbf{F}_1$ on Γ_1 and vertical forces $\mathbf{F}_2$ on Γ_2. For simplicity, the volume forces are neglected. With $\mathbf{F}_1 = (P,\, 0)$, $\mathbf{F}_2 = (0, -P)$, this gives the linear form

$$\langle \mathbf{g}, \mathbf{v} \rangle := P\Big(\int_{\Gamma_1} v_1 \, ds - \int_{\Gamma_2} v_2 \, ds \Big). \tag{24}$$

Further, let $\mathbf{n}$ be the outer unit normal vector on the boundary. The stress vector on the surface is decomposed into the normal, respectively, the tangential stress:

$$\sigma_n := \sigma(u)\mathbf{n} \cdot \mathbf{n}, \quad \boldsymbol{\sigma_t} := \sigma(u)\mathbf{n} - \sigma_n \mathbf{n}.$$

Likewise $u_n(s)$, respecitvely $u_t(s)$ denotes the normal, respectively, the tangential component of the displacement vector $\mathbf{u}$ on Γ_c.

On Γ_c we assume unilateral contact and non-monotone friction:

$$\begin{cases} u_n(s) = -u_2(s) \le 0 & s \in \Gamma_c \\ -\sigma_t(s) \in \partial j(s, u_t(s)) & \text{for a.a. } s \in \Gamma_c. \end{cases}$$

The latter condition, written as a differential inclusion by means of the Clarke's subdifferential [16] ∂j, describes the physical law between the tangential component σ_t of the stress boundary vector and the tangential component u_t of the displacement vector $\mathbf{u}$ on Γ_c. A typical zig-zagged non-monotone multivalued friction law ∂j is shown in Figure 5.1(right) that results from the nested max-min superpotential j, $j = \max\{g_1, g_2, \min\{g_3, g_4\}\}$, depicted in Figure 5.1(left) with a red bold line where g_1, g_2 are quadratic functions and g_3, g_4 are linear ones.

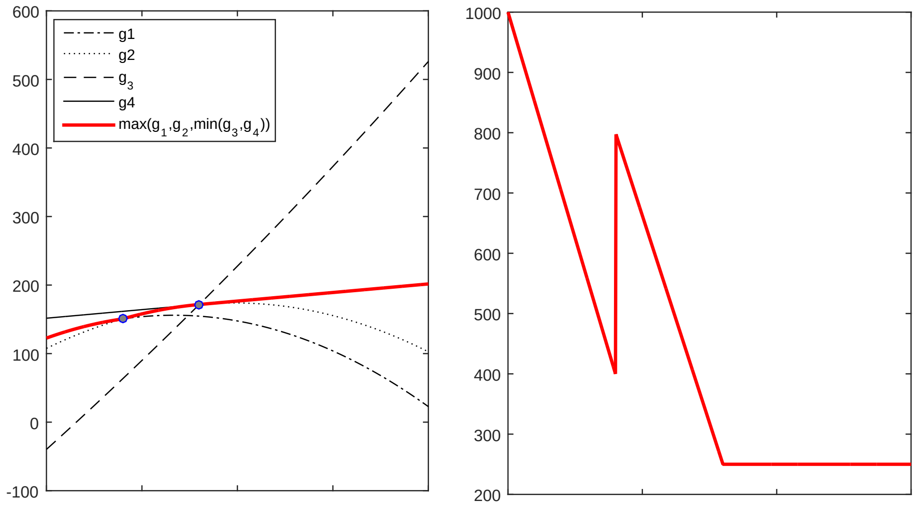


Figure 5.1: Max-min superpotential and the corresponding non-monotone friction law.

To give a variational formulation of the above boundary problem we define

$$V = H^1(\Omega, \mathbb{R}^2),\ K = \{\mathbf{v} \in V \ : \ v_2 \geq 0 \text{ on } \Gamma_c\}$$

and following the standard notations from linear elasticity [35], we introduce the bilinear form of linear elasticity

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \sigma_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{u})\, dx \tag{25}$$

with the stress tensor σ and the strain tensor ε related by the linear Hooke's law

$$\sigma_{ij}(\mathbf{u}) = \frac{E\nu}{1-\nu^2}\delta_{ij}\, tr\big(\varepsilon(\mathbf{u})\big) + \frac{E}{1+\nu}\varepsilon_{ij}(\mathbf{u}), \quad i, j = 1, 2, \tag{26}$$

with the Kronecker symbol δ_{ij} and the trace $tr\big(\varepsilon(\mathbf{u})\big) := \varepsilon_{11}(\mathbf{u}) + \varepsilon_{22}(\mathbf{u})$.

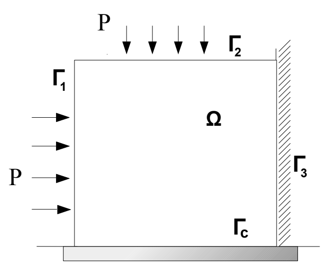


Figure 5.2: 2D benchmark examples with force distribution and boundary decomposition: $u_1 = 0$ on Γ_3, $u_2 =$ free on Γ_3.

As variational formulation of the above boundary problem we arrive at the following HVI: Find $\mathbf{u} \in K$ such that

$$a(\mathbf{u}, \mathbf{v}-\mathbf{u}) + \int_{\Gamma_c} j^0(s, u_t(s); v_t(s) - u_t(s))\, ds \geq \langle \mathbf{g}, \mathbf{v}-\mathbf{u}\rangle \quad \forall\, \mathbf{v} \in K. \tag{27}$$

Here, the notation $j^0(x;z)$ stands for the generalized directional derivative of the superpotential j at x in direction z defined by

$$j^0(x;z) = \limsup_{y\to x, t\to 0^+} \frac{j(y+tz)-j(y)}{t}.$$

The existence of a solution follows from Theorem 4.1 in Section 4.1 as the functional $\varphi\colon V\times V \to \mathbb{R}$ defined by

$$\varphi(\mathbf{u},\mathbf{v}) = \int_{\Gamma_c} j^0(s, u_t(s); v_t(s) - u_t(s))\, ds$$

is continuous with respect to the second argument by [16, Prop. 2.1.1], satisfies the assumption (A^{φ}) (for details see [45]) as well as the condition (A^s). The latter condition (A^s) demands, together with the Cauchy-Schwarz inequality, that for all $y \in K^\infty \cap Y$ with $\|y\| = 1$, it holds

$$-1 \le \frac{1}{\|g\|_*}\, \langle g, y\rangle < -\frac{c_\varphi}{\|g\|_*}.$$

This means that the directions y of escape should stay in a given angle range with the applied force g, and moreover g should be large enough, $\|g\|_* > c_\varphi$. Therefore, in our numerical experiments the applied boundary load P is of order 10^{10}.

To conclude this subsection, let us note that in the *coercive* situation (i.e. with a Dirichlet condition on some boundary part) the assumption of one-sided Lipschitz continuity (see [42]): There exists a constant α_0, small enough compared to the constant of coercivity of the bilinear form of linear elasticity, such that for any $\mathbf{u}, \mathbf{v} \in V$ it holds

$$\varphi(\mathbf{u},\mathbf{v}) + \varphi(\mathbf{v},\mathbf{u}) \le \alpha_0 \|\mathbf{u}-\mathbf{v}\|^2, \tag{28}$$

guarantees uniqueness. In [42] has been shown that if the multivalued friction law includes only non-negative jumps at the points of discontinuity, as ∂j plotted in Figure 5.1(right), the sufficient condition for uniqueness (28) is globally satisfied.

5.2. Approximation of a nested max-min function

Now we extend the smoothing procedure from [41, 43, 44] to provide smoothing approximations of nested max-min functions. We describe a class of smoothing methods, which can be applied to overcome the non-differentiabilities in nested functions. As a result we obtain a smoothing approximation based on a single regularized parameter. Let

$$\mathbb{R}_+ = \{\varepsilon \in \mathbb{R} : \varepsilon \ge 0\}, \quad \mathbb{R}_{++} = \{\varepsilon \in \mathbb{R} : \varepsilon > 0\}.$$

We start with the maximum function $f\colon \mathbb{R}^n \to \mathbb{R}$,

$$f(x) = \max\{g_1(x), g_2(x), \dots, g_m(x)\}$$

of m continuous functions g_i.

This function is locally Lipschitz continuous and can be expressed by means of the plus function $\mathrm{p}\colon \mathbb{R} \to \mathbb{R}_+$, $p(x) = x^+ = \max(x, 0)$ following [10] as

$$f(x) = g_1(x) + \mathrm{p}[g_2(x) - g_1(x) + \ldots + \mathrm{p}[g_m(x) - g_{m-1}(x)]]. \tag{29}$$

Let $P\colon \mathbb{R}_{++} \times \mathbb{R} \to \mathbb{R}$ be the smoothing function via convolution for the plus function p:

$$P(\varepsilon, t) = \int_{\mathbb{R}} \mathrm{p}(t - \varepsilon s)\rho(s)\, ds,$$

where $\rho\colon \mathbb{R} \to \mathbb{R}_+$ is a density function such that

$$k := \int_{\mathbb{R}} |s|\rho(s)\, ds < \infty. \tag{30}$$

Replacing p by its approximation $P(\varepsilon, t)$ in (29) we obtain the smoothing approximation $S_0\colon \mathbb{R}^n \times \mathbb{R}_{++} \to \mathbb{R}$ for the maximum function defined by

$$S_0(x, \varepsilon) := g_1(x) + P[\varepsilon, g_2(x) - g_1(x) + \ldots + P[\varepsilon, g_m(x) - g_{m-1}(x)]]. \tag{31}$$

Using (31), the minimum of m continuous functions can be expressed similarly:

$$\begin{aligned}
&\min\{g_1(x), g_2(x), \ldots, g_m(x)\} = -\max\{-g_1(x), -g_2(x), \ldots, -g_m(x)\}\\
&\approx -[-g_1(x) + P[\varepsilon, -g_2(x) + g_1(x) + \ldots + P[\varepsilon, -g_m(x) + g_{m-1}(x)]]\\
&=: S_1(x, \varepsilon).
\end{aligned} \tag{32}$$

Now, we extend this procedure to the nested max-min functions. Let

$$j(x) = \max\{g_1(x), g_2(x), \ldots, g_r(x), \min\{g_{r+1}(x), \ldots, g_m(x)\}\}.$$

By (32), we have first $\min\{g_{r+1}(x), \ldots, g_m(x)\} \approx S_1(x, \varepsilon)$ and then by (31),

$$\begin{aligned}
j(x) &\approx \max\{g_1(x), g_2(x), \ldots, g_r(x), S_1(x, \varepsilon)\}\\
&\approx g_1(x) + P[\varepsilon, g_2(x) - g_1(x) + \ldots + P[\varepsilon, S_1(x, \varepsilon) - g_r(x)]].
\end{aligned} \tag{33}$$

The resulting approximation (33) is implemented recursively.

In our numerical examples we use the Zang probability density function [56]

$$\rho(t) = \begin{cases} 1 & \text{if } -\frac{1}{2} \le t \le \frac{1}{2} \\ 0 & \text{otherwise}, \end{cases}$$

which leads to

$$P(\varepsilon, t) = \int_{\mathbb{R}} \mathrm{p}(t - \varepsilon s)\rho(s)\, ds = \begin{cases} 0 & \text{if} \quad t < -\frac{\varepsilon}{2} \\ \frac{1}{2\varepsilon}(t + \frac{\varepsilon}{2})^2 & \text{if } -\frac{\varepsilon}{2} \le t \le \frac{\varepsilon}{2} \\ t & \text{if} \quad t > \frac{\varepsilon}{2}. \end{cases} \tag{34}$$

The regularized problem of (27) is given now as follows: Find $\mathbf{u}_\varepsilon \in K$ such that

$$a(\mathbf{u}_\varepsilon, \mathbf{v} - \mathbf{u}_\varepsilon) + \langle DJ_\varepsilon(\mathbf{u}_\varepsilon), \mathbf{v} - \mathbf{u}_\varepsilon \rangle \geq \langle \mathbf{g}, \mathbf{v} - \mathbf{u}_\varepsilon \rangle \quad \forall \mathbf{v} \in K, \tag{35}$$

where $DJ_\varepsilon \colon V \to V^*$ is the Gâteaux derivative of

$$J_\varepsilon(\mathbf{v}) = \int_{\Gamma_c} S(s, v_t(s), \varepsilon)\, ds$$

defined by
$$\langle DJ_\varepsilon(\mathbf{u}), \mathbf{v} \rangle = \int_{\Gamma_c} \frac{\partial S}{\partial \xi}(s, u_t(s), \varepsilon) v_t(s)\, ds.$$

Here $S(s, \xi, \varepsilon) \colon \Gamma_c \times \mathbb{R} \times \mathbb{R}_{++} \to \mathbb{R}$ is the smoothing approximation of the nested max-min function j as defined in (33) based on (34).

5.3. Discretization by finite element methods

We consider the standard triangulation $\{\mathcal{T}_h\}$ of Ω: first we divide Ω into small squares of the size h and then each square by its diagonal (from left to the right) into two triangles. Let Σ_h be the set of all vertices of the triangles of $\{\mathcal{T}_h\}$ and $\mathcal{P}_h^c$ the set of nodes on $\overline{\Gamma}_c$, i.e.

$$\mathcal{P}_h^c = \{x_i \in \Sigma_h \,:\, x_i \in \overline{\Gamma}_c\}.$$

Using the simplest finite element ansatz functions, namely continuous piecewise linear functions, the space V and the set of admissible displacements K are approximated, respectively, by

$$\begin{aligned} V_h &= \{v_h \in C(\overline{\Omega}; \mathbb{R}^2) \,:\, v_{h|_T} \in (\mathbb{P}_1)^2 \;\forall T \in \mathcal{T}_h\}, \\ K_h &= \{v_h \in V_h \,:\, v_{h2}(x_i) \geq 0 \;\forall x_i \in \mathcal{P}_h^c\}. \end{aligned}$$

Let us note that piecewise polynomial approximation of higher than linear order leads to nonconforming approximation; see [28, 42] for more details.

Thus the finite element discretization of the regularized problem (35) reads as follows: Find $u_{\varepsilon,h} \in K_h$ such that

$$a(u_{\varepsilon,h}, v_h - u_{\varepsilon,h}) + \langle DJ_h(u_{\varepsilon,h}), v_h - u_{\varepsilon,h} \rangle \geq \langle g, v_h - u_{\varepsilon,h} \rangle \quad \forall v_h \in K_h,$$

where using the trapezoidal rule for numerical quadrature

$$\begin{aligned} \langle DJ_h(u_{\varepsilon,h}), v_h \rangle = \frac{1}{2} \sum_{x_i \in \overline{\Gamma}_c} |x_i x_{i+1}| \big[&\frac{\partial S}{\partial \xi}(x_i, u_{\varepsilon,h1}(x_i), \varepsilon) v_{h1}(x_i) \\ &+ \frac{\partial S}{\partial \xi}(x_{i+1}, u_{\varepsilon,h1}(x_{i+1}), \varepsilon) v_{h1}(x_{i+1}) \big]. \end{aligned} \tag{36}$$

Convergence is guaranteed by Theorem 4.3 as $t = (\varepsilon, h) \to (0, 0)$.

For the numerical solution of the discrete problem we need its algebraic representation. Let n be the number of all nodes of triangulation and q the number of all contact nodes lying on the boundary $\overline{\Gamma}_c$. The space V_h is isomorphic with $\mathbb{R}^s$, where $s = 2n$, and the set K_h can be identified with $\mathcal{K}$ defined as

$$\mathcal{K} = \{v \in \mathbb{R}^s \, : \, \mathcal{A}v \geq 0\}.$$

Here $\mathcal{A} \in \mathbb{R}^{q,s}$ is given by $\mathcal{A} = (\underbrace{\mathcal{O}}_{q\times(s-q)} | \underbrace{I}_{q\times q})$ with the zero matrix $\mathcal{O}$ and the identity matrix I.

Let $v \in \mathbb{R}^s$ be the nodal value vector of $v_h = (v_{h1}, v_{h2}) \in V_h$ splitted into horizontal and vertical components. Moreover, let $v_i \in \mathbb{R}^{s-r}$, $v_c \in \mathbb{R}^r$, $r = 2q$ be subvectors of v containing the nodal values of v_h at the internal nodes x_i and at the nodes $x_i \in \overline{\Gamma}_c$, respectively. Assume also that the components of v_c are listed last in v, so $v = (v_i, v_c)^T$ and $v_c = (v_t^c|_{\Gamma_c}, v_n^c|_{\Gamma_c})^T$. Thus, we arrive at the following finite-dimensional variational inequality: Find $u \in \mathcal{K}$ such that

$$\langle \mathrm{A}u - \mathrm{g}, v - u\rangle + \langle \mathrm{C}(u_t^c), v_t^c - u_t^c\rangle \geq 0 \quad \forall v \in \mathcal{K}, \tag{37}$$

where $\mathrm{A} \in \mathbb{R}^{s,s}$ and $\mathrm{g} \in \mathbb{R}^s$ and $\mathrm{C}\colon \mathbb{R}^q \to \mathbb{R}^q$ is defined by

$$\mathrm{C}(u_t) = \begin{pmatrix} \frac{h}{2}\,\frac{\partial S(x_1, u_t^{c1}, \varepsilon)}{\partial \xi} \\ h\,\frac{\partial S(x_2, u_t^{c2}, \varepsilon)}{\partial \xi} \\ \dots \\ \frac{h}{2}\,\frac{\partial S(x_q, u_t^{cq}, \varepsilon)}{\partial \xi} \end{pmatrix}.$$

Then we decompose A and g as follows: $\mathrm{A} = \begin{pmatrix} \mathrm{A}_{ii} & \mathrm{A}_{ic} \\ \mathrm{A}_{ci} & \mathrm{A}_{cc} \end{pmatrix}$, $\mathrm{g} = \begin{pmatrix} \mathrm{g}_i \\ \mathrm{g}_c \end{pmatrix}$, and obtain the following equivalent formulation: Find $(u_i, u_c) \in \mathbb{R}^{s-r} \times \mathbb{R}^r$ such that

$$\langle \mathrm{A}_{ci}u_i + \mathrm{A}_{cc}u_c + \begin{pmatrix} 0 \in \mathbb{R}^q \\ \mathrm{C}(u_t^c) \in \mathbb{R}^q \end{pmatrix}, v_c - u_c\rangle \geq \langle \mathrm{g}_c, v_c - u_c\rangle \tag{38}$$

and
$$\mathrm{A}_{ii}u_i + \mathrm{A}_{ic}u_c = \mathrm{g}_i.$$

Further, due to the condensation technique we obtain the following reduced finite-dimensional variational inequality problem formulated only in terms of the contact displacements u_c: Find $u_c \in \tilde{\mathcal{K}}$ such that

$$\langle F(u_c), v_c - u_c\rangle \geq 0 \quad \forall v_c \in \tilde{\mathcal{K}},$$

where $F\colon \mathbb{R}^r \to \mathbb{R}^r$ is defined by

$$F(u_c) := \tilde{\mathrm{A}}u_c + \begin{pmatrix} 0 \\ \mathrm{C}(u_t^c) \end{pmatrix} - \tilde{\mathrm{g}} \quad \text{and} \quad \tilde{\mathcal{K}} := \{v_c \in \mathbb{R}^r : \tilde{\mathcal{A}}v_c \geq 0\},$$

with
$$\tilde{\mathcal{A}} := (\underbrace{\mathcal{O}}_{q\times q} | \underbrace{I}_{q\times q}),$$

The last problem is further reformulated as a mixed complementarity problem and then, by means of the Fischer-Burmeister function $f(a, b) = \sqrt{a^2 + b^2} - (a+b)$, as a system of nonlinear equations. Finally, by applying the natural merit function $\frac{1}{2}\|\cdot\|^2$, we obtain an equivalent smooth unconstrained minimization problem.

5.4. Numerical results

In our numerical examples we have used four different triangulations corresponding to $h = 1/8, 1/16$, $1/32$ and $1/64$. Thus the number of the contact nodes is $q = 9, 17, 33$ and 65, respectively.

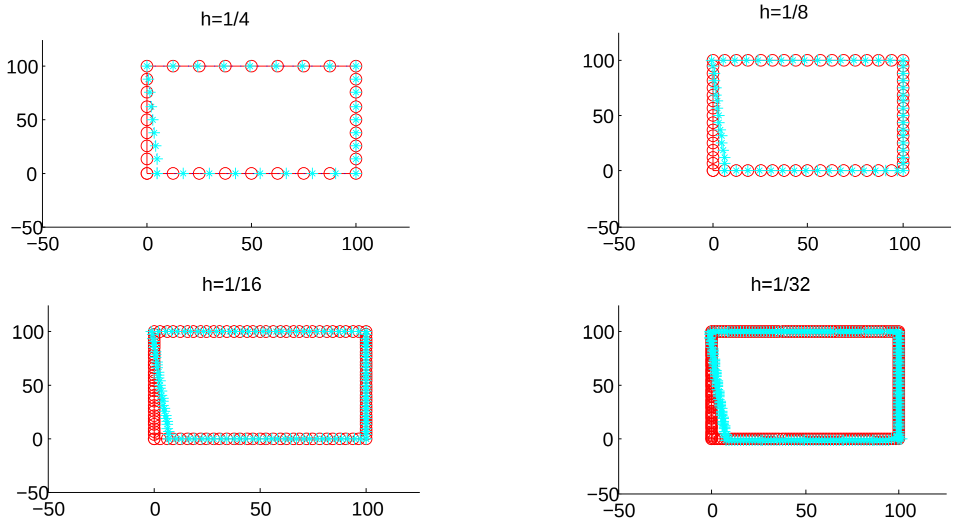
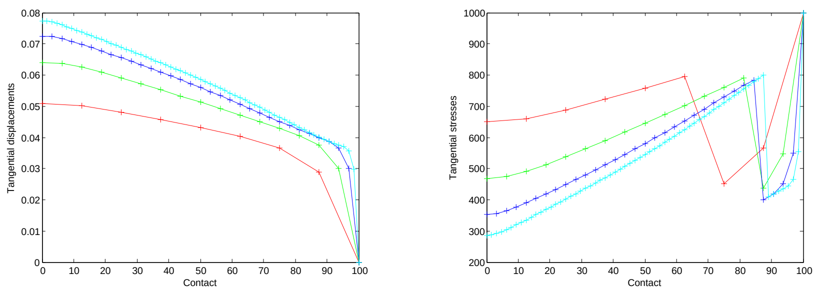


Figure 5.3: The complete displacement field on the whole boundary Γ.

The numerical experiments show that variations of ε in the range of $(10^{-5}, 10^{-1})$ do not have any noticeable influence on the numerical solution. The numerical results obtained with $\varepsilon = 0.1$.are collected in Figures 5.3 and 5.4. They show the behavior of the tangential component of the displacement vector and the distribution of the tangential component $-\sigma_t$ of the stress vector along the contact boundary Γ_c. In particular, Figure 5.3 displays the deformation of the benchmark model for four different mesh sizes $h = 1/4$, $1/8$, $1/16$ and $1/32$.



(a) u_1 on Γ_c. (b) $-\sigma_t$ on Γ_c.

Figure 5.4: The left image shows the tangential component u_1 on Γ_c for four discretization parameters $h{=}1/4$ (*red*), $h{=}1/8$ (*green*), $h{=}1/16$ (*dark blue*), $h{=}1/32$ (*light blue*). The right image shows the distribution of the tangential stress $-\sigma_t$ along Γ_c for the same four scenarios.

The tangential component u_1 along Γ_c is captured in Figure 5.4(a). The computed tangential stress $-\sigma_t$ along Γ_c is shown in Figure 5.4(b). The numerical results obtained reflect the non-monotone friction law from Figure 5.1(right) well.

6. Conclusions – an outlook

A problem of its own right is to follow the stream of generalized convexity and relax the convexity of functions/sets in semi-coercive linear variational inequalities, see [34] for recent work in finite dimensions.

Here we worked in the primal form of the variational problems that comes from an energy formulation. There are also mixed forms involving Lagrangians, see [29], [30, section 11.4], what is unexplored in the semi/no-coercive situation.

In our paper we focused to problems of linear elasticity. But often nonlinear materials arise in the applications that can be modelled by monotone operators with non quadratic growths. This leads to more general semi-coercive VI-HVIs in reflexive Banach spaces, see [15]. However, there only weak convergence results for a related approximation procedure are obtained. So it remains to establish strong convergence under appropriate conditions.

As we have seen in our approximation results, res ipsa loquitur norm convergence hinges on uniqueness. For uniqueness in specially structured semi-coercive problems with linear VIs of first kind, we can refer to [51]. So uniqueness in nonlinear semi-coercive unilateral friction problems is a challenging research direction.

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