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Mountain Pass Solutions for Nonsmooth Elliptic Problems

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We study the existence of solutions for degenerate nonlinear elliptic equations using nonsmooth mountain pass arguments both with and without symmetry, where the weak slope is taken to define critical points. First we consider a class of problems where the leading elliptic term $A(x, Du)$ satisfies usual p-growth conditions (for $p > 1$) but A is merely convex and not necessarily differentiable in the gradient. Secondly, we show the existence of a sequence of solutions for a class of problems involving the highly singular 1-Laplacian as leading term. The results demonstrate that differentiablility of the leading term is not needed for typical results of elliptic problems.

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1. Introduction

A typical problem in mountain pass theory is the determination of critical points of indefinite functionals of the type

$$E(u) = \int_\Omega A(x, Du)\, dx - \int_\Omega F(x, u)\, dx \tag{1}$$

where $A(x, \cdot)$ is convex and

$$F(x, u) := \int_0^u f(x, s)\, ds\,.$$

First investigations due to Ambrosetti and Rabinowitz considered $A(x, Du) = |Du|^2$ (cf. [1], [22]). The extension to $A(x, Du) = |Du|^p$ (e.g. by Mawhin et al. [13])

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provided solutions of the elliptic equation involving the p-Laplace operator

$$-\Delta_p u := -\operatorname{div} |Du|^{p-2} Du = f(x,u)\,. \tag{2}$$

For more general cases convexity of $A(x,\cdot)$ is important, since it is related to ellipticity, but also differentiability of $A(x,\cdot)$ was assumed previously. In this way classical smooth methods have been extended to get solutions of

$$-\operatorname{div} a(x,u) = f(x,u)$$

where $a(x,u) = D_u A(x,u)$ (cf. Colasuonno et al. [7], Kristály et al. [16], Mihăilescu [20], De Nápoli and Mariani [9]). In Filippakis et al. [14] the case of the p-Laplacian is treated where, however, the lower order term in (1) is merely a nonsmooth potential. A quasilinear version of the previous problem can be found in Canino and Degiovanni [4] and in Canino [3].

In the present paper we want to demonstrate that smoothness of the leading term $A(x,\cdot)$ is not needed for the mountain pass arguments and that also the usually assumed continuity of $f(x,\cdot)$ can be dropped. Using the concept of weak slope (cf. Degiovanni and Marzocchi [10]) and other tools from nonsmooth analysis, we obtain critical points of E for a wide class of nonsmooth integrands A and F. In particular, the two functionals appearing in E in (1) are merely locally Lipschitz continuous and we obtain weak solutions of the Euler-Lagrange equation

$$-\operatorname{div} a(x) = \mathsf{f}(x) \tag{3}$$

where a and f are selections related to u by

$$a(x) \in \partial A(x, Du(x))\,, \quad \mathsf{f}(x) \in [\underline{f}(x,u(x)), \overline{f}(x,u(x))] \quad \text{a.e. on } \Omega$$

with ∂A denoting the convex subdifferential of $A(x,\cdot)$ and

$$\underline{f}(x,t) = \operatorname{ess}\liminf_{s\to t} f(x,s)\,, \quad \overline{f}(x,t) = \operatorname{ess}\limsup_{s\to t} f(x,s)\,.$$

First we consider general $A(x,\cdot)$ having some typical p-growth for $p > 1$. We obtain a classical mountain pass solution $u \in W_0^{1,p}(\Omega)$ in Theorem 4.1 and, in the symmetric case, we verify the existence of a sequence of mountain pass solutions $u_k \in W_0^{1,p}(\Omega)$ in Theorem 4.3. Here, despite the possible nonsmoothness of $A(x,\cdot)$, we also weaken convexity conditions used in previous treatments.

Finally we also study the degenerate case $A(x,Du) = |Du|$ related to $p = 1$. Here it turns out that we have to work in $BV(\Omega)$ instead of $W_0^{1,1}(\Omega)$ and that we have to consider homogeneous boundary conditions in a more general way. In the symmetric case we obtain the existence of mountain pass solutions $u_k \in BV(\Omega)$ in Theorem 5.1. This gives in particular solutions of (3) with the 1-Laplace operator $\operatorname{div} \frac{Du}{|Du|}$ on the left hand side, that needs however a very careful interpretation. For this case let us still refer to Marzocchi [19] who considered critical points for smooth F, and Degiovanni et al. [11] who obtained multiple solutions of hemivariational inequalities.

We always define critical points by means of the weak slope $|dE|(u)$ (cf. Degiovanni and Marzocchi [10]) which turns out to be a very powerful tool to treat the generality encountered in our problems. For the treatment of the problem involving the 1-Laplace operator, where E is the sum of a convex lower semicontinuous and a locally Lipschitz continuous function, it seems to be the only reasonable method currently available. The other problems, where E is locally Lipschitz continuous, could also be treated by methods developed in Chang [5] on the basis of Clarke's generalized gradient $\partial E(u)$. We however refrain from doing that, since $0 \in \partial E(u)$ provides more critical points than $|dE|(u) = 0$ in general and $0 \in \partial E(u)$ might be satisfied for u that shouldn't be considered as critical (cf. Example 1.4 in Ribarska et al. [23]). Moreover the weak slope, that is designed for the treatment of critical points, will be still available for natural generalizations of the leading term A where E is not locally Lipschitz continuous anymore.

Notation. For a Banach space X let X^* be its dual and $\langle \cdot, \cdot \rangle$ the corresponding duality form. $B_\delta(u)$ is the open ball of radius δ centered at u. We write $\operatorname{sgn} s$ for the real sign function, $|x|$ for the Euclidean norm, $x \cdot y$ for the scalar product on $\mathbb{R}^n$, and $|\Omega|$ for the Lebesgue measure of $\Omega \subset \mathbb{R}^n$. By $L^p(\Omega)$ we denote the usual space of p-integrable functions and by $\|u\|_p$ its norm. $W^{1,p}(\Omega)$ stands for the p-integrable functions having p-integrable weak derivatives and $W_0^{1,p}(\Omega)$ is the closure of smooth functions with compact support in $W^{1,p}(\Omega)$. $BV(\Omega)$ denotes the space of functions of bounded variation. $u = u^+ - u^-$ is the decomposition of function u into positive and negative part. We set for $f \colon \mathbb{R} \to \mathbb{R}$

$$\operatorname*{ess\,lim\,inf}_{s \to t} f(s) := \lim_{\delta \to 0} \operatorname*{ess\,inf}_{|s-t|<\delta} f(s) , \quad \operatorname*{ess\,lim\,sup}_{s \to t} f(s) := \lim_{\delta \to 0} \operatorname*{ess\,sup}_{|s-t|<\delta} f(s) .$$

For an extended valued function E we use $\mathcal{D}(E)$ for its effective domain of definition. $\partial E(u)$ denotes the convex subdifferential for a convex function E and Clarke's generalized gradient for a locally Lipschitz continuous function E.

2. Assumptions and preliminary material

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $W_0^{1,p}(\Omega)$ be the usual Sobolev space equipped with the norm $\|u\|^p = \int_\Omega |Du|^p \, dx$. We consider functionals of the form

$$E(u) := E_0(u) - E_1(u) := \int_\Omega A(x, Du) \, dx - \int_\Omega F(x, u) \, dx$$

where

$$F(x, s) := \int_0^s f(x, t) \, dt . \tag{4}$$

Let us first formulate typical assumptions and basic consequences we want to use for the analysis of E.

(A) $A \colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ with $A(x, \cdot)$ convex for a.e. $x \in \Omega$, and $A(\cdot, g)$ measurable for all $g \in \mathbb{R}^n$. (Notice that $A(x, \cdot)$ is continuous for a.e. $x \in \Omega$ and that the subdifferential $\partial A(x, g)$ of $A(x, \cdot)$ at g is nonempty for a.e. $x \in \Omega$ and all $g \in \mathbb{R}^n$.)

(A1) There are $c_1, c_2 > 0$ and $p > 1$ such that for a.e. $x \in \Omega$

(a) $A(x, 0) = 0\,,$

(b) $0 \le \langle a, g\rangle \le pA(x, g)$ for all $g \in \mathbb{R}^n$, $a \in \partial A(x, g)\,,$

(c) $A(x, g) \ge c_1|g|^p$ for all $g \in \mathbb{R}^n$,

(d) $|a| \le c_2(1 + |g|^{p-1})$ for all $a \in \partial A(x, g)$, $g \in \mathbb{R}^n$.

(A2) $A(x, \cdot)$ is strictly convex on $\mathbb{R}^n$ for a.e. $x \in \Omega\,.$

(A3) Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable such that for some $c_3 > 0$

(a) $|f(x, s)| \le c_3(1 + |s|^{q-1})$ for a.e. $x \in \Omega$ and a.e. $s \in \mathbb{R}$ where $p < q < p^* = \frac{np}{n-p}$ if $p < n$ or $p < q < \infty$ if $p \ge n\,,$

(b) for $F(x, s) = \int_0^s f(x, t)\,dt$ there is $s_0 > 0$ and $\vartheta > p$ such that $0 < \vartheta F(x, s) \le f(x, s)s$ for a.e. $|s| \ge s_0$ and a.e. $x \in \Omega\,.$

(A4) There is some $\lambda < p\, c_1 \lambda_{1,p}$ with

$$\operatorname{ess}\limsup_{s\to 0} \frac{f(x, s)}{|s|^{p-2}s} \le \lambda \quad \text{uniformly for a.e. } x \in \Omega$$

where $\lambda_{1,p}$ is the first eigenvalue of the p-Laplace operator in $W_0^{1,p}(\Omega)$,

i.e.
$$\lambda_{1,p} = \inf_{\substack{u\in W_0^{1,p}(\Omega)\\ u\neq 0}} \frac{\int_\Omega |Du|^p\,dx}{\int_\Omega |u|^p\,dx}\,. \tag{5}$$

Remark 2.1. Let us mention that in previous works instead of strict convexity as in (A2) the much stronger condition of p-uniform convexity was used, i.e. that there is some $c_4 > 0$ such that

$$A\Big(x, \frac{g_1 + g_2}{2}\Big) \le \frac{1}{2}A(x, g_1) + \frac{1}{2}A(x, g_2) - c_4|g_1 - g_2|^p \tag{6}$$

for all $x \in \Omega$, $g_1, g_2 \in \mathbb{R}^n$ (cf. Nápoli and Mariani [9]).

Let us now present some *examples* for functions A that meet our assumptions. Denoting the q-norm of $g \in \mathbb{R}^n$ by $|g|_q$, we have that

$$A(x, g) = |g|_1^p \ \text{ or } \ A(x, g) = |g|_\infty^p$$

are nonsmooth examples satisfying the conditions (A) and (A1). Strictly convex nonsmooth examples are given by

$$A(x, g) = (|g|_1 + |g|_2)^p \ \text{ or } \ A(x, g) = (|g|_\infty + |g|_2)^p$$

(cf. Lucia and Schuricht [18] for further details).

Lemma 2.2. *If A satisfies* (A), (A1a), (A1d), *then we have for all $g \in \mathbb{R}^n$, $a_g \in \partial A(x, g)$, and a.e. $x \in \Omega$:*

(1) $A(x, g) \le \langle a_g, g\rangle \le c_2(|g| + |g|^p)$ *and*

(2) $|A(x, g)| \le c_2(|g| + |g|^p)$.

Proof. By convexity we have for fixed $g \in \mathbb{R}^n$ and all $a_g \in \partial A(x,g)$ that

$$A(x,0) - A(x,g) \geq \langle a_g, 0-g \rangle \quad \text{and} \quad A(x,g) - A(x,0) \geq \langle a_0, g-0 \rangle\,.$$

Hence $$\langle a_0, g \rangle \leq A(x,g) \leq \langle a_g, g \rangle\,.$$

Using (A1d) we readily get

$$A(x,g) \leq \langle a_g, g \rangle \leq c_2(1+|g|^{p-1})|g| = c_2(|g|+|g|^p)\,.$$

Analogously, assertion (2) follows. □

The next lemma generalizes Colasuonno et al. [7, Lemma 2.4].

Lemma 2.3. *Assume that A satisfies* (A), (A1a), (A1c), (A1d), (A2) *for $x \in \Omega$. Moreover let $g_n, g \in \mathbb{R}^n$ and $a_n \in \partial A(x,g_n)$, $a \in \partial A(x,g)$ be such that we have*

$$\langle a_n - a, g_n - g \rangle \to 0 \ \text{ as } \ n \to \infty\,.$$

Then $g_n \to g$.

Proof. Using Lemma 2.2 we have

$$\begin{aligned} \langle a_n - a, g_n - g \rangle &= \langle a_n, g_n \rangle + \langle a, g \rangle - \langle a_n, g \rangle - \langle a, g_n \rangle \\ &\geq A(x,g_n) + \langle a, g \rangle - |a_n|\,|g| - |a|\,|g_n| \\ &\geq c_1|g_n|^p + \langle a, g \rangle - c_2(1+|g_n|^{p-1})|g| - |a|\,|g_n|\,. \end{aligned}$$

If g_n would be unbounded, the right hand side would go to infinity (up to a subsequence) which contradicts the assumption. Hence, $\{g_n\}$ is bounded and, by (A1d), also $\{a_n\}$. Thus there is a subsequence (denoted the same way) such that $g_n \to \tilde{g}$ and $a_n \to \tilde{a}$. We have that $\tilde{a} \in \partial A(x,\tilde{g})$ (cf. Clarke [6, Propositions 2.1.5, 2.2.6, 2.2.7]) and

$$\langle a_n - a, g_n - g \rangle \to \langle \tilde{a} - a, \tilde{g} - g \rangle = 0\,.$$

By strict convexity we obtain that $\tilde{g} = g$. With the subsequence principle we finally get the assertion. □

Lemma 2.4. *We have:*

(1) *If A satisfies* (A) *and* (A1b)*, then one has for a.e. $x \in \Omega$*

$$A(x,tg) \leq t^p A(x,g) \quad \text{for all } g \in \mathbb{R}^n,\ t \geq 1\,.$$

(2) *Let f satisfy* (A3a)*. Then $F(\cdot,\cdot)$ defined by* (4) *is a Carathéodory function (i.e. $F(\cdot,s)$ Lebesgue measurable for all $s \in \mathbb{R}$ and $F(x,\cdot)$ continuous for a.e. $x \in \Omega$). Moreover $E_1 : L^q(\Omega) \to \mathbb{R}$ is Lipschitz continuous on bounded sets.*

(3) *Let f satisfy* (A3b) *and let F be as in* (4)*. Then one has for a.e. $x \in \Omega$*

$$F(x,ts) \geq t^{\vartheta} F(x,s) \quad \text{for all } |s| \geq s_0,\ t \geq 1\,.$$

(4) *Let f satisfy* (A3) *and let F be as in* (4). *Then for any $w \in L^q(\Omega)$, $w \neq 0$, there are $t_0 \geq 1$ and γ_1, $\gamma_2 > 0$ such that*

$$E_1(tw) = \int_\Omega F(x, tw)\, dx \geq t^\vartheta \gamma_1 - \gamma_2 \quad \textit{for all } t \geq t_0\,.$$

Notice that (A3a) combined with assertion (2) implies $\vartheta \leq q$ and that t_0 and γ_1 in assertion (3) depend on w.

Proof. For (1) we consider $\varphi(t) := A(x, tg)$ with $g \neq 0$ (for $g = 0$ the assertion is trivial). Since $A(x, \cdot)$ is locally Lipschitz continuous, φ is Lipschitz continuous on bounded intervals and, thus, absolutely continuous. We have

$$\varphi'(t) = \langle a_{tg}, g\rangle = \frac{1}{t}\,\langle a_{tg}, tg\rangle \overset{\text{(A1b)}}{\leq} \frac{p}{t}\, A(x, tg) = \frac{p}{t}\,\varphi(t) \quad \text{for a.e. } t \geq 1$$

where $a_{tg} \in \partial A(x, tg)$. Hence $\frac{\varphi'(t)}{\varphi(t)} \leq \frac{p}{t}$ a.e. and a simple integration implies the statement.

For (2) we have that $F(\cdot, s)$ is Lebesgue measurable for all $s \in \mathbb{R}$ and $F(x, \cdot)$ is absolutely continuous, thus F is a Carathéodory function. Obviously $E_1(0)$ is finite by (A3a) and for $\|u\|_q, \|v\|_q \leq R$ we get by using Hölders inequality

$$\begin{aligned} |E_1(u) - E_1(v)| &\leq \int_\Omega \Big| \int_{v(x)}^{u(x)} |f(x,t)|\, dt \Big|\, dx \;\leq\; c_3 \int_\Omega \Big| \int_{v(x)}^{u(x)} 1 + |t|^{q-1}\, dt \Big|\, dx \\ &\leq c_3 \int_\Omega |u(x) - v(x)| + |u(x) - v(x)| \big(|u(x)| + |v(x)|\big)^{q-1}\, dx \\ &\leq c_3 \|u - v\|_1 + c_3 \big\| |u| + |v| \big\|_q^{q-1} \|u - v\|_q\, dx \leq \tilde{c} \|u - v\|_q \end{aligned} \tag{7}$$

for some $\tilde{c} > 0$. Hence E_1 is well-defined on $L^q(\Omega)$ and Lipschitz continuous on bounded sets (cf. also Chang [5] and Littig and Schuricht [17]).

For (3) we use $\varphi(t) := F(x, ts)$ which is absolutely continuous and, thus, differentiable almost everywhere with $\varphi'(t) = f(x, ts)s$. By (A3b) we have that $\frac{\varphi'(t)}{\varphi(t)} \geq \frac{\vartheta}{t}$ and a simple integration implies the assertion.

For (4) we first derive from (A3a) that $|f(x, s)| \leq \beta$ for a.e. $|s| \leq s_0$ and some $\beta > 0$. Hence, by the definition of F,

$$|F(x, s)| \leq |f(x, s)s| \leq \beta s_0 \quad \text{for a.e. } |s| \leq s_0\,. \tag{8}$$

Now, for fixed $w \in L^q(\Omega) \setminus \{0\}$, there is some $t_0 \geq 1$ such that $\Omega_0 := \{|t_0 w| \geq s_0\}$ has positive measure and we have $\int_{\Omega_0} F(x, t_0 w)\, dx =: \gamma > 0$. By $\Omega_0 \subset \{|tw| \geq s_0\}$ for $t \geq t_0$, we obtain

$$\begin{aligned} E_1(tw) &= \int_{\{|tw| \geq s_0\}} F(x, tw)\, dx + \int_{\{0 < |tw| < s_0\}} F(x, tw)\, dx \\ &\geq \int_{\Omega_0} F(x, \tfrac{t}{t_0} t_0 w)\, dx - \beta s_0\, |\Omega| \geq \frac{t^\vartheta}{t_0^\vartheta} \gamma - \beta s_0\, |\Omega| \quad \text{for all } t \geq t_0 \end{aligned}$$

and the assertion follows with $\gamma_1 := \frac{\gamma}{t_0^\vartheta}$, $\gamma_2 := \beta s_0 |\Omega|$. □

We now consider properties of

$$E_0(u) = \int_\Omega A(x, Du(x))\,dx\,.$$

Lemma 2.5. *Let* (A), (A1a), (A1d) *be satisfied. Then* $E_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$ *is convex and locally Lipschitz continuous. We have* $E^* \in \partial E_0(u)$ *if and only if there is some* $a \in L^{p'}(\Omega, \mathbb{R}^n)$ *with*

$$a(x) \in \partial A(x, Du(x)) \;\; a.e. \;\; in \;\; \Omega$$

such that
$$\langle E^*, v\rangle = \int_\Omega a(x)\cdot Dv(x)\,dx \quad for\ all\ \ v \in W_0^{1,p}(\Omega) \tag{9}$$

(∂A *denotes the convex subdifferential of* $A(x,\cdot)$*).*

Proof. Function E_0 is convex by (A). By Lemma 2.2 it is well defined on $W_0^{1,p}(\Omega)$ and it is bounded on bounded sets. Thus it is locally Lipschitz continuous on $W_0^{1,p}(\Omega)$. For the convex subdifferential see Lucia and Schuricht [18]. ☐

We say that ∂E_0 satisfies condition (S$_+$) if for any sequence $\{u_n\}$ in $W_0^{1,p}(\Omega)$

$$u_n \rightharpoonup u\,,\;\; E_n^* \in \partial E_0(u_n)\,,\;\; \limsup_{n\to\infty}\langle E_n^*, u_n - u\rangle \le 0 \quad\Longrightarrow\quad u_n \to u\,. \tag{S$_+$}$$

Lemma 2.6. *We have that* ∂E_0 *satisfies condition* (S$_+$) *if one of the following conditions is satisfied:*

(1) *A satisfies* (A), (A1a), (A1c), (A1d), *and* (A2).

(2) *A satisfies* (A), (A1a), (A1d), (6).

Remark 2.7. The first part of the lemma extends a result of Colasuonno, Pucci and Varga [7, Lemma 2.5] to the nonsmooth setting. Moreover the symmetry condition $A(x,g) = A(x,-g)$ can be removed and (A1d) relaxes the growth condition $\frac{\partial}{\partial g}A(x,g) \le \tilde{c}|g|^{p-1}$. The second assertion generalizes results of Nápoli and Mariani [9, Proposition 2.1] to the nonsmooth setting. Notice that the conditions in (2) do not imply the conditions in (1) in general.

Proof. We consider $\{u_n\}$ in $W_0^{1,p}(\Omega)$ with $u_n \rightharpoonup u$, and we let $E_n^* \in \partial E_0(u_n)$ be such that

$$\limsup_{n\to\infty}\langle E_n^*, u_n - u\rangle \le 0\,.$$

For the proof of (1) notice that E_0 is well-defined on $W_0^{1,p}(\Omega)$ by Lemma 2.2 and we have $E_0(u) \le \liminf_{n\to\infty} E_0(u_n)$ by weak lower semicontinuity. Convexity implies

$$\limsup_{n\to\infty} E_0(u_n) \le E_0(u) + \limsup_{n\to\infty}\langle E_n^*, u_n - u\rangle \le E_0(u)\,.$$

Thus $E_0(u_n) \to E_0(u)$ and, consequently, $\lim\limits_{n\to\infty}\langle E_n^*, u_n - u\rangle = 0\,.$

Clearly $\langle E^*, u_n - u\rangle \to 0$ for $E^* \in \partial E_0(u)$. With $a_n(\cdot)$, $a(\cdot)$ corresponding to E_n^*, E^*, respectively, we have

$$0 = \lim_{n\to\infty} \langle E_n^* - E^*, u_n - u\rangle = \lim_{n\to\infty} \int_\Omega \langle a_n - a, Du_n - Du\rangle \, dx \, .$$

The integrand on the right hand side is nonnegative by the monotonicity due to convexity of $A(x,\cdot)$. Therefore we obtain $\langle a_n - a, Du_n - Du\rangle \to 0$ in $L^1(\Omega)$. Hence, up to a subsequence,

$$\langle a_n(x) - a(x), Du_n(x) - Du(x)\rangle \to 0 \quad \text{a.e. on } \Omega \, .$$

Lemma 2.3 implies that $Du_n(x) - Du(x) \to 0$ a.e. on Ω. By convexity,

$$0 \overset{\text{(A1c)}}{\leq} A\Big(x, \frac{Du_n(x) - Du(x)}{2}\Big) \leq \frac{A(x, Du_n(x)) + A(x, -Du(x))}{2}$$

for a.e. $x \in \Omega$. Since $E_0(u_n) \to E_0(u)$, generalized dominated convergence and (A1a) yield

$$0 = \lim_{n\to\infty} \int_\Omega A\Big(x, \frac{Du_n(x) - Du(x)}{2}\Big) \, dx \overset{\text{(A1c)}}{\geq} \frac{c_1}{2^p} \lim_{n\to\infty} \|u_n - u\|^p \, .$$

But this implies $u_n \to u$ and, using the subsequence principle, the assertion follows.

For the proof of (2) we argue as in the previous part to get $E_0(u_n) \to E_0(u)$. Since also $\frac{u+u_n}{2} \rightharpoonup u$, weak lower semicontinuity gives

$$E_0(u) \leq \liminf_{n\to\infty} E_0\Big(\frac{u+u_n}{2}\Big) \, .$$

From (6) we obtain

$$\limsup_{n\to\infty} c_4 \|u - u_n\|^p \leq \lim_{n\to\infty} \frac{E_0(u) + E_0(u_n)}{2} - \liminf_{n\to\infty} E_0\Big(\frac{u+u_n}{2}\Big) \leq 0$$

which implies $u_n \to u$ and the proof is complete. □

Finally we consider the generalized gradient of

$$E_1(u) = \int_\Omega F(x,u) \, dx$$

with F according to (4) and we use the notation

$$\underline{f}(x,t) := \text{ess} \liminf_{s\to t} f(x,s) \, , \quad \overline{f}(x,t) := \text{ess} \limsup_{s\to t} f(x,s) \, .$$

Lemma 2.8. *Let* (A3a) *be satisfied. Then, both as function on $L^q(\Omega)$ and on $W^{1,p}(\Omega)$, we have that E_1 is locally Lipschitz continuous and that for any $E^* \in \partial E_1(u)$ there is some $\mathsf{f} \in L^{q'}(\Omega)$ with*

$$\mathsf{f}(x) \in [\underline{f}(x,u(x)), \overline{f}(x,u(x))] \quad \textit{for a.e. } x \in \Omega \tag{10}$$

such that
$$\langle E^*, v\rangle = \int_\Omega \mathsf{f}(x)v(x)\,dx \quad \textit{for all } v \in L^q(\Omega)\,. \tag{11}$$

If $f(x,\cdot)$ is continuous, then $\partial E_1(u) = \{f(\cdot,u(\cdot))\}$ and E_1 is even continuously differentiable on $L^q(\Omega)$ and $W^{1,p}(\Omega)$ with

$$\langle E_1'(u), v\rangle = \int_\Omega f(x,u)\,v\,dx \quad \textit{for all } u,v \in L^q(\Omega) \textit{ or } W^{1,p}(\Omega)\,.$$

The last statement is included for completeness and can be used to specialize our general results to this case.

Proof. E_1 is locally Lipschitz continuous on $L^q(\Omega)$ by Lemma 2.4 and, by the continuous embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, it is also locally Lipschitz continuous on $W^{1,p}(\Omega)$.

From (A3a) and Example 2.2.5 in Clarke [6] we obtain that $F(x,\cdot)$ is locally Lipschitz continuous for a.e. $x \in \Omega$ and

$$\partial F(x,t) = [\underline{f}(x,t), \overline{f}(x,t)] \quad \text{for all } t \in \mathbb{R} \text{ and a.e. } x \in \Omega$$

where $\partial F(x,t)$ denotes the generalized gradient of $F(x,\cdot)$. Thus, for all $t \in \mathbb{R}$ and a.e. $x \in \Omega$,

$$|F^*| \le c_3(1 + |t|^{q-1}) \quad \text{for all } F^* \in \partial F(x,t)\,.$$

Recalling that $F(\cdot,t)$ is Lebesgue measurable for all $t \in \mathbb{R}$, we can apply Theorem 2.7.5 from [6] to get the structural assertion about $E^* \in \partial E_1(u)$ on $L^q(\Omega)$ (cf. also Chang [5] and Littig & Schuricht [17]).

Recall that the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous and notice that $W^{1,p}(\Omega)$ is a dense subset of $L^q(\Omega)$ (since $C_0^\infty(\Omega) \subset W^{1,p}(\Omega)$ is dense in $L^q(\Omega)$). Then, for $\tilde{E}_1 := E_{1|W^{1,p}}$, any $\tilde{E}^* \in \partial\tilde{E}_1(u) \subset W^{1,p}(\Omega)^*$ can be uniquely extended to $E^* \in \partial E_1(u) \subset L^q(\Omega)^* = L^{q'}(\Omega)$ by Proposition 5.1 in Aubin & Clarke [2]. But this verifies the structural assertion about $\partial E_1(u)$ on $W^{1,p}(\Omega)$.

For $f(x,\cdot)$ being continuous the assertion is basically known, but let us provide a brief proof for completeness. In this case $\partial E_1(u)$ reduces to a singleton. Hence E_1 is strictly differentiable with

$$\langle E_1'(u), v\rangle = \int_\Omega f(x,u)\,v\,dx \quad \text{for all } u,v \in L^q(\Omega) \text{ or } W^{1,p}(\Omega)$$

by Proposition 2.2.4 in [6].

In particular we have $f(\cdot, u(\cdot)) \in L^{\frac{q}{q-1}}(\Omega)$ and, with some $\tilde{c} > 0$,

$$\int_\Omega |f(x,u)|^{\frac{q}{q-1}}\, dx \le c_3 \int_\Omega \big(1 + |u|^{q-1}\big)^{\frac{q}{q-1}}\, dx \le \tilde{c} \int_\Omega 1 + |u|^q \, dx\,.$$

For continuity of E_1' we consider $u_n \to u$ in $L^q(\Omega)$. Then, up to a subsequence, $u_n(x) \to u(x)$ a.e. on Ω. As above we find some $\hat{c} > 0$ with

$$\int_\Omega |f(x,u_n) - f(x,u)|^{\frac{q}{q-1}}\, dx \le \hat{c} \int_\Omega 1 + |u_n|^q + |u|^q \, dx$$

and generalized dominated convergence implies

$$u(\cdot) \to f(\cdot, u(\cdot)) \in C(L^q(\Omega), L^{\frac{q}{q-1}}(\Omega))\,.$$

Thus $E_1 \in C^1(L^q(\Omega))$ (cf. also Rabinowitz [22, Proposition B.1]). Using the continuity of the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ we finally get $E_1 \in C^1(W^{1,p}(\Omega))$. $\square$

3. Nonsmooth critical point theory

In most previous investigations E is assumed to be differentiable and critical points are defined to be points u with $E'(u) = 0$. In our general setting where E is merely continuous or only lower semicontinuous (cf. Section 5 below), the notion of weak slope is a suitable tool to define and detect critical points. Let us briefly introduce the material as needed for our analysis (cf. Degiovanni and Marzocchi [10]).

Let X be a metric space endowed with metric d and let $E\colon X \to \mathbb{R}$ be continuous. For any $u \in X$ the *weak slope* $|dE|(u)$ is the supremum of all $\sigma \in [0, +\infty]$ such that there exists $\delta > 0$ and a continuous map $H\colon B_\delta(u) \times [0,\delta] \to X$ with the properties

$$d(H(v,t), v) \le t \quad \text{for all } \; v \in B_\delta(u),\; t \in [0,\delta]\,,$$

$$E(H(v,t)) \le E(v) - \sigma t \quad \text{for all } \; v \in B_\delta(u),\; t \in [0,\delta]\,.$$

Notice that $|dE| : X \to [0, +\infty]$ is lower semicontinuous.

Now, for a lower semicontinuous $E : X \to \mathbb{R} \cup \{+\infty\}$ we consider the epigraph

$$\mathrm{epi}(E) := \{(u,\xi) \in X \times \mathbb{R} \mid E(u) \le \xi\}$$

as a metric space with distance

$$d((u,\xi),(v,\zeta)) := \big(\, d(u,v)^2 + (\xi - \zeta)^2\,\big)^{1/2}\,.$$

Furthermore we consider the continuous function

$$\mathcal{G}_E : \mathrm{epi}(E) \to \mathbb{R} \quad \text{with} \quad \mathcal{G}_E(u,\xi) := \xi\,.$$

For any $u \in \mathcal{D}(E) := \{v \in X \mid E(v) < \infty\}$, the domain of definition, we define the weak slope as

$$|dE|(u) := \begin{cases} \dfrac{|d\mathcal{G}_E|(u, E(u))}{\sqrt{1 - |d\mathcal{G}_E|(u, E(u))^2}} & \text{if } |d\mathcal{G}_E|(u, E(u)) < 1\,, \\ +\infty & \text{if } |d\mathcal{G}_E|(u, E(u)) = 1\,. \end{cases}$$

If E is finite and continuous this definition is consistent with the previous one.

We call $u \in \mathcal{D}(E)$ a *critical point* of E if $|dE|(u) = 0$ and $E(u)$ is called the corresponding *critical value*. In general $\mathcal{G}_E$ might have critical points (v, ξ) with $E(v) < \xi$ that should not be critical points of E. This difficulty disappears if the *epigraph condition*

$$\inf\{|d\mathcal{G}_E|(v, \xi) \mid E(v) < \xi\} > 0 \tag{12}$$

is satisfied (cf. the discussion in [8, p. 163/164] or [4, p. 29/30] and notice that (12) is always satisfied for continuous functions). A sequence $\{u_k\}_{k\in\mathbb{N}}$ in $\mathcal{D}(E)$ is said to be a *Palais-Smale sequence* at level $c \in \mathbb{R}$ if

$$E(u_k) \to c \quad \text{and} \quad |dE|(u_k) \to 0\,.$$

We say that E satisfies a *Palais-Smale condition* at level c if every Palais-Smale sequence $\{u_k\}_{k\in\mathbb{N}}$ at level c for E has a convergent subsequence in X.

In our applications we consider functionals $E = E_0 - E_1$ on a Banach space X where E_0 is convex and lower semicontinuous and E_1 is locally Lipschitz continuous.

Lemma 3.1. *Let X be a Banach space, $E_0 : X \to \mathbb{R} \cup \{+\infty\}$ a convex lower semicontinuous function, $E_1 : X \to \mathbb{R}$ a locally Lipschitz continuous function, $E = E_0 - E_1$, and $u \in \mathcal{D}(E)$.*

(1) *If $E(u) < \xi$, then $|d\mathcal{G}_E|(u, \xi) = 1$.*

(2) *E satisfies the epigraph condition* (12).

(3) *We have that*

$$|dE|(u) \geq \min\left\{\|E_0^* - E_1^*\| \,\middle|\, E_0^* \in \partial E_0(u),\ E_1^* \in \partial E_1(u)\right\} \tag{13}$$

if the convex subdifferential $\partial E_0(u) \neq \emptyset$, otherwise $|dE|(u) = +\infty$.

For a critical point $u \in X$ of E we directly obtain that $\partial E_0(u) \neq \emptyset$ and that there are $E_0^* \in \partial E_0(u)$ and $E_1^* \in \partial E_1(u)$ such that

$$E_0^* - E_1^* = 0\,. \tag{14}$$

Proof. For (1) it is sufficient to consider $E_1 = 0$ by Proposition 1.6 in [12]. But this is a special case of Theorem 3.13 in [10] where $|d\mathcal{G}_E|(u, \xi) = 1$ is shown for E_1 being continuously differentiable. (2) directly follows from (1) and for (3) we refer to Remark 3.2 and Theorem 3.5 in [12]. □

4. Mountain pass solutions

In this section we study the existence of critical points of the functional

$$E(u) = E_0(u) - E_1(u) = \int_\Omega A(x, Du(x))\,dx - \int_\Omega F(x, u(x))\,dx$$

on $W_0^{1,p}(\Omega)$ where E_0 is convex and continuous and E_1 is locally Lipschitz continuous. We first apply a general version of the classical mountain pass result and then we apply a symmetric mountain pass result.

Theorem 4.1. *Assume that* (A), (A1), (A2), (A3), (A4) *are satisfied. Then E has a nontrivial critical point $u \in W_0^{1,p}(\Omega)$ and u is a weak solution of the Dirichlet problem*

$$-\operatorname{div} a(x) = \mathsf{f}(x) \quad on\ \Omega\,, \quad u = 0 \quad on\ \partial\Omega\,, \tag{15}$$

i.e. there are $a \in L^{p'}(\Omega, \mathbb{R}^n)$ and $\mathsf{f} \in L^{q'}(\Omega)$ related to u by

$$a(x) \in \partial A(x, Du(x))\,, \quad \mathsf{f}(x) \in [\underline{f}(x, u(x)), \overline{f}(x, u(x))] \quad a.e.\ on\ \Omega$$

such that (15) *is satisfied in the weak sense.*

The proof generalizes the ideas used in Nápoli and Mariani [9] where, in particular, the much stronger condition (6) is used instead of (A2) (cf. also Lemma 2.6 above). To handle our nonsmooth setting we apply the following abstract mountain pass theorem for continuous functions, that is a special case of Theorem 3.7 in Corvellec, Degiovanni and Marzocchi [8] or of Theorem 1.3.1 in Canino and Degiovanni [4].

Proposition 4.2. (Mountain Pass) *Let X be a complete metric space and let $E : X \to \mathbb{R}$ be continuous with $E(0) = 0$. Assume that there are constants $r, \alpha > 0$ and some $w \in X \setminus \overline{B_r(0)}$ such that E satisfies the Palais-Smale condition at any level $c \geq \alpha$ and that*

$$E(w) \leq 0\,, \quad \inf_{\partial B_r(0)} E \geq \alpha\,.$$

Then E has a critical point $u \in X$ with $E(u) \geq \alpha$.

Proof of Theorem 4.1. By Lemma 2.5 we have that $E_0 \colon W_0^{1,p}(\Omega) \to \mathbb{R}$ is convex and continuous with the subdifferential given by (9). By Lemma 2.8 functional $E_1 : W_0^{1,p}(\Omega) \to \mathbb{R}$ is locally Lipschitz continuous with generalized gradient according to (11). Clearly $E(0) = 0$.

For the application of Proposition 4.2 we have to check the Palais-Smale condition and the mountain pass geometry. We first show that E satisfies the Palais-Smale condition for any $c \in \mathbb{R}$. For that we fix a sequence $\{u_n\}$ in $W_0^{1,p}(\Omega)$ with

$$E(u_n) \to c \quad \text{and} \quad |dE|(u_n) \to 0\,.$$

By (13) there are $E^*_{0,n} \in \partial E_0(u_n)$ and $E^*_{1,n} \in \partial E_1(u_n)$ such that

$$|dE|(u_n) \geq \|E^*_{0,n} - E^*_{1,n}\| \to 0\,.$$

According to the Lemmas 2.5 and 2.8, $E^*_{0,n}$ and $E^*_{1,n}$ correspond to functions

$$a_n(x) \in \partial A(x, Du_n(x)) \quad \text{and} \quad \mathsf{f}_n(x) \in [\underline{f}(x, u_n(x)), \overline{f}(x, u_n(x))] \text{ a.e. on } \Omega\,,$$

respectively. With ϑ from (A3b) we now have

$$\begin{aligned} c_1\Big(1-\frac{p}{\vartheta}\Big)\|u_n\|^p = c_1\Big(1-\frac{p}{\vartheta}\Big)\int_\Omega |Du_n|^p\,dx &\overset{\text{(A1c)}}{\leq} \Big(1-\frac{p}{\vartheta}\Big)\int_\Omega A(x, Du_n)\,dx \\ &\overset{\text{(A1b)}}{\leq} \int_\Omega A(x, Du_n) - \frac{1}{\vartheta}\langle a_n, Du_n\rangle\,dx \\ &= E(u_n) - \frac{1}{\vartheta}\langle E^*_{0,n} - E^*_{1,n}, u_n\rangle + \int_\Omega F(x,u_n) - \frac{1}{\vartheta}\mathsf{f}_n(x)u_n\,dx\,. \end{aligned}$$

By (8) there is
$$M := \operatorname*{ess\,sup}_{x\in\Omega,\ |s|\leq s_0} \Big|F(x,s) - \frac{1}{\vartheta}f(x,s)s\Big| \tag{16}$$

and we have

$$\int\limits_\Omega F(x,u_n) - \frac{1}{\vartheta}\mathsf{f}_n(x)u_n\,dx \leq \int\limits_{|u_n(x)|\geq s_0} F(x,u_n) - \frac{1}{\vartheta}\mathsf{f}_n(x)u_n\,dx + M|\Omega| \overset{\text{(A3b)}}{\leq} M\,|\Omega|\,. \tag{17}$$

Therefore
$$c_1\Big(1-\frac{p}{\vartheta}\Big)\|u_n\|^p \leq E(u_n) + \frac{1}{\vartheta}\|E^*_{0,n} - E^*_{1,n}\|\,\|u_n\| + M\,|\Omega|\,. \tag{18}$$

If $\|u_n\|$ were unbounded, then $\|u_n\| \to \infty$ at least for a subsequence and we get a contradiction by dividing (18) by $\|u_n\|$. Thus $\|u_n\|$ is bounded and, by reflexivity, we have (up to a subsequence) that $u_n \rightharpoonup u$.

Let us now consider the identity

$$\langle E^*_{0,n}, u_n - u\rangle = \langle E^*_{0,n} - E^*_{1,n}, u_n - u\rangle + \int_\Omega \mathsf{f}_n(u_n - u)\,dx\,.$$

By the compact embedding $W^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega)$ with q according to (A3a), we have $u_n \to u$ in $L^q(\Omega)$. Since $\{\mathsf{f}_n\}$ is bounded in $L^{q'}(\Omega)$ by (A3a), we get

$$\int_\Omega \mathsf{f}_n(u_n - u)\,dx \to 0\,. \tag{19}$$

By $\|E^*_{0,n} - E^*_{1,n}\| \to 0$ and the boundedness of $\{u_n\}$ in $W^{1,p}_0(\Omega)$, we have that

$$\langle E^*_{0,n} - E^*_{1,n}, u_n - u\rangle \to 0\,.$$

With (19) we obtain $\langle E^*_{0,n}, u_n - u\rangle \to 0$. Thus $u_n \to u$ by Lemma 2.6 which verifies the Palais-Smale condition.

Let us now verify the mountain pass geometry by showing that there is some $r > 0$ and some $w \in W_0^{1,p}(\Omega)$ such that

$$\inf_{\|v\|=r} E(v) =: \alpha > 0\,, \quad \lim_{t\to\infty} E(tw) = -\infty\,.$$

By (A4) we can choose some small $\varepsilon > 0$ such that

$$c_1 > \frac{\lambda+\varepsilon}{p\lambda_{1,p}} \tag{20}$$

and there is some $\delta > 0$ such that

$$\frac{f(x,s)}{|s|^{p-2}s} \le \lambda + \varepsilon \quad \text{for a.e. } |s| \le \delta\,.$$

Hence $$f(x,s) \begin{cases} \le (\lambda+\varepsilon)s^{p-1} & \text{for a.e. } 0 < s \le \delta\,, \\ \ge -(\lambda+\varepsilon)|s|^{p-1} & \text{for a.e. } 0 > s \ge -\delta\,. \end{cases}$$

By integration we get

$$F(x,s) \le \frac{1}{p}(\lambda+\varepsilon)|s|^p \quad \text{for all } |s| \le \delta\,.$$

Using (A3) and the definition of F we find some $c > 0$ such that

$$|F(x,s)| \le |f(x,s)s| \le c_3(|s| + |s|^q) \le c|s|^q \quad \text{for a.e. } |s| \ge \delta\,.$$

With (5) and the continuous embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ we obtain some $\tilde{c} > 0$ such that

$$\begin{aligned} E(u) &\ge \int_\Omega A(x,Du)\,dx - \int_{\{|u(x)|\le\delta\}} \frac{\lambda+\varepsilon}{p}|u|^p\,dx - \int_{\{|u(x)|>\delta\}} c|u|^q\,dx \\ &\ge c_1\int_\Omega |Du|^p\,dx - \frac{\lambda+\varepsilon}{p\lambda_{1,p}}\int_\Omega |Du|^p\,dx - \tilde{c}\Big(\int_\Omega |Du|^p\,dx\Big)^{q/p}\,. \end{aligned}$$

With $r = \int_\Omega |Du|^p\,dx$, the right hand side can be considered as a function $\varphi(r)$. By $q > p$ and (20), we find some small $r > 0$ such that $\varphi(r) > 0$. Consequently $E(u) \ge \varphi(r) > 0$ for all u with $\|u\|^p = r$ and, therefore, one assumption about the mountain pass geometry is satisfied.

We now fix some $\tilde{w} \in W_0^{1,p}(\Omega)$. By Lemma 2.4, there are $t_0 \ge 1$, γ_1, $\gamma_2 > 0$ with

$$\begin{aligned} E(t\tilde{w}) &= \int_\Omega A(x,tD\tilde{w})\,dx - \int_\Omega F(x,t\tilde{w})\,dx \\ &\le t^p E_0(\tilde{w}) - t^\vartheta\gamma_1 + \gamma_2 \quad \text{for all } t \ge t_0\,. \end{aligned}$$

By $\vartheta > p$ we get $E(t\tilde{w}) \to -\infty$ as $t \to \infty$. Hence we have $E(w) \le 0$ for some $w \in W_0^{1,p}(\Omega) \setminus \overline{B_r(0)}$ and the mountain pass geometry is verified. Now the existence of a nontrivial critical point $u \in W_0^{1,p}(\Omega)$ follows directly from Proposition 4.2.

By Lemmas 2.5, 2.8, and (14) we readily see that u is weak solution of (15). $\square$

In the presence of some symmetry we can verify the existence of a sequence of critical points. In particular we extend some result from Nápoli and Mariani [9, Theorem 4.1] to our more general setting.

Theorem 4.3. *Assume that* (A), (A1), (A2), (A3), (A4) *are satisfied and that*

$$A(x,g)=A(x,-g),\ f(x,s)=-f(x,-s)\ \textit{for all } g\in\mathbb{R}^n,\ s\in\mathbb{R},\ \textit{a.e. } x\in\Omega\,.$$

Then E has a sequence of critical points $u_k\in W_0^{1,p}(\Omega)$ with corresponding critical levels $E(u_k)\to\infty$. Moreover the u_k are weak solutions of the Dirichlet problem

$$-\operatorname{div} a_k(x)=\mathsf{f}_k(x)\quad \textit{on } \Omega,\quad u=0\quad \textit{on } \partial\Omega\,, \tag{21}$$

i.e. there are $a_k\in L^{p'}(\Omega,\mathbb{R}^n)$ and $\mathsf{f}_k\in L^{q'}(\Omega)$ with

$$a_k(x)\in\partial A(x,Du_k(x))\,,\ \ \mathsf{f}_k(x)\in[\underline{f}(x,u_k(x)),\overline{f}(x,u_k(x))]\quad \textit{a.e. on } \Omega$$

such that (21) *is satisfied in the weak sense.*

The proof uses the following general critical point result due to Canino and Degiovanni [4, Theorem 1.3.3] (cf. also Canino [3, Theorem 1.4]).

Proposition 4.4. *Let X be an infinite-dimensional Banach space and assume that $E\colon X\to\mathbb{R}$ is continuous, even, and satisfies the Palais-Smale condition for any level $c\in\mathbb{R}$. Assume that there are $r>0$ and $\alpha>E(0)$ such that*

$$E(v)\ge\alpha\quad \textit{for all } \|v\|=r\,. \tag{22}$$

Moreover, for any finite dimensional subspace $\tilde X\subset X$ the set

$$\{v\in\tilde X\mid E(v)\ge E(0)\} \tag{23}$$

is assumed to be bounded. Then E has a sequence of critical points $u_k\in X$ with critical values $E(u_k)\to\infty$.

Proof of Theorem 4.3. The assumptions ensure that E_0 and E_1 are even. In the proof of Theorem 4.1 we have already seen that E is continuous, that it satisfies the Palais-Smale condition at any level $c\in\mathbb{R}$, and that (22) is satisfied for suitable $r>0$ and $\alpha>E(0)=0$. So we still have to show that the set in (23) is bounded. But here we can argue exactly as in the proof of Nápoli and Mariani [9, Lemma 4.1], since the continuity of $f(\cdot,\cdot)$ assumed there is not relevant for that. Then Proposition 4.4 provides the existence of a sequence of critical points $u_k\in W_0^{1,p}(\Omega)$ with $E(u_k)\to\infty$. By Lemma 2.5, Lemma 2.8, and (14) we get that the u_k are weak solutions of (21). □

5. 1-Laplace problems

Now we consider $p=1$ and we want to study the degenerate 1-homogeneous case $A(x,g)=|g|$. Instead of $W_0^{1,1}(\Omega)$ we have to work in $BV(\Omega)$. More precisely we consider

$$E_0(u):=\int_\Omega d|Du|+\int_{\partial\Omega}|u|\,d\mathcal{H}^{n-1}\quad \text{for } u\in BV(\Omega)$$

where the boundary term replaces the homogeneous boundary conditions in $BV(\Omega)$ (cf. [15]). With

$$E_1(u) = \int_\Omega F(x, u(x))\, dx$$

as before we are looking for critical points of $E = E_0 - E_1$ to verify the existence of weak solutions of

$$-\Delta_1 u = \mathsf{f}(x)$$

where $\Delta_1 u$ denotes the 1-Laplace operator (that is formally given by div $\frac{Du}{|Du|}$) and

$$\mathsf{f}(x) \in [\underline{f}(x, u(x)), \overline{f}(x, u(x))] \quad \text{a.e. on } \Omega\,.$$

For $\Omega \subset \mathbb{R}^n$ open and bounded with Lipschitz boundary, we formulate the problem in $Y := L^q(\Omega)$ with $1 < q < \frac{n}{n-1}$. With $X := BV(\Omega)$ we consider the extension

$$E_0(u) := \begin{cases} \int_\Omega d|Du| + \int_{\partial\Omega} |u|\, d\mathcal{H}^{n-1} & \text{for } u \in X\,, \\ \infty & \text{for } u \in Y \setminus X \end{cases}$$

which is convex and lower semicontinuous on $L^q(\Omega)$. For $u \in BV(\Omega)$ we have that $E^* \in \partial E_0(u)$ if and only if there is some $z \in L^\infty(\Omega, \mathbb{R}^n)$ with

$$\|z\|_\infty \le 1, \;\; E^* = -\operatorname{div} z \in L^n(\Omega), \;\; E_0(u) = \langle E^*, u\rangle = -\int_\Omega u \operatorname{div} z\, dx \tag{24}$$

where $\|z\|_\infty = 1$ for $u \neq 0$ (cf. Kawohl and Schuricht [15], Milbers and Schuricht [21] and notice that the vector field z replaces the possibly undefined expression $\frac{Du}{|Du|}$). Recall that E_1 is continuously differentiable on $L^q(\Omega)$ according to Lemma 2.8.

Theorem 5.1. *Let f satisfy* (A3) *with $p = 1$, let $f(x, \cdot)$ be odd for a.e. $x \in \Omega$, and let $c_5 > 0$ be such that*

$$F(x, \pm s_0) \ge c_5 \quad \textit{for a.e. } x \in \Omega \tag{25}$$

and let

$$\operatorname*{ess\,lim}_{s \to 0} f(x, s) = 0 \tag{26}$$

(*hence $\underline{f}(x, 0) = \overline{f}(x, 0) = 0$*)*. Then E has a sequence of critical points $u_k \in BV(\Omega)$ with $E(u_k) \to \infty$. Moreover, the u_k are weak solutions of*

$$-\Delta_1 u = \mathsf{f}(x) \quad \textit{on } \Omega\,,$$

i.e. there are $z_k \in L^\infty(\Omega, \mathbb{R}^n)$ and $\mathsf{f}_k(x) \in L^{q'}(\Omega)$ related to u_k by

$$\operatorname{div} z_k \in L^n(\Omega), \quad E_0(u_k) = -\int_\Omega u_k \operatorname{div} z_k\, dx, \quad \mathsf{f}_k(x) \in [\underline{f}(x, u_k(x)), \overline{f}(x, u_k(x))]$$

a.e. on Ω such that, in the weak sense,

$$-\operatorname{div} z_k(x) = \mathsf{f}_k(x) \quad \textit{on } \Omega. \tag{27}$$

For the proof of the theorem we first provide a discontinuous version of a symmetric mountain pass theorem extending former versions as e.g. given in Rabinowitz [22, Theorem 9.12] for smooth functionals and in Canino and Degiovanni [4, Theorem 1.3.3] for continuous functions.

Proposition 5.2. *Let* $Y = X \oplus X_0 \oplus X_\infty$ *be a Banach space where* X_0 *has finite dimension and* X *is infinite dimensional. Let* $E : Y \to \mathbb{R} \cup \{+\infty\}$ *be even and lower semicontinuous. Moreover, assume that*

(a) E *is finite on* $X \oplus X_0$ *and* $E(u) = \infty$ *on* $Y \setminus (X \oplus X_0)$ *,*

(b) $E = E_0 + E_1$ *with* E_0 *convex and lower semicontinuous and* E_1 *locally Lipschitz continuous on* Y*,*

(c) *there are* $\varrho > 0$ *and* $\alpha \in \mathbb{R}$ *such that*

$$E(v) \geq \alpha \quad \textit{for all } v \in \partial B_\varrho(0) \cap X$$

(d) *there is* $\beta < \alpha$ *such that for any finite dimensional subspace* $V \subset X \oplus X_0$ *we find* $0 < r < R$ *with*

$$E(v) \leq \beta \ \textit{ on } \ (B_r(0) \cap V) \cup (V \setminus B_R(0))$$

and, moreover, $\sup_{v \in V} E(u)$ *be finite,*

(e) E *satisfies the Palais-Smale condition at all levels* $c \geq \alpha$*.*

Then there is a sequence $\{u_k\}$ *of (nontrivial) critical points of* E *with* $E(u_k) \to \infty$*.*

Proof. We consider the epigraph

$$Z := \mathrm{epi}(E) = \{(v, \xi) \in Y \times \mathbb{R} \mid E(v) \leq \xi\}$$

which is a complete metric space if it is endowed with the distance

$$d\big((u,\xi),(v,\mu)\big) := \big(\, \|u - v\|^2 + (\xi - \mu)^2\, \big)^{1/2}.$$

Since E is even, Z is symmetric in the sense that it is invariant under the action $(u, \xi) \to (-u, \xi)$. By $\mathcal{S}$ we denote the collection of symmetric closed sets in $Z \setminus (\{0\} \times \mathbb{R})$. For $M \in \mathcal{S}$ we introduce the genus $\gamma(M)$ as

$$\gamma(M) := \gamma(M_Y) \ \text{ where } \ M_Y := \{v \in Y \mid (v, \xi) \in M\}$$

is the projection of M onto Y and $\gamma(M_Y)$ is the usual genus of symmetric sets (cf. Zeidler [24]). On Z we consider the Lipschitz continuous function $\mathcal{G}_E : Z \to \mathbb{R}$ given by $\mathcal{G}_E(u, \xi) := \xi$, which is invariant under the action $(u, \xi) \to (-u, \xi)$.

Assume that X_0 has dimension k and $X_0 = \mathrm{span}\{e_1, \ldots, e_k\}$. For $m \geq k$ inductively choose $e_{m+1} \in X$ with $e_{m+1} \notin \mathrm{span}\{e_1, \ldots, e_m\} =: X_m$. According to (d) we set

$$r_m = r_m(X_m), \quad R_m = R_m(X_m),$$

$$D_m = (B_{R_m}(0) \cap X_m) \setminus B_{r_m}(0), \ \text{ and } \ \alpha_m := \sup_{v \in X_m} E(v).$$

For $m \in \mathbb{N}$ we consider the compact sets

$$D_m^Z := D_m \times \{\alpha_m\} \cup \partial B_{R_m}(0) \times [\beta, \alpha_m] \cup \partial B_{r_m}(0) \times [\beta, \alpha_m] \cup (B_{r_m}(0) \cap X_m) \times \{\beta\}$$

and, clearly, $D_m^Z \subset Z$. Let $C_e(D_m^Z, Z)$ denote the set of continuous mappings $\varphi : D_m^Z \to Z$ that are equivariant, i.e. $\varphi(u, \xi) = (v, \zeta)$ implies $\varphi(-u, \xi) = (-v, \zeta)$ for all (u, ξ). Then

$$G_m := \{\varphi \in C_e(D_m^Z, Z) \mid \varphi(u, \xi) = (u, \xi) \text{ for } \xi \leq \beta\} \neq \emptyset\,,$$

since id $\in G_m$ for all $m \in \mathbb{N}$. With the classes

$$\Gamma_j := \{\varphi(\overline{D_m^Z \setminus S}) \mid \varphi \in G_m,\ m \geq j,\ S \in \mathcal{S},\ \gamma(S) \leq m - j\}\,.$$

we study the minimax values

$$c_j := \inf_{M \in \Gamma_j} \max_{(u,\xi) \in M} \mathcal{G}_E(u, \xi)\,, \quad j \in \mathbb{N}\,.$$

Now, with some simple adaptions, we can argue as in the proof of [22, Theorem 9.12]. In particular we always consider deformations of Z that leave $Z_\beta := \{(u, \xi) \in Z \mid \xi \leq \beta\}$ unchanged. Instead of [22, Proposition 9.23] we obtain that, for $j > k$ and $M \in \Gamma_j$,

$$M \ \cap \ (\partial B_\varrho \times \mathbb{R}) \ \cap \ (X \oplus X_\infty) \times \mathbb{R} \ \neq \ \emptyset\,.$$

Then we apply the equivariant deformation theorem [4, Theorem 1.2.5] to the continuous function $\mathcal{G}_E$ on Z. Notice that $(PS)_c$ for E satisfying (b) implies $(PS)_c$ for $\mathcal{G}_E$, since E satisfies the epigraph condition (12). Clearly this also implies the equivariant version of the Palais-Smale condition used in [4, Theorem 1.2.5]. Moreover this deformation theorem provides deformations that leave Z_β unchanged. This way we obtain a sequence $(u_n, \xi_n) \in Z$ of critical points of $\mathcal{G}_E$, i.e. $|d\mathcal{G}_E|(u_n, \xi_n) = 0$, with critical values $\mathcal{G}_E(u_n, \xi_n) = \xi_n = E(u_n) \to \infty$. By definition, the u_n are critical points of E, i.e. $|dE|(u_n) = 0$, which verifies the theorem. □

For the proof of Theorem 5.1 we apply Proposition 5.2. Notice that the verification of the Palais-Smale condition in former treatments (and also in our previous results) essentially uses the strong ellipticity of the operator. These arguments brake down in our 1-homogeneous case and new arguments are needed.

Proof of Theorem 5.1. We want to apply Proposition 5.2.
We choose $Y := L^q(\Omega)$ $(1 < q < \frac{n}{n-1}, \|\cdot\| = \|\cdot\|_q)$, $X := BV(\Omega)$, and $E := E_0 - E_1$. With $X_0 := \{0\}$ and a suitable subspace X_∞ we have $Y = X \oplus X_\infty$ and (a) of Proposition 5.2 is satisfied. Clearly, E_0 is convex and lower semicontinuous on Y. By Lemma 2.8 function E_1 is continuously differentiable on $L^q(\Omega)$ which provides (b). Obviously E is even.

By Poincaré's inequality in $BV(\mathbb{R}^n)$ there is $c > 0$ with

$$E_0(u) \geq c\|u\| \quad \text{for all } u \in X\,. \tag{28}$$

Obviously $F(x, 0) = 0$ and, hence, $E_1(0) = 0$. Let us show that there is $\varrho > 0$ such that

$$E_1(u) \leq \frac{c}{2}\|u\| \quad \text{for all } \|u\| \leq \varrho\,. \tag{29}$$

Otherwise there is a sequence $u_k \to 0$ with $E_1(u_k) > \frac{c}{2}\|u_k\|$ for all k. Then, up to a subsequence, $u_k(x) \to 0$ a.e. on Ω. By Lebourg's mean value theorem (cf. Theorem 2.3.7 in [6]) there are $t_k \in (0, 1)$ and $\mathsf{f}_k \in \partial E_1(t_k u_k)$ such that

$$E_1(u_k) = E_1(u_k) - E_1(0) = \int_\Omega \mathsf{f}_k u_k\, dx \leq \|\mathsf{f}_k\|_{q'}\|u_k\|$$

where $\mathsf{f}_k(x) \in [\underline{f}(x, u_k(x)), \overline{f}(x, u_k(x))]$ for a.e. $x \in \Omega$. From (26) we get $\mathsf{f}_k(x) \to 0$ for a.e. $x \in \Omega$. By (A3a) there is $\tilde{c} > 0$ such that

$$\|\mathsf{f}_k\|_{q'}^{q'} = \int_\Omega |\mathsf{f}_k(x)|^{\frac{q}{q-1}}\, dx \leq c_3 \int_\Omega \big(1 + |t_k u_k|^{q-1}\big)^{\frac{q}{q-1}}\, dx \leq \tilde{c}\int_\Omega 1 + |t_k u_k|^q\, dx$$

and dominated convergence implies $\|\mathsf{f}_k\|_{q'} \to 0$. But this contradicts $\frac{c}{2}\|u_k\| < \|\mathsf{f}_k\|_{q'}\|u_k\|$ and (29) follows. From (28) and (29) we now readily get

$$E(u) = E_0(u) - E_1(u) \geq \frac{c\varrho}{2} =: \alpha > 0 \quad \text{for all } u \in \partial B_\varrho(0)$$

which verifies (c).

For (d) we assume that $\{e_1, \dots, e_n\}$ is a basis in V. By our construction V is a subspace of $BV(\Omega)$. Therefore $E_0(\pm e_k)$ is finite for all k and, by convexity, E_0 is bounded on the polyhedron spanned by the $\pm e_k$, $k = 1, \dots, n$. Since E_0 is 1-homogeneous, we find some $r > 0$ such that

$$E_0(v) \leq \frac{\beta}{2} \quad \text{for all } v \in \overline{B_r(0)} \cap V \quad \text{with} \quad \beta := \frac{\alpha}{2}\,.$$

With some possibly smaller $r > 0$ we have the same inequality for E_1 by continuity. Thus we get $E(v) \leq \beta$ on $B_r(0) \cap V$.

By Lemma 2.4 and (25), we find some $\alpha_1 > 0$ with $F(x, s) \geq \alpha_1 |s|^\vartheta$ for $|s| \geq s_0$. Then, by the boundedness stated in (8), there is some $\alpha_2 > 0$ with

$$F(x, s) \geq \alpha_1 |s|^\vartheta - \alpha_2 \quad \text{for all } s \in \mathbb{R}, \ \text{a.e. } x \in \Omega\,.$$

Now we consider $w \in V$ with $\|w\| = r$. With our previous estimates we obtain

$$\begin{aligned} E(tw) &= E_0(tw) - E_1(tw) \\ &\leq tE_0(w) - \alpha_1 t^\vartheta \int_\Omega |w|^\vartheta\, dx + \alpha_2\, |\Omega| \\ &\leq t\big(E_0(w) - t^{\vartheta-1}\alpha_1 \|w\|_\vartheta^\vartheta\big) + \alpha_2\, |\Omega| \quad \text{for all } t > 0\,. \end{aligned}$$

Since all norms are equivalent on the finite dimensional subspace V, there is some $\tilde{c} > 0$ such that

$$E(tw) \leq t\big(E_0(w) - t^{\vartheta-1}\tilde{c}\|w\|^{\vartheta}\big) + \tilde{c} \leq t\big(\beta - t^{\vartheta-1}\tilde{c}r^{\vartheta}\big) + \tilde{c}\,.$$

By $\vartheta > 1$ the right hand side becomes negative for all t larger than some $t_0 > 0$. Hence

$$E(v) \leq 0 < \beta \text{ for all } v \in V \text{ with } \|v\| \geq t_0 r := R\,, \text{ i.e. for all } v \in V \setminus B_R(0)\,.$$

Obviously the continuous function E is bounded from above on $\overline{B_R(0) \cap V}$ and thus also on V. But this verifies (d).

We now show that E satisfies the Palais-Smale condition $(PS)_c$ for any $c \geq \alpha$. Consider $\{u_n\} \in Y$ with $E(u_n) \to c$ and $|dE|(u_n) \to 0$. According to (13) we have

$$\text{there are } E^*_{0,n} \in \partial E_0(u_n)\,,\ \ E^*_{1,n} \in \partial E_1(u_n) \text{ such that } \|E^*_{0,n} - E^*_{1,n}(u_n)\| \to 0\,.$$

Let f_n be related to $E^*_{1,n}$ by satisfying (10) with u_k. Using $E_0(u_n) = \langle E^*_{0,n}, u_n\rangle$, the Poincaré inequality in $BV(\mathbb{R}^n)$, and arguing as in (17) with $M > 0$ from (16), we get

$$\begin{aligned}\Big(1 - \frac{1}{\vartheta}\Big)E_0(u_n) &= E(u_n) - \frac{1}{\vartheta}E_0(u_n) + E_1(u_n)\\ &= E(u_n) - \frac{1}{\vartheta}\langle E^*_{0,n} - E^*_{1,n}, u_n\rangle + \int_\Omega F(x,u_n)\,dx - \int_\Omega \frac{1}{\vartheta}\mathsf{f}_n u_n\,dx\\ &\leq E(u_n) + \frac{1}{\vartheta}\|E^*_{0,n} - E^*_{1,n}\|\,\|u_n\| + M|\Omega|\\ &\leq E(u_n) + c\|E^*_{0,n} - E_{1,n}\|\,E_0(u_n) + M|\Omega|\end{aligned}$$

for some $c > 0$. If $E_0(u_n)$ were unbounded, $E_0(u_n) \to \infty$ at least for a subsequence. Then we divide by $E_0(u_n)$ in the previous inequality and obtain a contradiction. Therefore $E_0(u_n)$ has to be bounded and, by Poincaré's inequality, also $\|u_n\|_1$ is bounded. Hence $\{u_n\}$ is bounded in $BV(\Omega)$. By the compact embedding $BV(\Omega) \hookrightarrow L^q(\Omega)$ the sequence $\{u_n\}$ is relatively compact in $L^q(\Omega)$. Hence there is a subsequence converging in $L^q(\Omega)$ which verifies $(PS)_c$.

Now we can apply Proposition 5.2 and obtain the desired sequence of critical points u_k. By (24), Lemma 2.8, and (14), we readily see that the u_k are weak solutions of (27). ☐

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