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# Hopf Formulas for Nonlinear Obstacle Problems*

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A Hopf formula is derived for $\max\{u_t+H(Du),\ h(t,x)-u\}=0$, $u(T,x)=g(x)\geq h(t,x)$, where $g$ is assumed convex and $x\mapsto h(t,x)$ is also convex. This generalizes a formula without time dependent obstacle due to Subbotin. A Hopf formula for a concave obstacle is also derived. In addition, the Hopf formula for the obstacle problem with quasiconvex $g$ is established. Next we consider the double obstacle problem. Assume the two obstacles $g_1(x)\leq g_2(x)$ are given functions, both convex or both concave. The nonlinear double obstacle variational inequality $\max\{\min\{u_t+H(Du),g_2-u\},g_1-u\}=0$ on $(-\infty,T)\times\mathbb{R}^n$, with terminal data either $g_2$ in the convex case and $g_1$ in the concave case has a viscosity solution given by a Hopf type formula. These formulas are derived by using differential games with stopping times.

*Keywords:* Differential games, stopping times, Hopf formula, double obstacle.

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## 1. Introduction

The Hopf formula for the explicit solution of the Hamilton-Jacobi equation

$$u_t+H(Du)=0,\ (t,x)\in(-\infty,T)\times\mathbb{R}^n,\ u(T,x)=g(x),\ g\text{ convex},\tag{1}$$

is given by
$$u(t,x)=\sup_{y\in\mathbb{R}^n} y\cdot x-g^*(y)+(T-t)H(y),\tag{2}$$

where $g^*(y)=\sup_x y\cdot x-g(x)$ is the Fenchel conjugate of $g$. In 1984, Bardi and Evans [3] proved this Hopf formula by constructing a differential game for which the given problem (1) is the Isaacs equation and with a function $V(t,x)$ as the associated value function. Then they explicitly calculated the value function and showed that $V=u$ is given by the Hopf formula (2). This proved that (2) is the viscosity solution of (1). In the ensuing years, Hopf's formula for (1) has been extended in several ways. For example, $g$ need only be lower semicontinuous

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(possibly infinite), and equations of the form $u_t + H(u, Du) = 0$ with $u(T, x) = g(x)$, and $g$ merely lower semicontinuous and quasiconvex (i.e., having convex level sets) also have a Hopf formula. With $u$ dependence in $H$ and quasiconvex $g$ additional assumptions are needed. Direct proofs not relying on differential games were also devised to verify that the Hopf formulas are indeed viscosity solutions of the associated equations. Refer to [1] for details and the most general results. See also [16].

In 1995 Subbotin [15] considered the obstacle problem

$$\min\{u_t + H(Du), g(x) - u\} = 0, \quad (t, x) \in (-\infty, T) \times \mathbb{R}^n, \quad u(T, x) = g(x), \tag{3}$$

with $g$ a convex function. Subbotin proved, by using his theory of minimax solutions, that the function $u$ is given by a Hopf formula

$$u(t, x) = \sup_{y \in \mathbb{R}^n} y \cdot x - g^*(y) + (T - t)H(y) \wedge 0. \tag{4}$$

Since minimax solutions are equivalent to viscosity solutions, $u$ in (4) gives the unique viscosity solution of (3). This formula was later extended in [5] to quasiconvex $g's$ and the equation $\min\{u_t + H(u, Du), g(x) - u\} = 0$, but the proof there relied on a portion of Subbotin's lengthy proof which is avoided in this paper. See Theorem 2.3.

In 1974 Bensoussan and Friedman [6] were the first to consider differential games with stopping times. In 2009 Ghosh, Mallikarjuna, and Sheetal [11, 12] also considered infinite horizon differential games with stopping times using the Elliott-Kalton [8] formulation of differential games and viscosity solutions of first order equations. In these papers it was proved that the upper and lower values of such a game is a solution to a double obstacle problem. In particular Bensoussan and Friedman proved, using Friedman's definition of differential game, that the value function is a Lipschitz solution of a double obstacle problem using a dynamic programming principle. It is straightforward and routine to adapt their proof to prove that the value function is actually a viscosity solution and one may use Elliott-Kalton strategies to simplify matters. This procedure is standard in differential games.

This connection with double obstacle problems and differential games with stopping times motivates the question of the possibility of deriving a Hopf formula for double obstacle problems of the form

$$\max\{\min\{u_t + H(Du), g_2 - u\},\ g_1 - u\} = 0, \quad (t, x) \in (-\infty, T) \times \mathbb{R}^n, \tag{5}$$

with terminal data either $u(T, x) = g_1(x)$ or $u(T, x) = g_2(x)$. That is one of the main goals of this paper. For given obstacles and hamiltonian $H = H(p)$, we construct a differential game with stopping times for which the value satisfies (5) and then we explicitly calculate the value function. The result is our Hopf formula

$$u(t, x) = g_1(x) \vee \left( \sup_{y \in \mathbb{R}^n} y \cdot x - g_2^*(y) + (T - t)H(y) \wedge 0 \right). \tag{6}$$

It is assumed that $g_1 \le g_2$ and both functions are convex and $u(T,x) = g_2(x)$. In the case when $g_1 \le g_2$ are concave, the Hopf formula solution of (5) is

$$u(t,x) = g_2(x) \wedge \left( \inf_{z \in \mathbb{R}^n} z \cdot x - g_{1*}(z) + (T-t)H(z) \vee 0 \right), \tag{7}$$

for $u(T,x) = g_1(x)$. Here we need the Fenchel concave conjugate given by $g_*(y) = \inf_x y \cdot x - g(x)$. Observe also that when $g_1 \le g_2$ we have

$$\max\{\min\{u_t + H(Du), g_2 - u\},\ g_1 - u\} = \min\{\max\{u_t + H(Du), g_1 - u\},\ g_2 - u\}.$$

In passing, we mention that differential games with stopping times are closely related to reach-avoid differential games. A reach-avoid differential game is a differential game model of trying to steer a trajectory into a target at some time in $[t,T]$, while at the same time avoiding a constraint set for all times up to the time of hitting the target. These games are undergoing extensive study in the engineering community for their obvious practical applications [10, 13, 14]. The simplest payoff of such a game is given by

$$P(\eta,\zeta) = \inf_{t \le \tau \le T} g_2(\xi(\tau)) \vee \sup_{t \le \sigma \le \tau} g_1(\xi(\sigma)) \tag{8}$$

where $\eta$ and $\zeta$ are the controls, and $\xi(\cdot)$ is the associated trajectory given by $\dot{\xi} = f(s,\xi,\eta,\zeta)$, $\xi(t) = x$. It is shown in [10] that the upper and lower value functions of a reach-avoid game are viscosity solutions of an associated double obstacle problem. For instance, the upper value, $V^+$ with the payoff in (8) is the viscosity solution of

$$\max\{\min\{V_t^+ + \min_z \max_y D_x V^+ \cdot f(t,x,y,z), g_2(x) - V^+\},\ g_1(x) - V^+\}. \tag{9}$$

The terminal value is $V^+(T,x) = g_2(x) \vee g_1(x)$. Notice that the payoff

$$P(\eta,\zeta) = \sup_{t \le \tau \le T} g_2(\xi(\tau)) \wedge \inf_{t \le \sigma \le \tau} g_1(\xi(\sigma)) \tag{10}$$

leads to an upper value solving the same equation in (9)(assuming $g_1 \le g_2$) but with terminal condition $V^+(T,x) = g_1(x) \wedge g_2(x)$. The connection with double stopping time problems arises from the following simple calculation

$$\begin{aligned} P(\eta,\zeta) &= \sup_{t \le \tau \le T} g_2(\xi(\tau)) \wedge \inf_{t \le \sigma \le \tau} g_1(\xi(\sigma)) \\ &= \sup_{t \le \tau \le T} \inf_{t \le \sigma \le \tau} \left( g_2(\xi(\tau))\chi_{\tau<\sigma} + g_1(\xi(\sigma))\chi_{\sigma \le \tau} \right) \\ &\qquad\qquad \wedge \inf_{\tau < \sigma \le T} \left( g_2(\xi(\tau))\chi_{\tau<\sigma} + g_1(\xi(\sigma))\chi_{\sigma \le \tau} \right) \\ &= \sup_{t \le \tau \le T} \inf_{t \le \sigma \le T} \left( g_2(\xi(\tau))\chi_{\tau<\sigma} + g_1(\xi(\sigma))\chi_{\sigma \le \tau} \right). \end{aligned}$$

Thus, in certain circumstances, the Hopf formulas of this paper give explicit solutions for some reach-avoid problems.

As mentioned, the formulas for the double obstacle problem are derived by using differential games which requires certain Lipschitz conditions on the data of the problem. A direct proof that these formulas provide a viscosity solution to the appropriate double obstacle problem would be useful and could extend the formulas. In addition, it certainly seems possible that formulas could be derived for hamiltonians of the form $H = H(u, p)$, but we do not consider that in this paper.

The paper begins by considering the single obstacle problem. We study both a convex obstacle and a concave obstacle. The direct proof we provide in the convex and concave cases does not rely on differential games. However, we also sketch the proof for the concave formula using differential games in Remark 3.4. The formula for

$$\max\{u_t + H(Du),\ g - u\} = 0,\ \ (t,x)\in(-\infty,T)\times\mathbb{R}^n,\ \ u(T,x) = g(x),\ \ x\in\mathbb{R}^n,$$

is $$u(t,x) = \sup_{p\in\mathbb{R}^n} p\cdot x - g^*(p) + H(p)\vee 0(T-t) = (g^* - H\vee 0(T-t))^*,$$

when $g$ is convex, and

$$u(t,x) = \inf_{p\in\mathbb{R}^n} p\cdot x - g_*(p) + H(p)\vee 0(T-t) = (g_* - H\vee 0(T-t))_*,$$

when $g$ is concave. Even though the formulas seem dual, the proofs are not because the inequalities for the concave subsolution case go in the wrong direction.

We also extend the formula derived by Subbotin to the case when the obstacle is not the terminal data but some given function $h\colon (-\infty,T]\times\mathbb{R}^n \to \mathbb{R}$ with $x \mapsto h(t,x)$ uniformly Lipschitz continuous and convex for each $t \in (-\infty,T]$. The terminal data is still the convex function $g$ and we assume $g(x) \geq h(t,x)$. Then, we prove that the solution of

$$\begin{cases} \max\{u_t + H(Du),\ h(t,x) - u\} = 0, \\ (t,x)\in(-\infty,T)\times\mathbb{R}^n,\ u(T,x) = g(x),\ x\in\mathbb{R}^n, \end{cases} \tag{11}$$

is $$u(t,x) = \sup_{y\in\mathbb{R}^n}(y\cdot x - g^*(y) + H(y)(T-t)) \vee(\sup_{t\leq\tau\leq T} y\cdot x - h^*(\tau,y) + H(y)(\tau-t)). \tag{12}$$

Therefore, the obstacle need only lie below the terminal function and it may be time dependent.

This paper does not exhaust the possibilities for deriving Hopf (or Lax ) formulas for obstacle problems but extends the method of differential games introduced by Bardi & Evans to derive and prove such formulas for obstacle problems. We also do not derive the formulas under the most general assumptions possible or to the quasiconvex obstacle cases with hamiltonians also depending on the solution. We leave this as open. In addition, it is an open problem to derive the Hopf formula for the convex double obstacle problem when $u(T,x) = g_1(x)$ or the concave double obstacle problem when $u(T,x) = g_2(x)$.

**Notation.** For any given function $f\colon \mathbb{R}^n \to \mathbb{R}$, if $f$ is uniformly Lipschitz continuous we let $Lip(f)$ denote the Lipschitz constant of $f$. Also, we use the notation $a \wedge b = \min\{a, b\}, a \vee b = \max\{a, b\}$, and $B(R)$ is the closed ball centered at the origin with radius $R > 0$. Also, since we take terminal data for the equations in this paper, a viscosity subsolution (resp. supersolution ) of, for example, (1) is a function $u$ such that if $(t_0, x_0) \in \arg\max(u-\varphi), \varphi \in C^1$, (resp. $(t_0, x_0) \in \arg\min(u-\varphi)$) then $\max\{\varphi_t(t_0, x_0) + H(D\varphi(t_0, x_0)),\ g-u\} \geq 0$ (resp. $\leq 0$). See [2, 7, 8, 9] for the theory of viscosity solutions, optimal control, and differential games.

## 2. Hopf formula convex or concave obstacle

In this section we consider the single obstacle problem

$$\begin{cases} \max\{u_t + H(D_x u),\ g(x) - u\} = 0, \\ (t, x) \in (-\infty, T) \times \mathbb{R}^n,\ u(T, x) = g(x) \in \mathbb{R}^n. \end{cases} \tag{13}$$

Throughout we assume $H\colon \mathbb{R}^n \to \mathbb{R}$ is continuous and $g\colon \mathbb{R}^n \to \mathbb{R}$ is continuous on $dom(g) = \{x \mid -\infty < g(x) < +\infty\}$.

**Theorem 2.1.** (i) *If $g\colon \mathbb{R}^n \to \mathbb{R}$ is convex, the unique viscosity solution of* (13) *is given by the Hopf formula*

$$\begin{aligned} u(t, x) &= \sup_{p \in \mathbb{R}^n} p \cdot x - g^*(p) + (T-t)H(p) \vee 0 \\ &= (g^*(p) - (T-t)H(p) \vee 0)^*(x) \end{aligned} \tag{14}$$

*where $g^*(p) = \sup_{y \in \mathbb{R}^n} p \cdot y - g(y)$ is the Fenchel convex conjugate of $g$,*

(ii) *If $g\colon \mathbb{R}^n \to \mathbb{R}$ is concave, the unique viscosity solution of* (13) *is given by the Hopf formula*

$$\begin{aligned} u(t, x) &= \inf_{p \in \mathbb{R}^n} p \cdot x - g_*(p) + (T-t)H(p) \vee 0 \\ &= (g_*(p) - (T-t)H(p) \vee 0)_*(x), \end{aligned} \tag{15}$$

*where $g_*(p) = \inf_{y \in \mathbb{R}^n} p \cdot y - g(y)$ is the Fenchel concave conjugate of $g$.*

**Remark 2.2.** Equivalently (14) can be written

$$\begin{aligned} u(t, x) &= \sup_{y \in \mathbb{R}^n, p \in \partial g(y)} \{p \cdot (x-y) + g(y) + H(p) \vee 0(T-t)\} \\ &= \sup_{y \in \mathbb{R}^n} \inf_{p \in \mathbb{R}^n} p \cdot (x-y) + g(p) + (T-t)H(y) \vee 0. \end{aligned}$$

where $\partial g(y)$ is the convex subdifferential of $g$ at the point $y$. Also, (18) is equivalent to

$$u(t, x) = \inf_{\substack{y \in \mathbb{R}^n \\ -p \in \partial(-g)(y)}} \{p \cdot (x-y) + g(y) + H(p) \vee 0(T-t)\}.$$

In the convex case, the condition

$$\lim_{|p|\to\infty} \frac{g^*(p) + (T-t)H(p) \vee 0}{|p|} = +\infty \tag{16}$$

will guarantee that the domain of $u$ is all of $(-\infty, T] \times \mathbb{R}^n$. A similar condition applies to the concave case.

It is easy to check that $u(T, x) = g(x)$ in both cases (i) and (ii) and we omit the proof. Furthermore, since $u$ in (14) is the supremum of affine functions in both $t$ and $x$, $u$ is a convex function of both variables. Similarly, $u$ in (18) is a concave function of both variables.

**Proof.** (i): First we verify that $u$ is a subsolution. Set for each $p \in \mathbb{R}^n$

$$v(t, x; p) = p \cdot x - g^*(p) + H(p) \vee 0(T-t) \in C^1((-\infty, T) \times \mathbb{R}^n),$$

and $u(t, x) = \sup_{p\in\mathbb{R}^n} v(t, x; p)$. For the fixed point $(t, x)$, we may assume $u(t, x) \geq g(x) + \delta/2$ for some $\delta > 0$ since if $u(t, x) \leq g(x)$ there is nothing to prove. Choose $\hat{p} \in \mathbb{R}^n$ such that $u(t, x) \leq v(t, x; \hat{p}) + \delta/4$. Suppose $H(\hat{p}) \leq 0$. Then

$$\begin{aligned} g(x) + \delta/2 \leq u(t, x) \leq v(t, x; \hat{p}) + \delta/4 &= \hat{p} \cdot x - g^*(\hat{p}) + H(\hat{p}) \vee 0(T-t) + \delta/4 \\ &= \hat{p} \cdot x - g^*(\hat{p}) + \delta/4 \leq g(x) + \delta/4, \end{aligned}$$

a contradiction. This shows that $u(t, x) = \sup\{v(t, x; p) \mid p \in \mathbb{R}^n, H(p) > 0\}$ whenever $u(t, x) > g(x)$. Therefore, since $v_t = -H(p), D_x v = p$, we have $v_t + H(D_x v) = 0$ on $\{(t, x) \mid u(t, x) > g(x)\}$. This says $v$ is a subsolution for each $p$ when $u > g$ and since the supremum of subsolutions is a subsolution, $u = \sup_p v$ is a subsolution.

Next, to show $u$ is a supersolution, we may use the fact that $u$ is convex.

Let $(t_0, x_0) \in (-\infty, T) \times \mathbb{R}^n$ and $u(t_0, x_0) < +\infty$. Since $u$ is convex for any $(p_t, p_x) \in D^- u(t_0, x_0) \subset \partial u(t_0, x_0)$ we have

$$u(t, x) \geq u(t_0, x_0) + p_t(t - t_0) + p_x(x - x_0), \quad \forall (t, x) \in (-\infty, T] \times \mathbb{R}^n. \tag{17}$$

The set $D^- u(t, x)$ is the superdiffferential of $u$ ([2], for example) and $\partial u$ is the convex subdifferential. Set $t = T$ and rearrange to get

$$p_x \cdot x_0 - p_x \cdot x + g(x) + (T - t_0)H(p_x) \vee 0 - u(t_0, x_0) \geq (p_t + H(p_x) \vee 0)(T - t_0).$$

Take the inf over $x \in \mathbb{R}^n$ to get

$$p_x \cdot x_0 - g^*(p_x) + (T - t_0)H(p_x) \vee 0 - u(t_0, x_0) \geq (p_t + H(p_x) \vee 0)(T - t_0).$$

Since
$$\begin{aligned} u(t_0, x_0) &= \sup_{p\in\mathbb{R}^n} p \cdot x_0 - g^*(p) + (T - t_0)H(p) \vee 0 \\ &\geq p_x \cdot x_0 - g^*(p_x) + (T - t_0)H(p_x) \vee 0 \end{aligned}$$

we conclude that $p_t + H(p_x) \vee 0 \leq 0$ and then $p_t + H(p_x) \leq 0$.

Next,
$$u(t_0,x_0)=\sup_{p\in\mathbb{R}^n} p\cdot x_0-g^*(p)+(T-t_0)H(p)\vee 0$$
$$\geq \sup_{p\in\mathbb{R}^n} p\cdot x_0-g^*(p)=g(x_0),$$

so that $g(x_0)-u(t_0,x_0)\leq 0$. We have proved

$$\max\{p_t+H(p_x),\, g(x_0)-u(t_0,x_0)\}\leq 0,\quad (p_t,p_x)\in D^-u(t_0,x_0),$$

which means $u$ is a supersolution of (13).

(ii): The first part of the proof is similar to that of part (i). To show that $u$ given in (18) is a supersolution, set

$$v(t,x;p)=p\cdot x-g_*(p)+H(p)\vee 0(T-t).$$

Then $v\in C^1$, $v_t=-H(p)\vee 0$, $D_xv=p$ and therefore $v_t+H(D_xv)\vee 0=0$. This says $v_t+H(D_xv)\leq 0$. In addition, $v(t,x;p)\geq p\cdot x-g_*(p)\geq g(x)$. We conclude that $\max\{v_t+H(D_xv),\, g-v\}\leq 0$, i.e., for each $p\in\mathbb{R}^n$, $v(t,x;p)$ is a supersolution of (14). Since the infimum of supersolutions is a supersolution, $u(t,x)=\inf_{p\in\mathbb{R}^n} v(t,x;p)$ is a supersolution of (13).

We have left to prove that $u$ is a subsolution of (13). We use the fact that $u$ is a concave function. Let $(t_0,x_0)\in(-\infty,T)\times\mathbb{R}^n$, $u(t_0,x_0)>-\infty$ and let $(p_t,p_x)\in D^+u(t_0,x_0)\subset\partial(-u(t_0,x_0))$. We may assume $u(t_0,x_0)>g(x_0)$ since otherwise there is nothing to prove. Then,

$$u(t,x)\leq u(t_0,x_0)+p_t(t-t_0)+p_x\cdot(x-x_0),\quad (t,x)\in(-\infty,T]\times\mathbb{R}^n.$$

Set $t=T$ to get, after rearranging and adding $(T-t_0)H(p_x)\vee 0$ to both sides,

$$p_x\cdot x_0+g(x)-p_x\cdot x+(T-t_0)H(p_x)\vee 0-u(t_0,x_0)\leq(p_t+H(p_x)\vee 0)(T-t_0).$$

Taking the sup over $x\in\mathbb{R}^n$ and using the Fenchel concave conjugate, we obtain

$$p_x\cdot x_0-g_*(p_x)+(T-t_0)H(p_x)\vee 0-u(t_0,x_0)\leq(p_t+H(p_x\vee 0))(T-t_0).$$

Since
$$u(t_0,x_0)=\inf_{p\in\mathbb{R}^n} p\cdot x_0-g_*(p)+(T-t_0)H(p)\vee 0$$
$$\leq p_x\cdot x_0-g_*(p_x)+(T-t_0)H(p_x)\vee 0$$

we conclude that $p_t+H(p_x)\vee 0\geq 0$. Of course it does not now follow that $p_t+H(p_x)\geq 0$ and the rest of the proof diverges from that of part (i).

If $H(p_x)\geq 0$ we immediately have that $u$ is a subsolution of (13), so we may assume $H(p_x)<0$.

Since $u(t_0,x_0)+p_x\cdot(x-x_0)+p_t(t-t_0)\geq u(t,x)\geq g(x)$, if $p_t>0$ by sending $t\to-\infty$ we get a contradiction. Therefore, we have $p_t=0$ and $H(p_x)<0$.

Next, since we have $u(t_0,x_0) = \inf_{p\in\mathbb{R}^n} p\cdot x_0 - g_*(p) + H(p)\vee 0(T-t_0)$ and $g_*(p) = \inf_{y\in\mathbb{R}^n} p\cdot y - g(y)$, there is $y\in\mathbb{R}^n$ such that $g_*(p_x) = p_x\cdot y - g(y)$, $-y\in\partial(-g_*)(p_x)$ and

$$u(t_0,x_0) \le p_x\cdot(x_0-y) + g(y) + H(p_x)\vee 0(T-t_0) = p_x\cdot(x_0-y)+g(y),$$

$$g(x_0) < g(y) + p_x\cdot(x_0-y) \;\text{ and }\; g(x)\le g(y)+p_x\cdot(x-y),\ \forall\, x\in\mathbb{R}^n.$$

Since $H(p_x)<0$, by continuity there is $\varepsilon>0$ such that $H(p)<0, \forall\, p\in B_\varepsilon(p_x)$. Now define the concave function

$$\eta(x) = g(x) - g(y) - p_x\cdot(x-y).$$

We have the properties of $\eta$, $\eta(x)\le 0\ \forall\ x\in\mathbb{R}^n$, $\eta(x_0)<0$, and $\eta(y)=0$. Consider the closed convex sets

$$K_\eta = \{(z,x)\in\mathbb{R}\times\mathbb{R}^n \mid z\le\eta(x)\} \;\text{ and }\; K_\varepsilon = \{(z,x)\in\mathbb{R}\times\mathbb{R}^n \mid \frac{\varepsilon}{2}|x-x_0|\le z\}.$$

Clearly $K_\eta\cap K_\varepsilon=\emptyset$ and by Hahn-Banach they are strictly separated by a hyperplane $L(x) = a + q\cdot(x-y)$ for some $q\in\mathbb{R}^n, a\in\mathbb{R}$. In fact $|q|\le\frac{\varepsilon}{2}$ and

$$\eta(x_0) < L(x_0) < 0,\;\; \eta(x)<L(x),\;\text{ and }\; L(y)=a.$$

Therefore $g(x)-g(y)-p_x\cdot(x-y) < a + q\cdot(x-y), \forall\, x\in\mathbb{R}^n$ and so

$$g(x_0) < g(y) + (p_x+q)\cdot(x_0-y) + a < g(y) + p_x\cdot(x_0-y),\;\text{ and }\; H(p_x+q)<0,$$

which contradicts the fact that $g(y)+p_x\cdot(x_0-y)$ is the smallest value (with $H(p_x)<0$) greater than $g(x_0)$. Thus either $u(t_0,x_0)\le g(x_0)$ or $H(p_x)>0$ and in either case this means $u$ is a subsolution of (13). □

In the next theorem we will show how the Hopf formula may be extended to quasiconvex obstacles in a similar way. Recall that a quasiconvex function is, by definition, a function with convex level sets. Such functions may not be continuous or convex. Let $f\colon\mathbb{R}^n\to\mathbb{R}$ and $E(\gamma,f) = \{x\in\mathbb{R}^n \mid f(x)\le\gamma\}$ be the $\gamma-$level set of $f,\gamma\in\mathbb{R}$. The theory of quasiconvex functions uses the conjugates

$$f^{\%}(\gamma,p) = \sup_{x\in E(\gamma,f)} p\cdot x - f(x), \quad f^{\%\%}(x) = \sup_{\gamma\in\mathbb{R},\ p\in\mathbb{R}^n} \big(p\cdot x - f^{\%}(\gamma,p)\big)\wedge\gamma.$$

See [1] where many results on quasiconvex duality can be found. Refer also to the references there for further results.

**Theorem 2.3.** *Let $H\colon\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ be continuous with $\gamma\mapsto H(\gamma,p)$ nondecreasing and $p\mapsto H(\gamma,p)$ homogeneous, degree one (i.e., $H(\gamma,\lambda p)=\lambda H(\gamma,p)$). If $g\colon\mathbb{R}^n\to\mathbb{R}$ is continuous and quasiconvex, the unique viscosity solution of* (13) *is given by the Hopf formula*

$$\begin{aligned} u(t,x) &= (g^{\%}(\gamma,p) - (T-t)H(\gamma,p)\vee 0)^{\%}(x) \\ &= \sup_{\gamma\in\mathbb{R},p\in\mathbb{R}^n} \big(p\cdot x - g^{\%}(\gamma,p) + H(\gamma,p)\vee 0(T-t)\big)\wedge\gamma. \end{aligned} \tag{18}$$

**Proof.** According to [1, Theorem 4.2] $u$ in (18) is quasiconvex and is a supersolution of

$$u_t + H(u, D_x u) \vee 0 \leq 0, \quad (t,x) \in (-\infty, T) \times \mathbb{R}^n, \quad u(T,x) = g(x).$$

Consequently, $u_t + H(u, D_x u) \leq 0$ and

$$\begin{aligned} u(t,x) &= \sup_{\gamma \in \mathbb{R}, p \in \mathbb{R}^n} \big(p \cdot x - g^{\%}(\gamma, p) + H(\gamma, p) \vee 0 (T-t)\big) \wedge \gamma \\ &\geq \sup_{\gamma \in \mathbb{R}, p \in \mathbb{R}^n} \big(p \cdot x - g^{\%}(\gamma, p)\big) \wedge \gamma = g(x), \end{aligned}$$

because $g$ is quasiconvex. Therefore, $\max\{u_t + H(u, D_x u),\ g(x) - u\} \leq 0$, and $u$ is a supersolution of (13).

To show that $u$ in (18) is a subsolution we set

$$v(t,x;\gamma,p) = (p \cdot x - g^{\%}(\gamma,p) + (T-t)H(\gamma,p) \vee 0) \wedge \gamma.$$

This function is piecewise $C^1$. Using the same argument as in Theorem 2.1, we get

$$u(t,x) = \sup\{v(t,x;\gamma,p) \mid H(\gamma,p) > 0\}, \quad \text{when } \ u(t,x) > g(x).$$

Thus we may set $v(t,x;\gamma,p) = (p \cdot x - g^{\%}(\gamma,p) + (T-t)H(\gamma,p)) \wedge \gamma$. Since $\gamma \mapsto H(\gamma,p)$ is non decreasing and $p \mapsto H(\gamma,p)$ is homogeneous degree one, it is not hard to verify that for fixed $\gamma, p$, when $u(t,x) > g(x)$, $v$ is a viscosity solution of $v_t + H(v, D_x v) = 0$. Indeed see [1, Corollary 6.2] and the proof of [1, Proposition 6.1]. Consequently, since the supremum of subsolutions is a subsolution, $u$ is a subsolution of $u_t + H(u, Du) = 0$ when $u > g$ and therefore $\max\{u_t + H(u, D_x u), g - u\} \geq 0$. Since $g$ is quasiconvex.

$$u(T,x) = \sup_{\gamma,p}(p \cdot x - g^{\%}(\gamma,p)) \wedge \gamma = g(x)$$

we conclude that $u$ is a subsolution of (13) as well. □

## 3. The differential game derivation for time dependent obstacle

We will use differential games to derive and prove Hopf formulas for double obstacle problems as well as to extend the formula in Theorem 2.1 to obstacles with time dependence. The basic plan for this approach is due to Bardi & Evans [3].

Set $L_g = Lip(g), L_H = Lip(H)$ and

$$Y[t,T] = \{\eta \colon [t,T] \to B(L_g) \mid \eta \text{ is Lebesgue measurable}\},$$

$$Z[t,T] = \{\zeta \colon [t,T] \to B(L_H) \mid \zeta \text{ is Lebesgue measurable}\}.$$

A map $\Delta \colon Y[t,T] \to Z[t,T]$ is a strategy (for the minimizer) if it is nonanticipating, i.e., $\eta(r) = \hat{\eta}(r)$, $\forall\ t \leq s \leq T$, $\forall t \leq r \leq s$, implies that $\Delta[\eta](r) = \Delta[\hat{\eta}](r)$, $t \leq r \leq s$. Let $\mathbb{Z}[t,T]$ denote the class of all such strategies $\Delta$. Similarly $\mathcal{Y}[t,T]$ is the set of all nonanticipating maps $\mathbb{G} \colon \mathbb{Z}[t,T] \to \mathcal{Y}[t,T]$.

The following representation lemma connects the hamiltonian $H$ with differential games. The proof is due to L. C. Evans.

**Lemma 3.1.** *Assume* $|H(p) - H(q)| \leq L_H|p - q|$. *Then for* $|p| \leq L_g$,

$$H(p) = \max_{|y|\leq L_g} \min_{|z|\leq L_H} p \cdot z + H(y) - y \cdot z \qquad \text{(Lower)}$$

*and*

$$H(p) = \min_{|z|\leq L_H} \max_{|y|\leq L_g} p \cdot y + H(z) - y \cdot z. \qquad \text{(Upper)}$$

**Proof.** We only prove the first part since the second is similar.

$$\begin{aligned} &H(p) - H(y) \geq -L_H|p - y| \\ &\implies H(p) \geq H(y) - L_H|p - y| = H(y) + \min_{|z|\leq L_H} z \cdot (p - y), \forall\, y \in \mathbb{R}^n \\ &\implies H(p) \geq \max_{y\in\mathbb{R}^n} \min_{|z|\leq L_H} p \cdot z + H(y) - y \cdot z \geq H(p). \end{aligned}$$

In particular, if $|p| \leq L_g$, then $H(p) = \max_{|y|\leq L_g} \min_{|z|\leq L_H} p \cdot z + H(y) - y \cdot z$. $\square$

In the sequel we set $Y = B(L_g), Z = B(L_H)$. We have the following extension of Theorem 2.1 to time dependent convex obstacles.

**Theorem 3.2.** *Assume that* $H: \mathbb{R}^n \to \mathbb{R}$ *is continuous and that* $g: \mathbb{R}^n \to \mathbb{R}$ *is uniformly Lipschitz continuous and convex. Also, let* $h: (-\infty, T] \times \mathbb{R}^n \to \mathbb{R}$ *be continuous, and* $x \mapsto h(t, x)$, *uniformly Lipschitz and convex for each* $t \in (-\infty, T]$, *and* $g(x) \geq h(t, x)$. *Then the Hopf formula solution of*

$$\begin{cases} \max\{u_t + H(Du),\, h(t, x) - u\} = 0, \\ (t, x) \in (-\infty, T) \times \mathbb{R}^n,\ u(T, x) = g(x),\ x \in \mathbb{R}^n \end{cases} \tag{19}$$

*is*

$$\begin{aligned} u(t, x) = \sup_{y\in\mathbb{R}^n} &(y \cdot x - g^*(y) + H(y)(T - t)) \\ &\vee\big(\sup_{t\leq\tau\leq T} y \cdot x - h^*(\tau, y) + H(y)(\tau - t)\big). \end{aligned} \tag{20}$$

**Remark 3.3.** Notice that if $h(t, x) \equiv g(x)$ the formula reduces to the time independent convex case. Also since $u$ is the maximum of convex functions, $u$ is convex in $(t, x)$. Each component of the two suprema are affine functions in $t$ and $x$ and the supremum of affine functions is convex.

**Proof.** The theorem can be proved either directly as in Theorem 2.1 or by using the representation of $u$ as the (lower) value of a differential game with stopping times. We first need to assume $H$ is uniformly Lipschitz.

By Lemma 3.1 equation (19) becomes equivalent to the Hamilton-Jacobi-Isaacs equation for a differential game, namely,

$$\max\{u_t + \max_{|y|\leq L_g} \min_{|z|\leq L_H} D_x u \cdot z + H(y) - y \cdot z,\, h(t, x) - u\} = 0, \tag{21}$$

where $(t, x) \in (-\infty, T) \times \mathbb{R}^n$ and $u(T, x) = g(x),\ x \in \mathbb{R}^n$. By standard uniqueness theorems in viscosity solutions (see [2]), (21) has a unique viscosity solution. It is shown in [6, Theorem 2.4](c.f. also [14]) that $u$ is the value function of a differential game with stopping times given by

$$u(t,x) = \sup_{t\le\tau\le T} \inf_{\Delta\in\mathbb{Z}[t,T]} \sup_{\eta\in Y[t,T]} g(\xi(\tau))\chi_{\tau=T} + h(\tau,\xi(\tau))\chi_{\tau<T} + \\ + \int_t^{\tau\wedge T} H(\eta(s)) - \eta(s)\cdot\zeta(s)\, ds.$$

where $\zeta(s) = \Delta[\eta](s)$ and $\dot\xi(s) = \zeta(s)$, $t < s \le T$, $\xi(t) = x \in \mathbb{R}^n$, and it is a calculation to show that $Lip(u) \le L_g$. The theory of differential games with stopping times was introduced by Bensoussan and Friedman [6] and more recently with viscosity solutions in [11, 12, 14]. We may rewrite this function $u$ as

$$\begin{aligned} u(t,x) &= \sup_{t\le\tau\le T} \inf_\Delta \sup_\eta\ g(\xi(\tau))\chi_{\tau=T} + h(\tau,\xi(\tau))\chi_{\tau<T} + \int_t^{\tau\wedge T} H(\eta(s)) - \eta(s)\cdot\zeta(s) ds \\ &= \inf_\Delta \sup_\eta \left( g(\xi(T)) + \int_t^T H(\eta(s)) - \eta(s)\cdot\zeta(s)\, ds \right) \\ &\qquad \vee \sup_{t\le\tau<T} \inf_\Delta \sup_\eta \left( h(\tau,\xi(\tau)) + \int_t^\tau H(\eta(s)) - \eta(s)\cdot\zeta(s)\, ds \right). \end{aligned}$$

Next, $\xi(\tau) = x + \int_t^\tau \zeta(s)\, ds$ and using Jensen's inequality

$$\begin{aligned} u(t,x) &\le \inf_\Delta \sup_\eta \left( \frac{1}{T-t} \int_t^T g(x+\zeta(s)(T-t)) + H(\eta(s)) - \eta(s)\cdot\zeta(s)\, ds \right) \\ &\quad \vee \sup_{t\le\tau<T} \inf_\Delta \sup_\eta \left( \frac{1}{\tau-t} \int_t^\tau h(\tau, x+\zeta(s)(\tau-t)) + H(\eta(s)) - \eta(s)\cdot\zeta(s)\, ds \right). \end{aligned}$$

Using the fact that (see [3, Lemma 3.3] and Lemma 3.5 below)

$$\inf_\Delta \sup_\eta \frac{1}{\tau-t} \int_t^\tau \Lambda(\tau,\eta(s),\Delta[\eta](s))\, ds = \max_{|y|\le L_g} \min_{|z|\le L_H} \Lambda(\tau,y,z) \tag{22}$$

for any function $\Lambda(\tau,y,z)$ continuous in $(y,z)$, we get after a change of variables (details are similar to those in [3, Theorem 3.1, p. 1378])

$$\begin{aligned} u(t,x) &\le \max_{y\in Y} \min_{z\in Z} g(x+z(T-t)) + H(y)(T-t) - y\cdot z(T-t) \\ &\qquad \vee \sup_{t\le\tau<T} \max_{y\in Y} \min_{z\in Z} h(\tau, x+z(\tau-t)) + H(y)(\tau-t) - y\cdot z(\tau-t) \\ &\le \sup_{y\in\mathbb{R}^n} \Big( y\cdot x - g^*(y) + H(y)(T-t) \vee \sup_{t\le\tau<T} y\cdot x - h^*(\tau,y) + H(y)(\tau-t) \Big) \\ &=: \sup_{y\in\mathbb{R}^n} \Big( v(t,x;y) \vee \sup_{t\le\tau<T} b(t,x;\tau,y) \Big) \end{aligned}$$

To show that the opposite inequality holds it is easy to prove $v_t + H(D_x v) = 0$ and $b_t + H(D_x b) = 0$ for each $y \in \mathbb{R}^n$, $t \le \tau < T$. Also, $v(T;x,y) \le g(x)$,

$\lim_{t \to T} \sup_{t \le \tau < T} b(t; x, y) \le h(T, x) \le g(x)$. Since the supremum of subsolutions is a subsolution, the function $\sup_{y \in \mathbb{R}^n} \big(v(t, x; y) \vee \sup_{t \le \tau < T} b(t, x; \tau, y)\big)$ is a subsolution. Standard comparison results [2] now give the result. Also, the uniform Lipschitz continuity assumption on $H$ may be relaxed by approximation just as in [3]. We omit the details. $\square$

As mentioned, the proof of the theorem can be carried out under weaker assumptions in a way similar to Theorem 2.1 giving a direct proof that (20) is the viscosity solution of (19). Here is that argument.

**Direct proof of Theorem 3.2.** First, it is easy to see that

$$a(t, x; \tau, y) = (y \cdot x - g^*(y) + H(y)(T - t)) \vee (y \cdot x - h^*(\tau, y) + H(y)(\tau - t)),$$

for each fixed $t \le \tau < T$, $y \in \mathbb{R}^n$, is a subsolution of (19). That is due to the fact $a_t + H(D_x a) = 0$, in the viscosity sense, and $a(T, x; T, y) \le g(x)$. Since the supremum of subsolutions is a subsolution, $u$ given in (20) is a subsolution. Next, by convexity of $u$ if $(p_t, p_x) \in D^- u(t_0, x_0) \subset \partial u(t_0, x_0)$ we have

$$u(t, x) \ge u(t_0, x_0) + p_t(t - t_0) + p_x \cdot (x - x_0), \quad (t, x) \in (-\infty, T] \times \mathbb{R}^n. \tag{23}$$

Choose $t = T$ in (23) to get

$$g(x) \ge u(t_0, x_0) + p_t(T - t_0) + p_x \cdot (x - x_0)$$

and rearranging gives

$$p_x \cdot x_0 - p_x \cdot x + g(x) + H(p_x)(T - t_0) - u(t_0, x_0) \ge (p_t + H(p_x))(T - t_0).$$

Take the infimum over $x \in \mathbb{R}^n$ to see that

$$p_x \cdot x_0 - g^*(p_x) + H(p_x)(T - t_0) - u(t_0, x_0) \ge (p_t + H(p_x))(T - t_0).$$

By definition of $u$, the left side of this inequality is nonpositive and we conclude $p_t + H(p_x) \le 0$. Since $u(t, x) \ge h(t, x)$ everywhere and $u(T, x) = g(x)$, we conclude that $u$ is a supersolution of (19).

**Remark 3.4.** The Hopf formula (18) in Theorem 2.1 for a concave obstacle can also be derived using differential games. Here is a sketch of the argument.

First, we represent the hamiltonian as

$$H(p) = \min_z \max_y p \cdot y + H(z) - y \cdot z,$$

and then the solution of (13) is given as the upper value of the differential game with trajectory $\dot{\xi}(s) = \eta(s) = \mathbb{G}[\zeta](s), t < s \le T, \xi(t) = x \in \mathbb{R}^n$, and

$$u(t, x) = \sup_{\mathbb{G}, \tau} \inf_{\zeta} g(\xi(\tau)) + \int_t^{\tau} H(\zeta(s)) - \eta(s) \cdot \zeta(s)) \, ds. \tag{24}$$

Notice that we require the upper value representation. Using Jensen's inequality for concave functions we arrive at

$$u(t,x) \geq \sup_{\mathbb{G},\tau} \inf_{\zeta} \frac{1}{\tau - t} \int_t^\tau g(x + \eta(s)(\tau - t)) + (H(\zeta(s)) - \eta(s) \cdot \zeta(s))(\tau - t)\, ds.$$

The key step is now to convert from strategies and controls to points. Here is the extension of Bardi and Evans' Lemma 3.3. For the convenience of the reader we sketch the proof.

**Lemma 3.5.** *Let $\Lambda\colon [t,T] \times Y \times Z \to \mathbb{R}$ be upper semicontinuous on $[t,T]$ and continuous on $Y \times Z$. Then*

$$\sup_{\mathbb{G},\tau} \inf_{\zeta} \frac{1}{\tau - t} \int_t^\tau \Lambda(\tau, \mathbb{G}[\zeta](s), \zeta(s))\, ds = \min_z \max_{y,\tau} \Lambda(\tau, y, z). \tag{25}$$

**Proof.** Let $z^* \in \arg\min_z \{\max_{y,\tau} \Lambda(\tau, y, z)\}$. Then

$$\begin{aligned} \sup_{\mathbb{G},\tau} \inf_{\zeta} \frac{1}{\tau - t} \int_t^\tau \Lambda(\tau, \mathbb{G}[\zeta](s), \zeta(s))\, ds &\leq \sup_{\mathbb{G},\tau} \frac{1}{\tau - t} \int_t^\tau \Lambda(\tau, \mathbb{G}[z^*](s), z^*)\, ds \\ &\leq \min_z \max_{y,\tau} \Lambda(\tau, y, z). \end{aligned}$$

For showing the reverse inequality we take some $\varepsilon > 0$. Then there is a nonanticipating map $\Phi\colon Z \to [t,T] \times Y$ such that

$$\max_{y,\tau} \Lambda(\tau, y, z) \leq \Lambda(\Phi(z), z) + \varepsilon$$

for each $z \in Z$ (see [3, Lemma 3.3] for details of the construction). Next, set $(\tau^*, \mathbb{G}^*) = \Phi$. Then we have

$$\begin{aligned} \sup_{\mathbb{G},\tau} \inf_{\zeta} \frac{1}{\tau - t} \int_t^\tau \Lambda(\tau, \mathbb{G}[\zeta](s), \zeta(s))\, ds &\geq \inf_{\zeta} \frac{1}{\tau^* - t} \int_t^{\tau^*} \Lambda(\tau^*, \mathbb{G}^*[\zeta](s), \zeta(s))\, ds \\ &\geq \inf_{\zeta} \frac{1}{\tau^* - t} \int_t^{\tau^*} \max_{y,\tau} \Lambda(\tau, y, \zeta(s)) - \varepsilon\, ds \\ &\geq \inf_{\zeta} \frac{1}{\tau^* - t} \int_t^{\tau^*} \min_z \max_{y,\tau} \Lambda(\tau, y, z) - \varepsilon\, ds = \min_z \max_{y,\tau} \Lambda(\tau, y, z) - \varepsilon. \qquad \square \end{aligned}$$

Applying the lemma we have

$$\begin{aligned} u(t,x) &\geq \sup_{\mathbb{G},\tau} \inf_{\zeta} \frac{1}{\tau - t} \int_t^\tau g(x + \eta(s)(\tau - t)) + (H(\zeta(s)) - \eta(s) \cdot \zeta(s))(\tau - t)\, ds \\ &= \min_z \max_{t \leq \tau \leq T} \max_y g(x + y(\tau - t)) + H(z)(\tau - t) - y \cdot z(\tau - t). \end{aligned}$$

Making the change of variables $w = x + y(\tau - t)$ we get

$$\begin{aligned} u(t,x) &\geq \min_z \max_{t\leq\tau\leq T} \max_w g(w) + H(z)(\tau - t) - z\cdot(w-x) \\ &\geq \min_z \max_{t\leq\tau\leq T} z\cdot x - g_*(z) + H(z)(\tau - t) \\ &= \min_z z\cdot x - g_*(z) + H(z)\vee 0(T-t). \end{aligned}$$

The reverse inequality follows from the fact that

$$v(t,x) = \min_z z\cdot x - g_*(z) + H(z)\vee 0(T-t)$$

is a supersolution of (13) and, by comparison, $u \leq v$. □

## 4. Hopf formula double obstacle problem

### 4.1. Both obstacles convex

When the problem involves two obstacles $g_1 \leq g_2$ the problem of finding a Hopf formula becomes more complicated but is still tractable using differential games with stopping times.

The next theorem considers the case that the terminal data is the upper obstacle.

**Theorem 4.1.** *Assume $H\colon \mathbb{R}^n \to \mathbb{R}$ is continuous and $g_i\colon \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2$, are uniformly Lipschitz continuous and convex with $g_1 \leq g_2$. The unique uniformly continuous viscosity solution of*

$$\begin{cases} \max\{\min\{u_t + H(Du),\ g_2 - u\},\ g_1 - u\} = 0, \\ (t,x) \in (-\infty, T)\times\mathbb{R}^n,\ u(T,x) = g_2(x),\ x\in\mathbb{R}^n \end{cases} \tag{26}$$

*is, for $(t,x) \in (-\infty, T]\times\mathbb{R}^n$, given by*

$$u(t,x) = g_1(x) \vee \sup_{y\in\mathbb{R}^n} \left(y\cdot x - g_2^*(y) + H(y)\wedge 0(T-t)\right). \tag{27}$$

**Proof.** Without loss of generality we assume $H$ is uniformly Lipschitz and use approximation at the conclusion to obtain the result for the general case. Let $L_g = Lip(g_1) \vee Lip(g_2)$. Assume that $L_H = Lip(H) > L_g$. Control functions for the maximizer will be $\eta\colon [t,T] \to B(L_g)$ and for the minimizer $\zeta\colon [t,T] \to B(L_H)$. The hamiltonian is represented as

$$H(p) = \max_{|y|\leq L_g} \min_{|z|\leq L_H} p\cdot z + H(y) - y\cdot z, \quad |p| \leq L_g.$$

We begin again with the representation of the solution of (26) as the lower value of a differential game with stopping times. Since $u(T,x) = g_2(x)$ the solution of (26) is given by (see [6, Theorem 2.4] for the proof),

$$u(t,x) = \inf_{t\leq\tau\leq T} \sup_{t\leq\sigma\leq T} \inf_{\Delta} \sup_{\eta} g_1(\xi(\sigma))\chi_{\sigma<\tau} + g_2(\xi(\tau))\chi_{\tau\leq\sigma} + \int_t^{\tau\wedge\sigma} H(\eta(s)) - \eta(s)\cdot\zeta(s)\, ds$$

with $\dot{\xi}(s) = \zeta(s)$, $t < s \le T$, $\xi(t) = x \in \mathbb{R}^n$. In addition (see [6, Theorem 2.1]) $u$ is Lipschitz continuous with $Lip(u) \le L_g$. Next we estimate $u$ from above using Jensen's inequality:

$$\begin{aligned} u(t,x) = \inf_{t\le\tau\le T} \sup_{t\le\sigma\le T} \inf_{\Delta} \sup_{\eta} g_1(\xi(\sigma))\chi_{\sigma<\tau} + g_2(\xi(\tau))\chi_{\tau\le\sigma} + \\ + \int_t^{\tau\wedge\sigma} H(\eta(s)) - \eta(s)\cdot\zeta(s)\, ds \\ \le \inf_{t\le\tau\le T} \sup_{t\le\sigma\le T} \inf_{\Delta} \sup_{\eta} \frac{1}{\tau\wedge\sigma - t} \int_t^{\tau\wedge\sigma} \Big[ g_1(x + \zeta(s)(\tau\wedge\sigma - t))\chi_{\sigma<\tau} + \\ + g_2(x + \zeta(s)(\tau\wedge\sigma - t))\chi_{\tau\le\sigma} + (H(\eta(s)) - \eta(s)\cdot\zeta(s))(\tau\wedge\sigma - t)\Big]\, ds. \end{aligned}$$

Now we need the following variant of Lemma 3.5 whose proof is a routine modification of [3, Lemma 3.3].

**Lemma 4.2.** *Let* $\Lambda\colon [t,T]\times[t,T]\times B(L_g)\times B(L_H)\to\mathbb{R}$ *be Lipschitz continuous in* $B(L_g)\times B(L_H)$. *Then, for each fixed* $\sigma,\tau\in[t,T]$ *we have*

$$\inf_{\Delta}\sup_{\eta} \frac{1}{\tau\wedge\sigma - t}\int_t^{\tau\wedge\sigma} \Lambda(\tau,\sigma,\eta(s),\Delta[\eta](s))\, ds = \max_{|y|\le L_g}\min_{|z|\le L_H} \Lambda(\tau,\sigma,y,z), \quad (28)$$

*and*

$$\sup_{\mathbb{G}}\inf_{\zeta} \frac{1}{\tau\wedge\sigma - t}\int_t^{\tau\wedge\sigma} \Lambda(\tau,\sigma,\mathbb{G}[\zeta](s),\zeta(s))\, ds = \min_{|z|\le L_H}\max_{|y|\le L_g} \Lambda(\tau,\sigma,y,z). \quad (29)$$

Using the lemma we have

$$\begin{aligned} u(t,x) \le \inf_{t\le\tau\le T}\sup_{t\le\sigma\le T}\max_y\min_z \chi_{\sigma<\tau} g_1(x + z(\sigma - t)) + \chi_{\tau\le\sigma} g_2(x + z(\tau - t)) \\ + H(y)(\tau\wedge\sigma - t) - y\cdot z(\tau\wedge\sigma - t) \\ = \inf_{t\le\tau\le T}\sup_{\sigma<\tau}\max_y\min_z (g_1(x + z(\sigma - t)) + H(y)(\sigma - t) - y\cdot z(\sigma - t)) \\ \vee \sup_{t\le\tau\le\sigma}\Big(\max_y\min_z g_2(x + z(\tau - t)) + H(y)(\tau - t) - y\cdot z(\tau - t)\Big). \end{aligned}$$

Next, by making a change of variables in each $\min_z$, we get

$$\begin{aligned} u(t,x) \le \inf_{t\le\tau\le T}\sup_{\sigma<\tau\le T}\max_y\min_z (g_1(x + z(\sigma - t)) + H(y)(\sigma - t) - y\cdot z(\sigma - t)) \\ \vee \sup_{t\le\tau\le\sigma}\Big(\max_y\min_z g_2(x + z(\tau - t)) + H(y)(\tau - t) - y\cdot z(\tau - t)\Big). \\ \le \inf_{t\le\tau\le T}\sup_{\sigma<\tau}\sup_y (y\cdot x - g_1^*(y) + H(y)(\sigma - t)) \vee (y\cdot x - g_2^*(y) + H(y)(\tau - t)) \\ = \inf_{t\le\tau\le T}\sup_y (y\cdot x - g_1^*(y) + H(y)\vee 0(\tau - t)) \vee (y\cdot x - g_2^*(y) + H(y)(\tau - t)) \end{aligned}$$

$$= \inf_{t \le \tau \le T} \sup_y (y \cdot x - g_1^*(y)) \vee (y \cdot x - g_2^*(y) + H(y)(\tau - t))$$

$$\left[\text{using } \; g_1(x) = \sup_y y \cdot x - g_1^*(y)\right]$$

$$= \inf_{t \le \tau \le T} g_1(x) \vee \sup_y (y \cdot x - g_2^*(y) + H(y)(\tau - t))$$

$$\le (g_1(x) \vee g_2(x)) \wedge \left( g_1(x) \vee \sup_y y \cdot x - g_2^*(y) + H(y) \wedge 0(T - t) \right)$$

$$= g_1(x) \vee \sup_y y \cdot x - g_2^*(y) + H(y) \wedge 0(T - t).$$

We conclude that

$$u(t, x) \le g_1(x) \vee \sup_y y \cdot x - g_2^*(y) + H(y) \wedge 0(T - t). \tag{30}$$

Next, by an argument we have used several times it is easy to prove that

$$w(t, x) = g_1(x) \vee \sup_y y \cdot x - g_2^*(y) + H(y) \wedge 0(T - t)$$

is a subsolution of (26) and can therefore be omitted. By comparison it follows that $u(t, x) \ge w(t, x)$ and the theorem is proved. □

**Example 4.3.** The data we consider are $g_1(x) = 0$, $g_2(x) = 1 + |x|$, and the hamiltonian $H(y) = 1 - 2|y|$. Calculation gives,

$$g_1^*(y) = \begin{cases} 0, & y = 0 \\ +\infty, & y \ne 0, \end{cases} \quad \text{and} \quad g_2^*(y) = \begin{cases} -1, & |y| \le 1 \\ +\infty, & |y| > 1. \end{cases}$$

We calculate the Hopf solution with $u(T, x) = 1 + |x|$,

$$\begin{aligned} u(t, x) &= g_1(x) \vee \max_y y \cdot x - g_2^*(y) + H(y) \wedge 0(T - t) \\ &= 0 \vee \max_{|y| \le 1} y \cdot x + 1 + (1 - 2|y|) \wedge 0(T - t) \\ &= 0 \vee \max_{1/2 \le |y| \le 1} y \cdot x + 1 + (1 - 2|y|)(T - t) \vee \max_{|y| \le 1/2} (y \cdot x + 1) \\ &= 0 \vee ((1 + |x| - (T - t))\chi_{|x| > 2(T-t)} + 1\chi_{|x| \le 2(T-t)}) \vee \left( \frac{1}{2}|x| + 1 \right) \\ &= ((1 + |x| - (T - t))\chi_{|x| > 2(T-t)} + \left( \frac{1}{2}|x| + 1 \right) \chi_{|x| \le 2(T-t)}) \\ &= \begin{cases} 1 + |x| - (T - t), & |x| > 2(T - t) \\ \frac{1}{2}|x| + 1, & |x| \le 2(T - t). \end{cases} \end{aligned}$$

## 4.2. Both obstacles concave

In this section we assume $g_i \colon \mathbb{R}^n \to \mathbb{R}, i = 1, 2, g_1(x) \le g_2(x)$, are concave and continuous. The double obstacle problem we consider is

$$
\begin{cases} \max\{\min\{u_t + H(Du),\ g_2 - u\},\ g_1 - u\} = 0 \\ (t,x) \in (-\infty, T) \times \mathbb{R}^n,\ u(T,x) = g_1(x),\ x \in \mathbb{R}^n. \end{cases} \tag{31}
$$

Assume

$$
\begin{cases} H\colon \mathbb{R}^n \to \mathbb{R} \ \text{ is continuous, and } \ g_i\colon \mathbb{R}^n \to \mathbb{R},\ i = 1, 2, \\ \text{are uniformly Lipschitz and concave and } \ g_1 \le g_2, x \in \mathbb{R}^n. \end{cases} \tag{32}
$$

**Theorem 4.4.** *Under the assumption* (32) *the function*

$$
u(t,x) = g_2(x) \wedge \left( \inf_{z \in \mathbb{R}^n} z \cdot x - g_{1*}(z) + (T-t)H(z) \vee 0 \right), \tag{33}
$$

*for* $x \in \mathbb{R}^n$, $0 \le t \le T$, *is the unique uniformly continuous viscosity solution of* (31).

The proof is similar to the convex case; so we omit it.

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