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Relaxation of a Dynamic Game of Guidance and Program Constructions of Control

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The natural relaxation of guidance problem is considered. Namely, for fixed closed sets considered as parameters (target set and the set defining state constraints), we consider the similar guidance problem for ε-neighborhoods of these sets. We are interested to find the smallest size ε of these neighborhoods for which the player I can solve his guidance problem in class of generalized set-valued non-anticipating strategies. For the construction of solution, the Program Iterations Method is used. We obtain the above-mentioned smallest size as a position function. For determination of this function, iterative procedure operating in the function space is used. Also, it is shown that desired function is the fixed point of operator defining the iterative procedure.

Keywords: Pursuit-evasion differential game, program iteration method, guaranteed result.

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Introduction

In this paper, we consider an important part of differential game theory, namely game problems of guidance. And what is more, such problems are considered to be classical in differential game theory. The informative examples are given in the known monograph [20]. One of the fundamental results in differential game theory is theorem of alternative established by N. N. Krasovskii and A. I. Subbotin (see [24, 25]). According to this theorem, the solvability conditions for pursuit-evasion differential game on a finite-time interval are identified. We will take into account some corollaries of this theorem. In addition, we consider a series of differential games defined by different pairs of closed sets used as parameters. For every such pair, the corresponding alternative of N. N. Krasovskii and A. I. Subbotin holds. This fact defines a logical foundation for considered constructions. Namely, in this paper we have the differential game results realized as the alternative partition or the value function.

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Our case of differential game is defined by two sets in position space, namely the target set and the one defining state constraints for the player I. We consider these sets as parameters of our guidance problem. The goal of Player I is to reach the target set without breaking state constraints. On the other hand, the goal of player II is the opposite one. According to the theorem of alternative, the position set is divided into the sum of two subsets: the solvability set of player I and solvability set of player II. This theorem has defined the existence of saddle point for many other settings of differential games with typical quality functionals. It is important to mention the works of N. N. Krasovskii, L. S. Pontryagin, B. N. Pshenichnii, and A. I. Subbotin regarding the development of differential games theory [21–23, 27–29, 32]. It is also worth to mention the important result of A. V. Kryazhimsky [26] devoted to the modification of theorem of alternative for systems which not satisfy Lipschitz condition with respect to state variable.

In this research, we shall consider the relaxation of our guidance problem (we keep in mind the corresponding weakening of conditions of the problem). Namely, we consider not only the original target set and state constraints set, but their neighborhoods. We are trying to find a smallest size of those neighborhoods, which will allow player I to solve his guidance problem with a guarantee. In this new problem, we strive to construct the position function, where every its value is the desired smallest size of above-mentioned neighborhoods. For this function, the corresponding representation in the form of guaranteed result for some natural quality functional is implemented.

Theorem of alternative defines the solution of differential games in classes of positional strategies of the corresponding type. In [7], it is shown that similar alternative is implemented in class of multi-valued generalized non-anticipating strategies. It is important to note the [5], where the possibility of a non-anticipating selection of the set-valued generalized non-anticipating strategies is established (moreover, see [12]). The application of one-valued non-anticipating strategies in the differential game theory can be seen, for example, in [17, 30, 31, 35].

For our own constructions, we shall use multi-valued generalized non-anticipating strategies considered as set-valued non-anticipating mappings on the spaces of generalized controls defined as strategic measures (see [2–4, 6]).

Traditionally, open-loop constructions are used to find a solution for positional differential game (see [21–23, 25]). For the general case of differential game, the Program Iteration Method (PIM) was proposed (see [2–4, 6, 15, 34]). In this paper, we use two modifications of PIM. Namely, the first one is implemented in the space of sets in the position space, and the second one in a subspace of the position space. We show that it is possible to merge these two modifications in a following way. Second modification can be used to find a fixed point of the natural open-loop operator. Further below we shall call such an operator as program one.

This research is a continuation of investigations made by the science school of Sverdlovsk-Ekaterinburg. Many wonderful results were obtained by the members of this community in the field of differential games. They also identified key directions in modern control theory. Those results are connected with generalized solutions of Hamilton-Jacobi equation. Such solutions arise in the the differential game theory as well in terms of stability functions. N.N. Krasovskii has indroduced the definition of ”stability” in [21]. This notation is used in the differential game theory widely. In particular, it was invaluable in the proof of alternative theorem itself [24, 25]. Furthermore, it was extended to the function space in connection with infinitesimal form of stability. Also, generalized solutions provided by A.I. Subbotin played an important role for constructions of stability functions. His ideas are carried on by N.N. Subbotina and her students.

In connection with the above-mentioned generalized solutions, we shall note the investigations of A.I. Subbotin and A.G. Chentsov [14, 33] where another modification of PIM was implemented. Thus, PIM has a natural connection with generalized solutions of A.I. Subbotin. Moreover, we shall note that PIM procedures realize fixed point for operator defining the corresponding procedure. Such fixed point has the property similar to stability (of sets or functions). So, by PIM are reproduced the important property similar to stability which is essential for proof of alternative by N.N. Krasovskii and A.I. Subbotin. Although it’s not used directly in the current paper, the alternative theorem provides a critical importance to such constructions.

In the given article main results are presented as follows. The position function defining limit possibilities of the player I under ”neighborhood”-type realization of guidance with given closed sets in position space is introduced. This is obtained by the PIM modification in terms of set space. For direct construction of the above-mentioned function, the special open-loop (program) operator was proposed. Also, it is established that values of such position function, which defines limit possibilities under ”neighborhood”-type realization of guidance for player I coincide with values of the guaranteed result in the class of set-valued generalized non-anticipating strategies.

Present article is a natural continuation of the research made in [10], where another modification of PIM was considered. Namely, it was related to stability iterations (see ex. [9]). Such iterations implemented the above-mentioned stability property in its limit. As a corollary, in [10], another iterated procedure was generated in terms of function space. However, the results obtained in this paper are different in a following way. We consider PIM adaptation in terms of function space and obtain the position function which defines limit possibilities of the player I under state constraints. Such function is constructed directly using special program operator. Finally, we show that constructed function coincides with the function of guaranteed result of player I in the class of set-valued generalized non-anticipating strategies. Also, it is worth to mention the interesting property of the obtained function. Namely, it is the fixed point of constructed open-loop operator in the function space.

1. Preliminaries

In this paper we use standard notation from set theory. A set is called a *family* if all its elements are sets. To each object z we assign a singleton $\{z\}$ which contains z: $z \in \{z\}$. We also assign to the set H a family $\mathcal{P}(H)$ of all its subsets and assume $\mathcal{P}'(H) \triangleq \mathcal{P}(H) \setminus \{\emptyset\}$. Thus, $\mathcal{P}'(H)$ is the family of all non-empty subsets of H; of course, $\mathcal{P}(\emptyset) = \{\emptyset\}$ and $\mathcal{P}'(\emptyset) = \emptyset$. For an arbitrary non-empty family $\mathcal{U}$ and a set V, the family

$$\mathcal{U}|_V \triangleq \{U \cap V : U \in \mathcal{U}\} \in \mathcal{P}'(\mathcal{P}(V)) \tag{1.1}$$

is the *trace* of $\mathcal{U}$ on the set V. If $\mathbb{H}$ is a set and $\mathcal{H} \in \mathcal{P}'(\mathcal{P}(\mathbb{H}))$, then

$$\mathbf{C}_{\mathbb{H}}[\mathcal{H}] \triangleq \{\mathbb{H} \setminus H : H \in \mathcal{H}\} \in \mathcal{P}'(\mathcal{P}(\mathbb{H})) \tag{1.2}$$

is the family dual with respect to $\mathcal{H}$. Note that in (1.1) and (1.2) the initial family can be a topology.

If A and B are non-empty sets, then by B^A we denote the set of all mappings from A into B. If $f \in B^A$ and $C \in \mathcal{P}'(A)$, then the restriction of f to the set C is defined as follows:

$$(f|C) \triangleq (f(x))_{x \in C} \in B^C.$$

Assume $\mathbb{N} \triangleq \{1; 2; \ldots\} \in \mathcal{P}'(\mathbb{R})$, and $\mathbb{N}_0 \triangleq \{0\} \cup \mathbb{N}$ ($\mathbb{N}$ is a non-empty subset of the real line $\mathbb{R}$). By $\mathcal{R}_+[S]$ we denote the set of all real-valued non-negative functions on non-empty set S. We define

$$\overline{\mathfrak{s}, \mathfrak{t}} \triangleq \{k \in \mathbb{N}_0 | (\mathfrak{s} \leq k) \ \& \ (k \leq \mathfrak{t})\}$$

for all $\mathfrak{s}, \mathfrak{t} \in \mathbb{N}_0$; moreover, $\overrightarrow{m, \infty} \triangleq \{k \in \mathbb{N}_0 | m \leq k\}$ for all $m \in \mathbb{N}_0$. We suppose that elements of $\mathbb{N}$ are not sets. Taking this circumstance into account, for a non-empty set H and a number $k \in \mathbb{N}$, we use H^k instead of $H^{\overline{1,k}}$ for the set of all collections $(h_i)_{i \in \overline{1,k}}$, where $h_i \in H \ \forall i \in \overline{1,k}$. By $H^{\mathbb{N}}$ we denote the set of all sequences in the set H. If $\mathcal{H}$ is a non-empty family, $(\mathbb{H}_i)_{i \in \mathbb{N}} \in \mathcal{H}^{\mathbb{N}}$, and $\mathbb{H}$ is a set, then, by definition

$$((\mathbb{H}_i)_{i \in \mathbb{N}} \downarrow \mathbb{H}) \iff ((\mathbb{H} = \bigcap_{i \in \mathbb{N}} \mathbb{H}_i) \& (\mathbb{H}_{k+1} \subset \mathbb{H}_k \ \forall k \in \mathbb{N})).$$

1.1. Required definitions and notation from measure theory

In this subsection we fix arbitrary set E. Then, by definition, $(\sigma - alg)[E]$ is the family of all σ-algebras of subsets of E. If $\hat{\mathcal{E}} \in (\sigma - alg)[E]$, then $(E, \hat{\mathcal{E}})$ is the measurable space with "unit" E. For $\mathfrak{E} \in \mathcal{P}'(\mathcal{P}(E))$, by $\sigma_E^o(\mathfrak{E})$ we denote σ-algebra of the family $(\sigma - alg)[E]$ generated by $\mathfrak{E}$; this is the weakest σ-algebra of non-empty family $\{\bar{\mathcal{E}} \in (\sigma - alg)[E] | \mathfrak{E} \subset \bar{\mathcal{E}}\}$.

Let us note for $H \in \mathcal{P}(E)$ and $\mathfrak{E} \in \mathcal{P}'(\mathcal{P}(E))$, the family $\mathfrak{E}|_H \in \mathcal{P}'(\mathcal{P}(H))$ generates σ-algebra $\sigma_H^o(\mathfrak{E}|_H) \in (\sigma - alg)[H]$ for which

$$\sigma_H^o(\mathfrak{E}|_H) = \sigma_E^o(\mathfrak{E})|_H;$$

for $H \in \sigma_E^o(\mathfrak{E})$, we obtain the corollary

$$\sigma_H^o(\mathfrak{E}|_H) = \{\Lambda \in \sigma_E^o(\mathfrak{E}) | \Lambda \subset H\}$$

and $(H, \sigma_H^o(\mathfrak{E}|_H)) = (H, \sigma_E^o(\mathfrak{E})|_H)$ is a measurable subspace of $(E, \sigma_E^o(\mathfrak{E}))$. If τ is a topology on E, then $\sigma_E^o(\tau) \in (\sigma - alg)[E]$ is σ-algebra of Borel subsets of E (see e.g. [18, Section 1.3]). For any $\tilde{\mathcal{E}} \in (\sigma - alg)[E]$, by $(\sigma - add)_+[\tilde{\mathcal{E}}]$ we denote the set of all real-valued non-negative countably additive measures on σ-algebra $\tilde{\mathcal{E}}$; for $\mu \in (\sigma - add)_+[\tilde{\mathcal{E}}]$, in the form of $(E, \tilde{\mathcal{E}}, \mu)$, we have standard measure space.

We consider Borel measures: if $\tilde{\mathcal{E}} = \sigma_E^o(\tau)$, where τ is a topology on E, then $\mu \in (\sigma - add)_+[\tilde{\mathcal{E}}]$ is called a *Borel measure*. We use properties of Borel measures noted in [1, Ch. 1 and, moreover, Additions I and II]. Let us note only following property (see [1, Ch. 1]): if (E, τ) is a metrizable topological space, then every measure $\mu \in (\sigma - add)_+[\sigma_E^o(\tau)]$ is regular.

2. The dynamic system: controls and trajectories

We fix $n \in \mathbb{N}$ and consider $\mathbb{R}^n$ as phase space of the system

$$\dot{x} = f(t, x, u, v). \tag{2.1}$$

In addition, we fix $t_0 \in \mathbb{R}$ and $\vartheta_0 \in \mathbb{R}$ such that $t_0 < \vartheta_0$. Let $T \triangleq [t_0, \vartheta_0]$. We consider two non-empty compacta P and Q in the spaces $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively ($p \in \mathbb{N}$ and $q \in \mathbb{N}$). Suppose that in (2.1), $u \in P$ and $v \in Q$. Finally, we suppose that

$$f : T \times \mathbb{R}^n \times P \times Q \to \mathbb{R}^n \tag{2.2}$$

is a continuous function. Thus, (2.1) is a control system on the finite time interval T. We suppose that $u \in P$ and $v \in Q$ are control parameters of players I and II respectively. We will consider player I as an ally; therefore, $u(t) \in P$ is useful control defined on some time interval in T. The control $v(t) \in Q$ of the player II is considered as noise. According to (2.1), the control functions $u_{t_*}(\cdot)_{\vartheta_0} = (u(t),\ t_* \le t \le \vartheta_0)$ and $v_{t_*}(\cdot)_{\vartheta_0} = (v(t),\ t_* \le t \le \vartheta_0)$ defined for simplicity as piecewise constant functions, act on the time interval $[t_*, \vartheta_0]$. If $x_* \in \mathbb{R}^n$, then, as a result, the unique trajectory $x_{t_*}(\cdot)_{\vartheta_0} = (x(t), t_* \le t \le \vartheta_0)$, $x(t_*) = x_*$ is implemented (now, we suppose our system to be equipped with the uniqueness property mentioned above). Later, we shall assume the employment of generalized controls defined as strategic measures. Then, the corresponding conditions will be introduced. At this moment, we consider only informative sense of the problem. Keep in mind that for every position $(t_*, x_*) \in T \times \mathbb{R}^n$, player I has some set $\mathfrak{U}\langle t_*, x_*\rangle$ of admissible procedures for formation of own controls on the time interval $[t_*, \vartheta_0]$. Since the player II can use "arbitrary" (measurable or measure-valued) controls, for every $U \in \mathfrak{U}\langle t_*, x_*\rangle$, some non-empty bundle $\mathcal{X}(t_*, x_*, U)$ of all possible trajectories on $[t_*, \vartheta_0]$ is defined. Let us consider U as a control procedure of $\mathfrak{U}\langle t_*, x_*\rangle$. In addition, it is supposed that all trajectories of $\mathcal{X}(t_*, x_*, U)$ are continuous and the aim of player I consists in guidance with the given target set $\mathbf{M}, \mathbf{M} \subset T \times \mathbb{R}^n$, under state constraints which are defined by

t-cross-sections of the $\mathbf{N}, \mathbf{N} \subset T \times \mathbb{R}^n$ (later, we shall suppose that $\mathbf{M}$ and $\mathbf{N}$ are closed sets). Let us define the goal of player I as follows: for $x(\cdot) \in \mathcal{X}(t_*, x_*, U)$

$$\exists \vartheta \in [t_*, \vartheta_0] : ((\vartheta, x(\vartheta)) \in \mathbf{M}) \ \& \ ((t, x(t)) \in \mathbf{N} \ \forall t \in [t_*, \vartheta]). \tag{2.3}$$

If (2.3) is not guaranteed for every $U \in \mathfrak{U}\langle t_*, x_* \rangle$, then we have another question: for which sets $\tilde{\mathbf{M}}$ and $\tilde{\mathbf{N}}$, where

$$(\mathbf{M} \subset \tilde{\mathbf{M}}) \ \& \ (\mathbf{N} \subset \tilde{\mathbf{N}}), \tag{2.4}$$

exists a procedure $U \in \mathfrak{U}\langle t_*, x_* \rangle$ which will guarantee condition (2.3) with following replacement: $\mathbf{M} \to \tilde{\mathbf{M}}$ and $\mathbf{N} \to \tilde{\mathbf{N}}$. Thus, according to (2.4), we consider a relaxation of the guidance problem. Of course, in many cases, this replacement must satisfy some rules. We shall define the neighborhoods of $\mathbf{M}$ and $\mathbf{N}$ as $\tilde{\mathbf{M}}$ and $\tilde{\mathbf{N}}$ respectively. In particular, we can use ε-neighborhoods, where $\varepsilon > 0$. Also, we define the metric

$$\rho : (T \times \mathbb{R}^n) \times (T \times \mathbb{R}^n) \to [0, \infty[$$

on $T \times \mathbb{R}^n$ as follows: for $t_1 \in T, x_1 \in \mathbb{R}^n, t_2 \in T$, and $x_2 \in \mathbb{R}^n$

$$\rho((t_1, x_1), (t_2, x_2)) \triangleq \sup(\{|t_1 - t_2|; \|x_1 - x_2\|\}), \tag{2.5}$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$. In (2.5) and below we shall use the designation "sup", not "max" under employment of unordered pair of real numbers (in this case, we obtain the greatest of two numbers). The designation "max" is used in connection with values of real-valued functions. Further, we shall follow (2.5). We denote by $\mathbf{t}$ the topology of coordinate-wise convergence on $T \times \mathbb{R}^n$ generated by metric ρ. Let $\mathcal{F}$ be the family of all subsets of $T \times \mathbb{R}^n$ closed in the sense of $(T \times \mathbb{R}^n, \mathbf{t})$:$\mathcal{F} \triangleq \mathbf{C}_{T \times \mathbb{R}^n}[\mathbf{t}]$. Suppose that $\mathcal{F}' \triangleq \mathcal{F} \setminus \{\emptyset\}$.

If $H \in \mathcal{P}'(T \times \mathbb{R}^n)$, then, for $z \in T \times \mathbb{R}^n$, we suppose that $\rho(z; H) \triangleq \inf(\{\rho(z, h) : h \in H\})$; by $\rho(\cdot, H)$ we denote the distance function by

$$z \longmapsto \rho(z; H) : T \times \mathbb{R}^n \to [0, \infty[. \tag{2.6}$$

Of course, $\mathcal{F}' \subset \mathcal{P}'(T \times \mathbb{R}^n)$ and, for $F \in \mathcal{F}'$, the distance function $\rho(\cdot, F)$ is defined. We note that

$$S_0(H, \varepsilon) \triangleq \{z \in T \times \mathbb{R}^n | \rho(z; H) \leq \varepsilon\} \in \mathcal{F}' \quad \forall H \in \mathcal{P}'(T \times \mathbb{R}^n) \ \forall \varepsilon \in [0, \infty[. \tag{2.7}$$

This corresponds the designations of [8]. In particular, for $F \in \mathcal{F}'$ and $\varepsilon \in]0, \infty[$, in the form of $S_0(F, \varepsilon)$, we obtain the closed ε-neighborhood of the set F; moreover, $S_0(F, 0) = F$.

In the following, we suppose that $\mathbf{M} \in \mathcal{F}', \mathbf{N} \in \mathcal{F}'$, and $\mathbf{M} \subset \mathbf{N}$. So, we consider the closed Differential Game. As a corollary, for $\varepsilon \in [0, \infty[$

$$(S_0(\mathbf{M}, \varepsilon) \in \mathcal{F}') \ \& \ (S_0(\mathbf{N}, \varepsilon) \in \mathcal{F}') \ \& \ (S_0(\mathbf{M}, \varepsilon) \subset S_0(\mathbf{N}, \varepsilon)). \tag{2.8}$$

Thus, in terms of (2.6) and (2.8), we introduce two parameterized families in $\mathcal{F}'$. By $\leqq$ we denote the point-wise order on the set $\mathcal{R}_+[T \times \mathbb{R}^n]$ of all real-valued nonnegative functions defined on $T \times \mathbb{R}^n$. From the inclusion $\mathbf{M} \subset \mathbf{N}$, the next relation

$$\rho(\cdot\,,\mathbf{N}) \leqq \rho(\cdot\,,\mathbf{M}) \tag{2.9}$$

holds. By (2.9) we show that for any $(t_*, x_*) \in T \times \mathbb{R}^n$, by the closed interval $[\rho((t_*, x_*), \mathbf{N}), \rho((t_*, x_*), \mathbf{M})]$ the natural (closed) interval of values $\varepsilon \in [0, \infty[$ is valid. We will show that this interval defines the natural boundaries for implementation of ε-weakened analogue of (2.3).

3. Generalized controls and trajectories

In this section we shall consider a very general variant of the system ((2.1),(2.2)), we introduce constructions connected with program controls. Separately, we select generalized controls implemented by strategic measures. Of course, usual controls can be immersed in the generalized control space as everywhere dense subset; in this connection, see [36, Ch. IV] and many other research works (see ex. [19] and [13,25]). Now, by the compactification concepts we are concentrated on the consideration of generalized controls and trajectories. Under $t \in T$, we consider metrizable compacta $[t, \vartheta_0], Z_t \triangleq [t, \vartheta_0] \times Q$, and $\Omega_t \triangleq [t, \vartheta_0] \times P \times Q$ with σ-algebras $\mathcal{T}_t$, $\mathcal{D}_t$, and $\mathcal{C}_t$ of Borel sets respectively; of course $([t, \vartheta_0], \mathcal{T}_t)$, $(Z_t, \mathcal{D}_t)$, and $(\Omega_t, \mathcal{C}_t)$ are measurable spaces. We select Borel cylinders

$$(I \times P \times Q \in \mathcal{C}_t \ \ \forall I \in \mathcal{T}_t) \ \ \& \ \ (I \times Q \in \mathcal{D}_t \ \ \forall I \in \mathcal{T}_t)$$
$$\& \ \ (D \underline{\times} P \triangleq \{(t, u, v) \in \Omega_t \mid (t, v) \in D\} \in \mathcal{C}_t \ \ \forall D \in \mathcal{D}_t).$$

By λ_t we denote the trace of the Lebesgue measure on σ-algebra $\mathcal{T}_t$. Let us introduce the following non-empty sets

$$(\mathcal{H}_t \triangleq \{\eta \in (\sigma - add)_+[\mathcal{C}_t] \mid \eta(I \times P \times Q) = \lambda_t(I) \ \forall I \in \mathcal{T}_t\})$$
$$\& \ (\mathcal{E}_t \triangleq \{\nu \in (\sigma - add)_+[\mathcal{D}_t] | \nu(I \times Q) = \lambda_t(I) \ \forall I \in \mathcal{T}_t\}). \tag{3.1}$$

In (3.1), $\mathcal{H}_t$ is the set of all combined generalized controls on $[t, \vartheta_0]$ and $\mathcal{E}_t$ is the set of generalized controls of player II. Finally, for $\nu \in \mathcal{E}_t$, we suppose that the program generated by corresponding control of player II is defined as follows:

$$\Pi_t(\nu) \triangleq \{\eta \in \mathcal{H}_t | \eta(D \underline{\times} P) = \nu(D) \ \forall D \in \mathcal{D}_t\}. \tag{3.2}$$

By $C(Z_t)$ and $C(\Omega_t)$ we denote the sets of all real-valued continuous functions defined on Z_t and Ω_t respectively. We equip $C(Z_t)$ and $C(\Omega_t)$ with corresponding norms of uniform convergence. As a result, we obtain separable Banach spaces. Let us consider the spaces $C^*(Z_t)$ and $C^*(\Omega_t)$ of all linear bounded functionals on $C(Z_t)$ and $C(\Omega_t)$ respectively. With employment of the Riesz theorem, we immerse $\mathcal{E}_t$ and $\mathcal{H}_t$ in $C^*(Z_t)$ and $C^*(\Omega_t)$ respectively. Of course, $\mathcal{E}_t$ and $\mathcal{H}_t$ are strongly bounded and closed sets in terms of weak-* topology. As a result, by the Banach-Alaoglu theorem we obtain two metrizable compacta (see [16, Ch. V]).

Therefore, in these spaces, compactness and sequential compactness are identified. Moreover, closedness and sequential closedness are identified as well. So, the required relative weak-* topologies are characterized in terms of weak-* convergent sequences. In a similar way the continuity property of mappings defined on $\mathcal{E}_t$ and $\mathcal{H}_t$ can be characterized as sequential continuity for all considered below cases. For weak-* convergence of sequences in $\mathcal{E}_t$ and $\mathcal{H}_t$, we shall use the following designation according to [8]: $\rightharpoonup$. Thus, in our constructions,

$$\mathfrak{F}_t \triangleq \{\mathbb{H} \in \mathcal{P}(\mathcal{H}_t) | \forall (\eta_j)_{j\in\mathbb{N}} \in \mathbb{H}^{\mathbb{N}} \ \forall \eta \in \mathcal{H}_t \ ((\eta_j)_{j\in\mathbb{N}} \rightharpoonup \eta) \Rightarrow (\eta \in \mathbb{H})\}$$

is the family of all weak-* closed subsets of $\mathcal{H}_t$. We note that $\mathcal{E}_t$, $\mathcal{H}_t$, and $\Pi_t(\nu)$ for $\nu \in \mathcal{E}_t$ are sequentially compact sets.

As usually, for $t_* \in T$, by $C_n([t_*, \vartheta_0])$ we denote the set of all continuous mappings from $[t_*, \vartheta_0]$ in $\mathbb{R}^n$. In addition, for $x(\cdot) = (x(t))_{t\in[t_*,\vartheta_0]} \in C_n([t_*, \vartheta_0])$

$$(t, u, v) \longmapsto f(t, x(t), u, v) : \Omega_{t_*} \to \mathbb{R}^n$$

is a continuous mapping; if $\eta \in \mathcal{H}_{t_*}$ and $\theta \in [t_*, \vartheta_0]$, then

$$\int_{[t_*,\theta[\times P\times Q} f(t, x(t), u, v)\eta(d(t, u, v)) \in \mathbb{R}^n$$

is defined component-wise. We suppose for any $(t_*, x_*) \in T \times \mathbb{R}^n$ and $\eta \in \mathcal{H}_{t_*}$

$$\begin{aligned}\Phi(t_*, x_*, \eta) \triangleq \{x(\cdot) \in C_n([t_*, \theta_0]) \mid x(t) \\ = x_* + \int_{[t_*,t[\times P\times Q} f(\xi, x(\xi), u, v)\eta(d(\xi, u, v)) \ \forall t \in [t_*, \theta_0]\}.\end{aligned}$$

In the following, it is supposed that, for every $(t_*, x_*) \in T \times \mathbb{R}^n$ and $\eta \in \mathcal{H}_{t_*}$,

$$\Phi(t_*, x_*, \eta) = \{\varphi(\cdot\,, t_*, x_*, \eta)\}, \tag{3.3}$$

where $\varphi(\cdot\,, t_*, x_*, \eta) = (\varphi(t, t_*, x_*, \eta))_{t\in[t_*,\vartheta_0]} \in C_n([t_*, \vartheta_0])$. This condition corresponds to the generalized uniqueness condition of A.V. Kryazhimsky (see [11]). In (3.3), $\varphi(\cdot\,, t_*, x_*, \eta)$ is generalized trajectory of system (2.1) corresponding to the position (t_*, x_*) and control-measure η. In the following, we suppose that $\forall a \in [0, \infty[\ \exists b \in [0, \infty[\ \forall x \in \mathbb{R}^n$

$$(\|x\| \leq a) \Rightarrow (\|\varphi(\theta, t, x, \eta)\| \leq b \ \forall t \in T \ \forall \eta \in \mathcal{H}_t \ \forall \theta \in [t, \vartheta_0]).$$

We note that for $t_* \in T$ the mapping

$$(x, \eta) \longmapsto \varphi(\cdot\,, t_*, x, \eta) : \mathbb{R}^n \times \mathcal{H}_{t_*} \to C_n([t_*, \vartheta_0]) \tag{3.4}$$

is continuous; in addition, $\mathbb{R}^n$ is equipped with usual $\|\cdot\|$-topology, $\mathcal{H}_{t_*}$ is equipped with relative weak-* topology, and $C_n([t_*, \vartheta_0])$ is equipped with usual topology of uniform convergence. In connection with this continuity property, see [13, Ch. IV] and [8, Section 4]. Of course, for $(t_*, x_*) \in T \times \mathbb{R}^n$ and $\nu \in \mathcal{E}_{t_*}$, the bundle

$$\mathcal{X}_\Pi(t_*, x_*, \nu) \triangleq \{\varphi(\cdot\,, t_*, x_*, \eta) : \eta \in \Pi_{t_*}(\nu)\}, \tag{3.5}$$

is a non-empty compactum in $C_n([t_*, \vartheta_0])$ with the topology of uniform convergence.

Let us restrict ourselves to a brief remark in respect to usual controls of the simplest types. Namely, these controls can be considered as variants of generalized controls (see ex. [36, Ch. IV]). Under $t_* \in T$, we consider now a piece-wise constant and continuous from the right control function $\bar{v}(\cdot) = \bar{v}_{t_*}(\cdot)_{\vartheta_0}$ with values in Q. Then

$$g \longmapsto \int_{t_*}^{\vartheta_0} g(t, \bar{v}(t))dt : C(Z_{t_*}) \to \mathbb{R}$$

is an element of $C^*(Z_{t_*})$ implemented by the unique measure $\nu[\bar{v}(\cdot)] \in \mathcal{E}_{t_*}$ in the form

$$g \longmapsto \int_{Z_{t_*}} g(t, v)\nu[\bar{v}(\cdot)](d(t, v)) : C(Z_{t_*}) \to \mathbb{R}.$$

Then, $v(\cdot) \longmapsto \nu[v(\cdot)]$ defines the required immersion of the simplest usual program controls in $\mathcal{E}_{t_*}$. In a similar way, the pair $(\bar{u}(\cdot), \bar{v}(\cdot))$ of usual piece-wise constant and continuous from the right controls is implemented by unique measure $\bar{\eta} = \eta[\bar{u}(\cdot), \bar{v}(\cdot)] \in \mathcal{H}_{t_*}$. We keep in mind that $\bar{\eta}$ is a realization of the functional

$$g \longmapsto \int_{t_*}^{\vartheta_0} g(t, \bar{u}(t), \bar{v}(t))dt : C(\Omega_{t_*}) \to \mathbb{R}.$$

More general representations of such type are reduced in [36, Ch. 4]. We restrict ourselves to representation of usual controls as linear bounded functionals on $C(Z_{t_*})$ and $C(\Omega_{t_*})$ respectively.

4. Generalized multi-valued non-anticipating strategies

In this section, we consider the required variant of admissible procedures for our control constructions. We suppose that these procedures are multi-valued operators in measure spaces. We employ non-anticipating control procedures. Let us provide some additional definitions.

If $t_1 \in T$ and $t_2 \in [t_1, \vartheta_0]$, then $\mathcal{D}_{t_2} = \{D \in \mathcal{D}_{t_1} | D \subset Z_{t_2}\}$ and $\mathcal{C}_{t_2} = \{C \in \mathcal{C}_{t_1} | C \subset \Omega_{t_2}\}$. In addition, $\mathcal{E}_{t_2} = \{(\nu|\mathcal{D}_{t_2}) : \nu \in \mathcal{E}_{t_1}\}$ and $\mathcal{H}_{t_2} = \{(\eta|\mathcal{C}_{t_2}) : \eta \in \mathcal{H}_{t_1}\}$. Moreover, for such t_1 and t_2, we have obtained the following σ-algebras of sets:

$$\mathcal{D}_{t_1}^{t_2} \triangleq \mathcal{D}_{t_1}|_{[t_1, t_2[\times Q} = \{D \in \mathcal{D}_{t_1} | D \subset [t_1, t_2[\times Q\},$$
$$\mathcal{C}_{t_1}^{t_2} \triangleq \mathcal{C}_{t_1}|_{[t_1, t_2[\times P \times Q} = \{H \in \mathcal{C}_{t_1} | H \subset [t_1, t_2[\times P \times Q\}.$$

Of course, for $\nu \in \mathcal{E}_{t_1}$, measure $(\nu|\mathcal{D}_{t_1}^{t_2}) \in (\sigma - add)_+[\mathcal{D}_{t_1}^{t_2}]$ is defined and, for $\eta \in \mathcal{H}_{t_1}$, measure $(\eta|\mathcal{C}_{t_1}^{t_2}) \in (\sigma - add)_+[\mathcal{C}_{t_1}^{t_2}]$ is defined too. With employment of these properties, we introduce multi-valued non-anticipating strategies.

Namely, according to [8], for $t_* \in T$.

$$\tilde{A}_{t_*} \triangleq \{\alpha \in \prod_{\nu \in \mathcal{E}_{t_*}} \mathcal{P}'(\Pi_{t_*}(\nu)) | \forall \nu_1 \in \mathcal{E}_{t_*}\ \forall \nu_2 \in \mathcal{E}_{t_*}\ \forall \theta \in [t_*, \vartheta_0]$$
$$((\nu_1 | \mathcal{D}^\theta_{t_*}) = (\nu_2 | \mathcal{D}^\theta_{t_*})) \Rightarrow (\{(\eta | \mathcal{C}^\theta_{t_*}) : \eta \in \alpha(\nu_1)\} = \{(\eta | \mathcal{C}^\theta_{t_*}) : \eta \in \alpha(\nu_2)\})\} \quad (4.1)$$

is the set of all generalized multi-valued non-anticipating strategies on time interval $[t_*, \vartheta_0]$. The employment of set-valued mappings for non-anticipating strategies allows to indicate the required solving strategy in a constructive way (see Section 6). We obtain (see [8])

$$\tilde{\Pi}_{t_*}(\alpha) \triangleq \bigcup_{\nu \in \mathcal{E}_{t_*}} \alpha(\nu) \in \mathcal{P}'(\mathcal{H}_{t_*})\ \forall \alpha \in \tilde{A}_{t_*}. \quad (4.2)$$

Among all non-anticipating strategies, we select quasi-programs (improved non-anticipating strategies, in particular, ones with more desirable topological properties), namely for $t_* \in T$

$$\tilde{A}^{\Pi}_{t_*} \triangleq \{\alpha \in \tilde{A}_{t_*} | \tilde{\Pi}_{t_*}(\alpha) \in \mathfrak{F}_{t_*}\} \quad (4.3)$$

is the set of all quasi-programs, in addition

$$\Pi_{t_*}(\cdot) \triangleq (\Pi_{t_*}(\nu))_{\nu \in \mathcal{E}_{t_*}} \in \tilde{A}^{\Pi}_{t_*}.$$

So, $\tilde{A}_{t_*}$ and $\tilde{A}^{\Pi}_{t_*}$ are non-empty sets. Thus, quasi-programs are indeed enhanced non-anticipating strategies (see (4.3)). We suppose for $(t_*, x_*) \in T \times \mathbb{R}^n$ and $\alpha \in \tilde{A}_{t_*}$

$$\mathbb{X}[t_*; x_*; \alpha] \triangleq \{\varphi(\cdot, t_*, x_*, \eta) : \eta \in \tilde{\Pi}_{t_*}(\alpha)\}, \quad (4.4)$$

where t_*, x_* and α are parameters (in this case, we use ";", not "," as a separation symbol). Thus, in (4.4), we obtain the trajectory bundle corresponding to (t_*, x_*) and α. In terms of our notation given in previous sections, we can use $\tilde{A}_{t_*}$ (or $\tilde{A}^{\Pi}_{t_*}$) as $\mathfrak{U}\langle t_*, x_* \rangle$ and $\mathbb{X}[t_*; x_*; \alpha]$ as $\mathcal{X}(t_*, x_*, U)$ (the last notation was introduced in the informative part of the paper), where $U = \alpha \in \tilde{A}_{t_*}$. As a result, for $(t_*, x_*) \in T \times \mathbb{R}^n$, we can introduce the next local problem. Namely, it is required to find the smallest number $\varepsilon_* \in [0, \infty[$ for which $\exists \alpha \in \tilde{A}_{t_*}\ \forall x(\cdot) \in \mathbb{X}[t_*; x_*; \alpha]\ \exists \vartheta \in [t_*, \vartheta_0]$:

$$((\vartheta, x(\vartheta)) \in S_0(\mathbf{M}, \varepsilon_*))\ \&\ ((t, x(t)) \in S_0(\mathbf{N}, \varepsilon_*)\ \forall t \in [t_*, \vartheta]); \quad (4.5)$$

of course, the problem of constructing this non-anticipating strategy $\alpha \in \tilde{A}_{t_*}$ is important, since it is implementing the property (4.5) for every $x(\cdot) \in \mathbb{X}[t_*; x_*; \alpha]$. However, in this study we shall focus more closely on finding that ε_*. Let us consider this problem more carefully. Namely, our goal is to find the number, which is very close to ε_* for every position (t_*, x_*). Thus, the corresponding function of position is being constructed with program iteration methods (PIM), similar to those in [2–4, 6, 8].

5. Program iteration method I

Here we consider a first modification of PIM. We restrict ourselves to the case of a closed differential game, where both target set and the set, defining the state constraints, are closed in $(T \times \mathbb{R}^n, \mathbf{t})$ (we note, that, in [8], a more general case was considered). Therefore, we can use only a particular variant of the constructions from [8]. So, if $M \in \mathcal{F}$, then we consider the operator

$$\mathbb{A}[M] : \mathcal{F} \to \mathcal{F} \tag{5.1}$$

defined by the following rule (see ex. [8, Section 7])

$$\mathbb{A}[M](F) \triangleq \big\{(t,x)\in F \mid \forall \nu \in \mathcal{E}_t \ \exists x(\cdot)\in \mathcal{X}_\Pi(t,x,\nu) \ \exists \vartheta \in [t,\vartheta_0] : ((\vartheta, x(\vartheta))\in M) \\ \&((\xi, x(\xi)) \in F \ \ \forall \xi \in [t,\vartheta])\big\}. \tag{5.2}$$

Of course, we use representation of [8, Section 5] and (3.5). Our operator (5.1) and (5.2) is the narrowing of operator defined in [8, (5.5)] to the family $\mathcal{F}$. Therefore, we use properties noted in [8, Section 5]. Now, we only note that according to (5.1) and (5.2), iteration procedures in $\mathcal{F}$ are implemented. Namely, for $M \in \mathcal{F}$ and $N \in \mathcal{F}$, the iterative sequence

$$(\mathcal{W}_s(M,N))_{s\in\mathbb{N}_0} : \mathbb{N}_0 \to \mathcal{F} \tag{5.3}$$

is implemented by the following rules:

$$(\mathcal{W}_0(M,N) \triangleq N) \ \& \ (\mathcal{W}_s(M,N) = \mathbb{A}[M](\mathcal{W}_{s-1}(M,N)) \ \forall s \in \mathbb{N}), \tag{5.4}$$

moreover, we obtain the limit set

$$\mathcal{W}(M,N) \triangleq \bigcap_{s\in\mathbb{N}_0} \mathcal{W}_s(M,N) \in \mathcal{F}. \tag{5.5}$$

In [8, Section 10], the representation of (5.4) as the solvability set in guidance problem corresponding to the pair (M,N) is indicated. Now, we note that for $M_1 \in \mathcal{F}$, $N_1 \in \mathcal{F}$, $M_2 \in \mathcal{F}$ and $N_2 \in \mathcal{F}$, implication

$$((M_1 \subset M_2)\&(N_1 \subset N_2)) \Rightarrow ((\mathcal{W}_s(M_1,N_1) \subset \mathcal{W}_s(M_2,N_2) \ \forall s \in \mathbb{N}_0) \\ \& \ (\mathcal{W}(M_1,N_1) \subset \mathcal{W}(M_2,N_2))) \tag{5.6}$$

is true. Of course, $M \cap N \subset \mathcal{W}(M,N) \ \forall M \in \mathcal{F} \ \forall N \in \mathcal{F}$. According to (2.8), for $\varepsilon \in [0,\infty[$ we have

$$(S_0(\mathbf{M},\varepsilon) \subset \mathcal{W}_s(S_0(\mathbf{M},\varepsilon), S_0(\mathbf{N},\varepsilon)) \ \forall s \in \mathbb{N}_0) \\ \& \ (S_0(\mathbf{M},\varepsilon) \subset \mathcal{W}(S_0(\mathbf{M},\varepsilon), S_0(\mathbf{N},\varepsilon))). \tag{5.7}$$

Since $\mathbf{M}$ is a non-empty set, we obtain, using (5.7)

$$\Big(T \times \mathbb{R}^n = \bigcup_{\varepsilon\in[0,\infty[} \mathcal{W}_s(S_0(\mathbf{M},\varepsilon), S_0(\mathbf{N},\varepsilon)) \ \forall s \in \mathbb{N}_0\Big) \\ \& \ \Big(T \times \mathbb{R}^n = \bigcup_{\varepsilon\in[0,\infty[} \mathcal{W}(S_0(\mathbf{M},\varepsilon), S_0(\mathbf{N},\varepsilon))\Big). \tag{5.8}$$

With the help of (5.8), we obtain for any $(t_*, x_*) \in T \times \mathbb{R}^n$

$$(\Sigma_0^{(s)}(t_*, x_*) \triangleq \{\varepsilon \in [0, \infty[| (t_*, x_*) \in \mathcal{W}_s(S_0(\mathbf{M}, \varepsilon), S_0(\mathbf{N}, \varepsilon))\} \neq \emptyset \ \forall s \in \mathbb{N}_0)$$
$$\& \ (\Sigma_0(t_*, x_*) \triangleq \{\varepsilon \in [0, \infty[| (t_*, x_*) \in \mathcal{W}(S_0(\mathbf{M}, \varepsilon), S_0(\mathbf{N}, \varepsilon))\} \neq \emptyset). \quad (5.9)$$

Here and below, given notation logically corresponds to [10]. By using (5.9), we suppose that $\varepsilon_0^{(s)} \in \mathcal{R}_+[T \times \mathbb{R}^n]$ with $s \in \mathbb{N}_0$ and $\varepsilon_0 \in \mathcal{R}_+[T \times \mathbb{R}^n]$ are defined by the following properties: for $(t, x) \in T \times \mathbb{R}^n$

$$(\varepsilon_0^{(s)}(t, x) \triangleq \inf(\Sigma_0^{(s)}(t, x))) \ \& \ (\varepsilon_0(t, x) \triangleq \inf(\Sigma_0(t, x))). \quad (5.10)$$

As a result, we obtain the sequence

$$(\varepsilon_0^{(s)})_{s \in \mathbb{N}_0} : \mathbb{N}_0 \to \mathcal{R}_+[T \times \mathbb{R}^n]$$

and a function ε_0. Thus, we have the following property, similar to one in [10, Proposition 1]:

$$\varepsilon_0(t, x) \in \Sigma_0(t, x) \ \forall (t, x) \in T \times \mathbb{R}^n. \quad (5.11)$$

Moreover, we obtain properties of $\varepsilon_0^{(s)}, s \in \mathbb{N}_0$, which are similar to (5.11) and correct, according to [8, Proposition 6.4]. Namely, next proposition takes place. Following proposition is similar with one in [10].

Proposition 5.1. *If $s \in \mathbb{N}_0$ and $(t_*, x_*) \in T \times \mathbb{R}^n$, then $\varepsilon_0^{(s)}(t_*, x_*) \in \Sigma_0^{(s)}(t_*, x_*)$.*

Proof. Using (5.10), we choose the sequence $(\beta_k)_{k \in \mathbb{N}} : \mathbb{N} \to \Sigma_0^{(s)}(t_*, x_*)$ for which

$$((\beta_k)_{k \in \mathbb{N}} \to \varepsilon_*) \ \& \ (\beta_{l+1} \leq \beta_l \quad \forall l \in \mathbb{N}),$$

where $\varepsilon_* \triangleq \varepsilon_0^{(s)}(t_*, x_*)$. Then, we obtain the following properties:

$$((S_0(\mathbf{M}, \beta_k))_{k \in \mathbb{N}} \downarrow S_0(\mathbf{M}, \varepsilon_*)) \ \& \ ((S_0(\mathbf{N}, \beta_k))_{k \in \mathbb{N}} \downarrow S_0(\mathbf{N}, \varepsilon_*)).$$

Therefore, by [8, Proposition 6.4], the following convergence

$$(\mathcal{W}_s(S_0(\mathbf{M}, \beta_k), S_0(\mathbf{N}, \beta_k)))_{k \in \mathbb{N}} \downarrow \mathcal{W}_s(S_0(\mathbf{M}, \varepsilon_*), S_0(\mathbf{N}, \varepsilon_*)) \quad (5.12)$$

holds. In particular, from (5.12) the next equality follows:

$$\mathcal{W}_s(S_0(\mathbf{M}, \varepsilon_*), S_0(\mathbf{N}, \varepsilon_*)) = \bigcap_{k \in \mathbb{N}} \mathcal{W}_s(S_0(\mathbf{M}, \beta_k), S_0(\mathbf{N}, \beta_k)). \quad (5.13)$$

By (5.9), $(t_*, x_*) \in \mathcal{W}_s(S_0(\mathbf{M}, \beta_k), S_0(\mathbf{N}, \beta_k)) \ \forall k \in \mathbb{N}$. Thus, according to (5.13),

$$(t_*, x_*) \in \mathcal{W}_s(S_0(\mathbf{M}, \varepsilon_*), S_0(\mathbf{N}, \varepsilon_*)).$$

Then, by using (5.9), we obtain $\varepsilon_* \in \Sigma_0^{(s)}(t_*, x_*)$. □

From (5.5), (5.9) and (5.10), we obtain $\Sigma_0(t,x) \subset \Sigma_0^{(s)}(t,x)\ \forall s \in \mathbb{N}_0\ \forall (t,x) \in T \times \mathbb{R}^n$. As a corollary,

$$\varepsilon_0^{(s)} \leqq \varepsilon_0\ \forall s \in \mathbb{N}_0. \tag{5.14}$$

From (5.14), we obtain $\{\varepsilon_0^{(s)}(t,x) : s \in \mathbb{N}_0\} \subset [0, \varepsilon_0(t,x)]\ \forall (t,x) \in T \times \mathbb{R}^n$. Therefore, the corresponding finite supremum is defined:

$$\sup_{k \in \mathbb{N}_0} \varepsilon_0^{(k)}(t,x) \in [0, \varepsilon_0(t,x)]\ \forall (t,x) \in T \times \mathbb{R}^n. \tag{5.15}$$

Proposition 5.2. *For every position* $(t_*, x_*) \in T \times \mathbb{R}^n$

$$\varepsilon_0(t_*, x_*) = \sup_{k \in \mathbb{N}_0} \varepsilon_0^{(k)}(t_*, x_*).$$

Proof. We suppose that $\Xi \triangleq \{\varepsilon_0^{(k)}(t_*, x_*) : k \in \mathbb{N}_0\}$. Of course, by (5.15), we have $\Xi \in \mathcal{P}'([0, \varepsilon^*])$, where $\varepsilon^* \triangleq \varepsilon_0(t_*, x_*)$. Then,

$$\varepsilon_* \triangleq \sup(\Xi) \leq \varepsilon^*. \tag{5.16}$$

Let us show that $\varepsilon_* = \varepsilon^*$. Indeed, let $\varepsilon_* \neq \varepsilon^*$. Then, by (5.16) $\varepsilon_* < \varepsilon^*$. By (5.9) and (5.10) we obtain $\varepsilon_* \notin \Sigma_0(t_*, x_*)$ and, as a corollary

$$(t_*, x_*) \notin \mathcal{W}(S_0(\mathbf{M}, \varepsilon_*), S_0(\mathbf{N}, \varepsilon_*)).$$

Then, by (5.5)

$$(t_*, x_*) \notin \mathcal{W}_r(S_0(\mathbf{M}, \varepsilon_*), S_0(\mathbf{N}, \varepsilon_*)) \tag{5.17}$$

for some $r \in \mathbb{N}_0$. In addition, $\varepsilon_*^{(r)} \triangleq \varepsilon_0^{(r)}(t_*, x_*) \in \Xi$ and, as a corollary, $\varepsilon_*^{(r)} \leq \varepsilon_*$.

Then

$$(S_0(\mathbf{M}, \varepsilon_*^{(r)}) \subset S_0(\mathbf{M}, \varepsilon_*))\ \&\ (S_0(\mathbf{N}, \varepsilon_*^{(r)}) \subset S_0(\mathbf{N}, \varepsilon_*)).$$

As a corollary, we obtain

$$\mathcal{W}_r(S_0(\mathbf{M}, \varepsilon_*^{(r)}), S_0(\mathbf{N}, \varepsilon_*^{(r)})) \subset \mathcal{W}_r(S_0(\mathbf{M}, \varepsilon_*), S_0(\mathbf{N}, \varepsilon_*)). \tag{5.18}$$

According to Proposition 5.1, $\varepsilon_*^{(r)} \in \Sigma_0^{(r)}(t_*, x_*)$, we have the following inclusion

$$(t_*, x_*) \in \mathcal{W}_r(S_0(\mathbf{M}, \varepsilon_*^{(r)}), S_0(\mathbf{N}, \varepsilon_*^{(r)})).$$

As a corollary, by (5.18), the inclusion

$$(t_*, x_*) \in \mathcal{W}_r(S_0(\mathbf{M}, \varepsilon_*), S_0(\mathbf{N}, \varepsilon_*))$$

holds, which clearly contradicts to (5.17). Thus, $\varepsilon_* = \varepsilon^*$. □

Proposition 5.3. *If* $(t_*, x_*) \in T \times \mathbb{R}^n$, *then*

$$(\Sigma_0^{(s)}(t_*, x_*) = [\varepsilon_0^{(s)}(t_*, x_*), \infty[\ \forall s \in \mathbb{N}_0)\ \&\ (\Sigma_0(t_*, x_*) = [\varepsilon_0(t_*, x_*), \infty[).$$

The corresponding proof is similar to the one in [10, Section 5]. We use [8, Proposition 6.4 and Theorem 6.1].

Proposition 5.4. *If* $b \in [0, \infty[$*, then*

$$((\varepsilon_0^{(k)})^{-1}([0, b]) = \mathcal{W}_k(S_0(\mathbf{M}, b), S_0(\mathbf{N}, b)) \ \forall k \in \mathbb{N}_0)$$
$$\& \ ((\varepsilon_0)^{-1}([0, b]) = \mathcal{W}(S_0(\mathbf{M}, b), S_0(\mathbf{N}, b))).$$

Proof. In this proof we will show that $(\varepsilon_0^{(k)})^{-1}([0, b]) = \mathcal{W}_k(S_0(\mathbf{M}, b), S_0(\mathbf{N}, b))$ $\forall k \in \mathbb{N}_0$, and the second part of the proposition can be shown in a similar way. We fix $b \in [0, \infty[$ and $k \in \mathbb{N}_0$. If $(t_*, x_*) \in \mathcal{W}_k(S_0(\mathbf{M}, b), S_0(\mathbf{N}, b))$, then $b \in \Sigma_0^{(k)}(t_*, x_*)$ and by (5.10) $\varepsilon_0^{(k)}(t_*, x_*) \le b$. Thus, $(t_*, x_*) \in (\varepsilon_0^{(k)})^{-1}([0, b])$. We have shown that

$$\mathcal{W}_k(S_0(\mathbf{M}, b), S_0(\mathbf{N}, b)) \subset (\varepsilon_0^{(k)})^{-1}([0, b]). \tag{5.19}$$

Now, we fix $(t^*, x^*) \in (\varepsilon_0^{(k)})^{-1}([0, b])$. Then $(t^*, x^*) \in T \times \mathbb{R}^n$ and $\varepsilon_0^{(k)}(t^*, x^*) \le b$. By Proposition 5.3, $b \in \Sigma_0^{(k)}(t^*, x^*)$. Then (see (5.9)),

$$(t^*, x^*) \in \mathcal{W}_k(S_0(\mathbf{M}, b), S_0(\mathbf{N}, b)).$$

Therefore, we have obtained the inclusion, which is opposite to (5.19). Proposition 5.4 is fully proved. □

Proposition 5.5. *If* $s \in \mathbb{N}_0$ *and* $(t_*, x_*) \in T \times \mathbb{R}^n$*, then* $\Sigma_0^{(s+1)}(t_*, x_*) \subset \Sigma_0^{(s)}(t_*, x_*)$.

Proof. Fix $s \in \mathbb{N}_0$ and $(t_*, x_*) \in T \times \mathbb{R}^n$. Moreover, let $\varepsilon_* \in \Sigma_0^{(s+1)}(t_*, x_*)$. Then, by (5.9), $(t_*, x_*) \in \mathcal{W}_{s+1}(S_0(\mathbf{M}, \varepsilon_*), S_0(\mathbf{N}, \varepsilon_*))$. In addition, by (5.2) and (5.4)

$$\mathcal{W}_{s+1}(S_0(\mathbf{M}, \varepsilon_*), S_0(\mathbf{N}, \varepsilon_*)) \subset \mathcal{W}_s(S_0(\mathbf{M}, \varepsilon_*), S_0(\mathbf{N}, \varepsilon_*)).$$

Thus, $(t_*, x_*) \in \mathcal{W}_s(S_0(\mathbf{M}, \varepsilon_*), S_0(\mathbf{N}, \varepsilon_*))$ and by (5.9), $\varepsilon_* \in \Sigma_0^{(s)}(t_*, x_*)$. □

From (5.10) and Proposition 5.5, we obtain

$$\varepsilon_0^{(s)} \leqq \varepsilon_0^{(s+1)} \ \forall s \in \mathbb{N}_0. \tag{5.20}$$

As a corollary, with (5.20) and Proposition 5.2, we show that

$$(\varepsilon_0^{(k)}(t, x))_{k \in \mathbb{N}} \uparrow \varepsilon_0(t, x) \ \forall (t, x) \in T \times \mathbb{R}^n. \tag{5.21}$$

Proposition 5.6. *The functions* $\varepsilon_0^{(0)}$ *and* $\rho(\cdot, \mathbf{N})$ *coincide:* $\varepsilon_0^{(0)} = \rho(\cdot, \mathbf{N})$.

Proof. From (5.4) and (5.9), we obtain

$$\Sigma_0^{(0)}(t, x) = \{\varepsilon \in [0, \infty[\,|\, (t, x) \in S_0(\mathbf{N}, \varepsilon)\} \ \forall (t, x) \in T \times \mathbb{R}^n. \tag{5.22}$$

Let $(t_*, x_*) \in T \times \mathbb{R}^n$. Then by (5.22) and Proposition 5.3

$$\Sigma_0^{(0)}(t_*, x_*) = \{\varepsilon \in [0, \infty[\,|\, (t_*, x_*) \in S_0(\mathbf{N}, \varepsilon)\} = [\varepsilon_0^{(0)}(t_*, x_*), \infty[. \tag{5.23}$$

Then, $(t_*, x_*) \in S_0(\mathbf{N}, \varepsilon_0^{(0)}(t_*, x_*))$ and by (2.7)

$$\rho((t_*, x_*), \mathbf{N}) \leq \varepsilon_0^{(0)}(t_*, x_*). \tag{5.24}$$

On the other hand, by (2.7), $(t_*, x_*) \in S_0(\mathbf{N}, \rho((t_*, x_*), \mathbf{N}))$. Then, by (5.23), $\rho((t_*, x_*), \mathbf{N}) \in \Sigma_0^{(0)}(t_*, x_*)$ and, as a corollary, $\varepsilon_0^{(0)}(t_*, x_*) \leq \rho((t_*, x_*), \mathbf{N})$. Thus, by (5.24) we have the required equality $\varepsilon_0^{(0)} = \rho(\cdot\,, \mathbf{N})$. □

Let us suppose that

$$\psi \triangleq \rho(\cdot\,, \mathbf{M}). \tag{5.25}$$

Proposition 5.7. *The function ψ is a majorant over ε_0: $\varepsilon_0 \leqq \psi$.*

Proof. Let $(t_*, x_*) \in T \times \mathbb{R}^n$ and $\varepsilon_* \triangleq \psi(t_*, x_*) = \rho((t_*, x_*), \mathbf{M})$.
Then $(t_*, x_*) \in S_0(\mathbf{M}, \varepsilon_*)$. By (5.7)

$$(t_*, x_*) \in \mathcal{W}(S_0(\mathbf{M}, \varepsilon_*), S_0(\mathbf{N}, \varepsilon_*)). \tag{5.26}$$

From (5.9) and (5.26), we obtain $\varepsilon_* \in \Sigma_0(t_*, x_*)$. Then, by (5.10) $\varepsilon_0(t_*, x_*) \leq \varepsilon_* = \psi(t_*, x_*)$. Since the selection of position (t_*, x_*) was arbitrary, the property $\varepsilon_0 \leqq \psi$ is established. □

From Proposition 5.2 and Proposition 5.7, we obtain

$$\varepsilon_0^{(s)} \leqq \psi \ \forall s \in \mathbb{N}_0. \tag{5.27}$$

Suppose that $C_+(T \times \mathbb{R}^n)$ is the set of all continuous real-valued non-negative functions defined on $T \times \mathbb{R}^n$. Then it is correct to say that $C_+(T \times \mathbb{R}^n) \subset \mathcal{R}_+[T \times \mathbb{R}^n]$ and

$$(\psi = \rho(\cdot\,, \mathbf{M}) \in C_+(T \times \mathbb{R}^n)) \ \& \ (\varepsilon_0^{(0)} = \rho(\cdot\,, \mathbf{N}) \in C_+(T \times \mathbb{R}^n)). \tag{5.28}$$

Let $\mathfrak{M}$ is the set of all lower semi-continuous non-negative real-valued functions on $T \times \mathbb{R}^n$, namely $\mathfrak{M} \triangleq \{g \in \mathcal{R}_+[T \times \mathbb{R}^n] | g^{-1}([0, b]) \in \mathcal{F} \ \forall b \in [0, \infty[\}$ and

$$\mathfrak{M}_\psi \triangleq \{g \in \mathfrak{M} | g \leqq \psi\}. \tag{5.29}$$

Then $C_+(T \times \mathbb{R}^n) \subset \mathfrak{M}$.

Proposition 5.8. *The following properties are valid:*

$$(\varepsilon_0^{(k)} \in \mathfrak{M}_\psi \ \forall k \in \mathbb{N}_0) \ \& \ (\varepsilon_0 \in \mathfrak{M}_\psi). \tag{5.30}$$

Proof. Fix $s \in \mathbb{N}_0$. Then, by (5.3) and Proposition 5.4

$$(\varepsilon_0^{(s)})^{-1}([0, b]) \in \mathcal{F} \ \forall b \in [0, \infty[.$$

Therefore, $\varepsilon_0^{(s)} \in \mathfrak{M}$. From (5.27), we obtain $\varepsilon_0^{(s)} \leqq \psi$. Then, by (5.29), $\varepsilon_0^{(s)} \in \mathfrak{M}_\psi$. Since the selection of s was arbitrary, we obtain the following property: $\varepsilon_0^{(k)} \in \mathfrak{M}_\psi \ \forall k \in \mathbb{N}_0$.

Moreover, from (5.5) and Proposition 5.4, we obtain $(\varepsilon_0)^{-1}([0,b]) \in \mathcal{F}\ \forall b \in [0,\infty[$. Therefore, $\varepsilon_0 \in \mathfrak{M}$. By using (5.29) and Proposition 5.7, we finally obtain $\varepsilon_0 \in \mathfrak{M}_\psi$. $\square$

Thus, we have the sequence $(\varepsilon_0^{(k)})_{k\in\mathbb{N}_0}$ in $\mathfrak{M}_\psi$ and the limit function ε_0. In the next section we will show that ε_0 is a function of guaranteed result in a class of non-anticipating strategies.

6. Guaranteed result in a class of non-anticipating strategies

First of all, we recall some statements of [8] connected with representation of solvability sets of player I. Those statements are considered in the general form for pursuit-evasion differential game. We consider generalized program (open-loop) controls $\eta \in \Pi_{t_*}(\nu)$ solving the problem of guidance to M under state constraints defined by $\mathcal{W}(M,N)$; namely (see [8, (10.2)]) for $M \in \mathcal{F}$, $N \in \mathcal{F}$ and $(t_*, x_*) \in \mathcal{W}(M,N)$

$$\pi^{(\mathcal{W})}_{t_*,x_*}\langle \nu | M, N\rangle \triangleq \{\eta \in \Pi_{t_*}(\nu) | \exists \vartheta \in [t_*, \vartheta_0] : ((\vartheta, \varphi(\vartheta, t_*, x_*, \eta)) \in M) \\ \& \ ((t, \varphi(t, t_*, x_*, \eta)) \in \mathcal{W}(M,N)\ \forall t \in [t_*, \vartheta])\}. \quad (6.1)$$

From (6.1) and [8, Proposition 10.3], we obtain

$$\pi^{(\mathcal{W})}_{t_*,x_*}\langle \cdot\, | M, N\rangle \in \tilde{A}^{\Pi}_{t_*}\ \forall M \in \mathcal{F}\ \forall N \in \mathcal{F}\ \forall (t_*, x_*) \in \mathcal{W}(M,N). \quad (6.2)$$

Thus, according to (6.1), a quasi-program is defined. Now, we note an important statement of [8, Theorem 10.1]. To accomplish this, for $M \in \mathcal{F}$, $N \in \mathcal{F}$, and $(t_*, x_*) \in N$, we introduce the set

$$\mathcal{S}_{M,N}(t_*, x_*) \triangleq \{\eta \in \mathcal{H}_{t_*} | \exists \vartheta \in [t_*, \vartheta_0] : ((\vartheta, \varphi(\vartheta, t_*, x_*, \eta)) \in M) \\ \& \ ((t, \varphi(t, t_*, x_*, \eta)) \in N\ \forall t \in [t_*, \vartheta])\} \quad (6.3)$$

of all generalized controls implementing the guidance to M under state constraints defined by N. Then, by [8, Theorem 10.1] for $M \in \mathcal{F}$ and $N \in \mathcal{F}$

$$\mathcal{W}(M,N) = \{(t,x) \in N | \exists \alpha \in \tilde{A}_t : \tilde{\Pi}_t(\alpha) \subset \mathcal{S}_{M,N}(t,x)\} \\ = \{(t,x) \in N | \exists \alpha \in \tilde{A}^{\Pi}_t : \tilde{\Pi}_t(\alpha) \subset \mathcal{S}_{M,N}(t,x)\} \quad (6.4)$$

and, moreover, with $(t_*, x_*) \in \mathcal{W}(M,N)$

$$\tilde{\Pi}_{t_*}(\pi^{(\mathcal{W})}_{t_*,x_*}\langle \cdot\, | M, N\rangle) \subset \mathcal{S}_{M,N}(t_*, x_*). \quad (6.5)$$

So, we note that (6.4), (6.5) defines, in fact, the main result of [8]. This result is invaluable for differential game theory. Namely, it defines solvability conditions in the class of non-anticipating strategies. We shall implement this crucial statement to represent guaranteed result in the class of non-anticipating strategies for special payoff functional. Such a functional will be introduced now. Besides, (6.4) defines the solvability set of player I and (6.2), (6.5) implements the exact non-anticipating strategy solving problem of guidance. Further, we shall use $S_0(\mathbf{M}, \varepsilon)$ as M and $S_0(\mathbf{N}, \varepsilon)$ as N where $\varepsilon \in [0, \infty[$. In particular, we consider an option $M = \mathbf{M}$ and $N = \mathbf{N}$.

Now, we introduce a type of functionals defined on the space of continuous n-vector-functions. For this purpose, we note that, while $t \in T$, $x(\cdot) \in C_n([t, \vartheta_0])$, and $\vartheta \in [t, \vartheta_0]$, the function

$$\tau \longmapsto \rho((\tau, x(\tau)), \mathbf{N}) : [t, \vartheta] \to [0, \infty[$$

is continuous and attains a maximum. Then

$$\omega(t, x(\cdot), \vartheta) \triangleq \sup(\{\rho((\vartheta, x(\vartheta)), \mathbf{M}); \max_{\tau \in [t, \vartheta]} \rho((\tau, x(\tau)), \mathbf{N})\}) \\ \in [0, \infty[\ \forall t \in T \ \forall x(\cdot) \in C_n([t, \vartheta_0]) \ \forall \vartheta \in [t, \vartheta_0]. \tag{6.6}$$

Under $t \in T$ and $x(\cdot) \in C_n([t, \vartheta_0])$, the function

$$\vartheta \longmapsto \omega(t, x(\cdot), \vartheta) : [t, \vartheta_0] \to [0, \infty[$$

is continuous; therefore the payoff

$$\gamma_t(x(\cdot)) \triangleq \min_{\vartheta \in [t, \vartheta_0]} \omega(t, x(\cdot), \vartheta) \in [0, \infty[\tag{6.7}$$

is defined correctly. For $t \in T$, by $\|\cdot\|_t^{(\mathbb{C})}$ we denote the norm of uniform convergence on $C_n([t, \vartheta_0])$:

$$\|x(\cdot)\|_t^{(\mathbb{C})} \triangleq \max_{\tau \in [t, \vartheta_0]} \|x(\tau)\| \ \forall x(\cdot) \in C_n([t, \vartheta_0]).$$

From (6.6) and (6.7), we obtain the next property:

$$|\gamma_t(x'(\cdot)) - \gamma_t(x''(\cdot))| \leq \|x'(\cdot) - x''(\cdot)\|_t^{(\mathbb{C})} \\ \forall t \in T \ \forall x'(\cdot) \in C_n([t, \vartheta_0]) \ \forall x''(\cdot) \in C_n([t, \vartheta_0]). \tag{6.8}$$

Thus, γ_t is a Lipschitz functional. For $(t_*, x_*) \in T \times \mathbb{R}^n$ the set $\{\gamma_{t_*}(\varphi(\cdot, t_*, x_*, \eta)) : \eta \in \mathcal{H}_{t_*}\}$ is a non-empty compactum in the sense of usual $|\cdot|$-topology of $\mathbb{R}$. Let us note that in this paper we use relative weak-* compactness of $\mathcal{H}_{t_*}$ and the continuity of mapping (3.4). Therefore, by (5.1), (5.2), and (5.4), we obtain for $(t_*, x_*) \in T \times \mathbb{R}^n$ and $\alpha \in \tilde{A}_{t_*}$ the following set $\{\gamma_{t_*}(x(\cdot)) : x(\cdot) \in \mathbb{X}[t_*; x_*; \alpha]\}$ is not empty and bounded. As a corollary, for $(t_*, x_*) \in T \times \mathbb{R}^n$ and $\alpha \in \tilde{A}_{t_*}$, following exact upper bound of our payoff functional is defined, namely:

$$\sup(\{\gamma_{t_*}(x(\cdot)) : x(\cdot) \in \mathbb{X}[t_*; x_*; \alpha]\}) = \sup(\{\gamma_{t_*}(\varphi(\cdot, t_*, x_*, \eta)) : \eta \in \tilde{\Pi}_{t_*}(\alpha)\}) \in [0, \infty[.$$

Also, for $(t_*, x_*) \in T \times \mathbb{R}^n$

$$\mathbf{v}(t_*, x_*) \triangleq \inf_{\alpha \in \tilde{A}_{t_*}} \sup_{\eta \in \tilde{\Pi}_{t_*}(\alpha)} \gamma_{t_*}(\varphi(\cdot, t_*, x_*, \eta)) \\ = \inf_{\alpha \in \tilde{A}_{t_*}} \sup_{x(\cdot) \in \mathbb{X}[t_*; x_*; \alpha]} \gamma_{t_*}(x(\cdot)) \in [0, \infty[. \tag{6.9}$$

Moreover, we note that for $\tilde{\alpha} \in \tilde{A}^{\Pi}_{t_*}$ the set $\mathbb{X}[t_*; x_*; \tilde{\alpha}]$ is a non-empty compactum in $C_n([t_*, \vartheta_0])$ with the norm $\|\cdot\|^{(\mathbb{C})}_{t_*}$ and in consequence

$$\max_{x(\cdot)\in\mathbb{X}[t_*;x_*;\alpha]} \gamma_{t_*}(x(\cdot)) \in [0, \infty[$$

is correct. From (6.5) we have the following lemma.

Lemma 6.1. *If* $(t_*, x_*) \in T \times \mathbb{R}^n$*, then for a non-anticipating strategy* $\alpha_* \triangleq \pi^{(\mathcal{W})}_{t_*,x_*}\langle \cdot\, | S_0(\mathbf{M}, \varepsilon_0(t_*, x_*)),\ S_0(\mathbf{N}, \varepsilon_0(t_*, x_*))\rangle$ *we have*

$$\max_{x(\cdot)\in\mathbb{X}[t_*;x_*;\alpha_*]} \gamma_{t_*}(x(\cdot)) \le \varepsilon_0(t_*, x_*). \tag{6.10}$$

Proof. First, we fix $(t_*, x_*) \in T \times \mathbb{R}^n$. Then, by (5.9) and (5.11) we obtain

$$(t_*, x_*) \in \mathcal{W}(S_0(\mathbf{M}, \varepsilon_*), S_0(\mathbf{N}, \varepsilon_*)), \tag{6.11}$$

where $\varepsilon_* \triangleq \varepsilon_0(t_*, x_*)$. By (6.2) we have the property $\alpha_* \in \tilde{A}^{\Pi}_{t_*}$. From (6.5), the following inclusion holds:

$$\tilde{\Pi}_{t_*}(\alpha_*) \subset \mathcal{S}_{S_0(\mathbf{M},\varepsilon_*),S_0(\mathbf{N},\varepsilon_*)}(t_*, x_*).$$

Then, by (5.4) and (6.3) $\forall x(\cdot) \in \mathbb{X}[t_*; x_*; \alpha_*] \exists \vartheta \in [t_*, \vartheta_0]$:

$$((\vartheta, x(\vartheta)) \in S_0(\mathbf{M}, \varepsilon_*))\ \&\ ((t, x(t)) \in S_0(\mathbf{N}, \varepsilon_*)\ \forall t \in [t_*, \vartheta]). \tag{6.12}$$

From (2.7) and (6.12), we obtain $\forall x(\cdot) \in \mathbb{X}[t_*; x_*; \alpha_*] \exists \vartheta \in [t_*, \vartheta_0]$:

$$(\rho((\vartheta, x(\vartheta)), \mathbf{M}) \le \varepsilon_*)\ \&\ (\rho((t, x(t)), \mathbf{N}) \le \varepsilon_*\ \forall t \in [t_*, \vartheta]). \tag{6.13}$$

By (6.6) and (6.13), we have $\forall x(\cdot) \in \mathbb{X}[t_*; x_*; \alpha_*]\ \exists \vartheta \in [t_*, \vartheta_0]$: $\omega(t_*, x(\cdot), \vartheta) \le \varepsilon_*$. From (6.7), we finally obtain

$$\gamma_{t_*}(x(\cdot)) \le \varepsilon_* \quad \forall x(\cdot) \in \mathbb{X}[t_*; x_*; \alpha_*].$$

As a result, property (6.10) is verified and Lemma 6.1 is fully proved. □

Corollary 6.2. *For* $(t_*, x_*) \in T \times \mathbb{R}^n$

$$\mathbf{v}(t_*, x_*) \le \inf_{\alpha\in\tilde{A}^{\Pi}_{t_*}} \max_{x(\cdot)\in\mathbb{X}[t_*;x_*;\alpha]} \gamma_{t_*}(x(\cdot))$$
$$\le \max_{x(\cdot)\in\mathbb{X}[t_*;x_*;\pi^{(\mathcal{W})}_{t_*,x_*}\langle\cdot|S_0(\mathbf{M},\varepsilon_0(t_*,x_*)),S_0(\mathbf{N},\varepsilon_0(t_*,x_*))\rangle]} \gamma_{t_*}(x(\cdot)) \le \varepsilon_0(t_*, x_*).$$

The corresponding proof can be obtained by combining (5.3) with (6.9) and Lemma 6.1.

Proposition 6.3. *Under* $(t_*, x_*) \in T \times \mathbb{R}^n$*, the inequality* $\varepsilon_0(t_*, x_*) \le \mathbf{v}(t_*, x_*)$ *holds.*

Proof. Assume the opposite: $\mathbf{v}(t_*, x_*) < \varepsilon_0(t_*, x_*)$.

Then by (6.9), for $\varepsilon_* \triangleq \varepsilon_0(t_*, x_*)$

$$b_* \triangleq \sup_{x(\cdot)\in\mathbb{X}[t_*;x_*;\alpha^*]} \gamma_{t_*}(x(\cdot)) = \sup_{\eta\in\tilde{\Pi}_{t_*}(\alpha^*)} \gamma_{t_*}(\varphi(\cdot\,, t_*, x_*, \eta)) < \varepsilon_*, \tag{6.14}$$

where $\alpha^* \in \tilde{A}_{t_*}$. Then, for $\eta^* \in \tilde{\Pi}_{t_*}(\alpha^*)$

$$\gamma_{t_*}(\varphi(\cdot\,, t_*, x_*, \eta^*)) \leq b_*.$$

By (6.7), $\omega(t_*, \varphi(\cdot\,, t_*, x_*, \eta^*), \vartheta^*) \leq b_*$ for some $\vartheta^* \in [t_*, \vartheta_0]$. Therefore, by (6.6)

$$(\rho((\vartheta^*, \varphi(\vartheta^*, t_*, x_*, \eta^*)), \mathbf{M}) \leq b^*) \;\&\; (\rho((t, \varphi(t, t_*, x_*, \eta^*)), \mathbf{N}) \leq b^* \tag{6.15}$$

for all $t \in [t_*, \vartheta^*]$). By (2.7) and (6.15) we obtain

$$((\vartheta^*, \varphi(\vartheta^*, t_*, x_*, \eta^*)) \in S_0(\mathbf{M}, b^*)) \;\&\; ((t, \varphi(t, t_*, x_*, \eta^*)) \in S_0(\mathbf{N}, b^*)$$

for all $t \in [t_*, \vartheta^*]$). Since the selection of η^* was arbitrary, we have successfully obtained the following: $\forall \eta \in \tilde{\Pi}_{t_*}(\alpha^*)\ \exists \vartheta \in [t_*, \vartheta_0]\ \forall t \in [t_*, \vartheta]$:

$$((\vartheta, \varphi(\vartheta, t_*, x_*, \eta)) \in S_0(\mathbf{M}, b^*)) \;\&\; ((t, \varphi(t, t_*, x_*, \eta)) \in S_0(\mathbf{N}, b^*)). \tag{6.16}$$

In other words, by (6.3) and (6.16)

$$\tilde{\Pi}_{t_*}(\alpha^*) \subset \mathcal{S}_{S_0(\mathbf{M},b^*),S_0(\mathbf{N},b^*)}(t_*, x_*). \tag{6.17}$$

Moreover, from (6.16) the inclusion $(t_*, x_*) \in S_0(\mathbf{N}, b^*)$ holds, indeed, $\tilde{\Pi}_{t_*}(\alpha^*) \neq \emptyset$, for $\eta \in \tilde{\Pi}_{t_*}(\alpha^*)$, $(t_*, x_*) = (t_*, \varphi(t_*, t_*, x_*, \eta))$. From (6.4) and (6.17), we obtain

$$(t_*, x_*) \in \mathcal{W}(S_0(\mathbf{M}, b^*), S_0(\mathbf{N}, b^*)).$$

By (5.9) $b_* \in \Sigma_0(t_*, x_*)$ and $\varepsilon_* = \varepsilon_0(t_*, x_*) \leq b_*$ (see (5.10)). We have a contradiction to (6.14). Thus, $\varepsilon_* \leq \mathbf{v}(t_*, x_*)$. □

From Corollary 6.2 and Proposition 6.3, we obtain for $(t_*, x_*) \in T \times \mathbb{R}^n$

$$\begin{aligned}\mathbf{v}(t_*, x_*) &= \inf_{\alpha\in\tilde{A}^{\Pi}_{t_*}} \max_{x(\cdot)\in\mathbb{X}[t_*;x_*;\alpha]} \gamma_{t_*}(x(\cdot)) \\ &= \max_{x(\cdot)\in\mathbb{X}[t_*;x_*;\pi^{(\mathcal{W})}_{t_*,x_*}\langle\cdot|S_0(\mathbf{M},\varepsilon_0(t_*,x_*)),S_0(\mathbf{N},\varepsilon_0(t_*,x_*))\rangle]} \gamma_{t_*}(x(\cdot)) = \varepsilon_0(t_*, x_*).\end{aligned} \tag{6.18}$$

Therefore, we have shown, that ε_0 is the function of guaranteed result. For every position $(t_*, x_*) \in T \times \mathbb{R}^n$ quasi-program (6.2) with

$$(M = S_0(\mathbf{M}, \varepsilon_0(t_*, x_*))) \;\&\; (N = S_0(\mathbf{N}, \varepsilon_0(t_*, x_*)))$$

is minimax in class of non-anticipating strategies, thus optimal. As a result, quasi-programs and non-anticipating strategies are equivalent by result. Therefore, the role of ε_0 is very essential and, in the next section, we consider "direct" approach for construction of this function. With respect to (6.18), we note the next property, which is connected with the alternative theorem.

Namely, under $(t_*, x_*) \in T \times \mathbb{R}^n$, $\varepsilon_* \triangleq \varepsilon_0(t_*, x_*)$, and $\varepsilon \in [0, \varepsilon_*[$

$$((t_*, x_*) \in \mathcal{W}(S_0(\mathbf{M}, \varepsilon_*), S_0(\mathbf{N}, \varepsilon_*))) \ \& \ ((t_*, x_*) \notin \mathcal{W}(S_0(\mathbf{M}, \varepsilon), S_0(\mathbf{N}, \varepsilon))). \quad (6.19)$$

According to the alternative theorem, its second statement indicates the possibilities of the player II, who is interested in evasion. One of such possibilities guarantees the evasion of the control trajectories from $S_0(\mathbf{M}, \varepsilon)$ under state constraints defined as $S_0(\mathbf{N}, \varepsilon)$. With respect to (6.7) and the second part of (6.19), this statement is transformed into an opportunity for player II to assure the following inequality $\gamma_{t_*}(x(\cdot)) > \varepsilon$. In this regard, we shall note the procedures implemented in [9] with stability iterations.

7. Program iteration method II

In this section we consider a direct procedure implementing the scheme

$$\varepsilon_0^{(0)} \to \varepsilon_0^{(1)} \to \ldots \to \varepsilon_0^{(k)} \to \ldots; \quad (7.1)$$

namely, for (7.1), the iterated procedure will be built. First, we consider some auxiliary constructions. For $t \in T$ and $x(\cdot) \in C_n([t, \vartheta_0])$

$$\tau \longmapsto \psi(\tau, x(\tau)) : [t, \vartheta_0] \to [0, \infty[$$

is a continuous function and attains maximum; therefore for $g \in \mathfrak{M}_\psi$ the function

$$\tau \longmapsto g(\tau, x(\tau)) : [t, \vartheta_0] \to [0, \infty[$$

is bounded above and following chain of inequalities holds:

$$0 \le \sup_{\tau \in [t, \vartheta]} g(\tau, x(\tau)) \le \max_{\tau \in [t, \vartheta]} \psi(\tau, x(\tau)) \quad \forall \vartheta \in [t, \vartheta_0]. \quad (7.2)$$

This property will be invaluable in the next constructions. Namely, for $g \in \mathfrak{M}_\psi$ and $t_* \in T$, we define

$$\mathfrak{H}[g; t_*] \in \mathcal{R}_+[C_n([t_*, \vartheta_0]) \times [t_*, \vartheta_0]] \quad (7.3)$$

as the functional

$$(x(\cdot), \vartheta) \longmapsto \sup(\{ \sup_{\tau \in [t_*, \vartheta]} g(\tau, x(\tau)); \psi(\vartheta, x(\vartheta))\}) : \quad (7.4)$$
$$C_n([t_*, \vartheta_0]) \times [t_*, \vartheta_0] \to [0, \infty[.$$

Now, we consider the Lebesgue sets of the functional (7.3), (7.4):

$$\tilde{\mathfrak{Y}}_b[g; t_*] \triangleq \mathfrak{H}[g; t_*]^{-1}([0, b]) \in \mathcal{P}(C_n([t_*, \vartheta_0]) \times [t_*, \vartheta_0]) \ \forall b \in [0, \infty[. \quad (7.5)$$

We use the following condition: for $t_* \in T$, the set $C_n([t_*, \vartheta_0]) \times [t_*, \vartheta_0]$ is equipped with the natural metrizable topology defined as the product of the uniform convergence topology on $C_n([t_*, \vartheta_0])$ and the usual $|\cdot|$-topology of $[t_*, \vartheta_0]$; for the set $C_n([t_*, \vartheta_0]) \times [t_*, \vartheta_0]$, we use only this topological structure.

Proposition 7.1. *If $g \in \mathfrak{M}_\psi$, $t_* \in T$, and $b \in [0, \infty[$, then set $\tilde{\mathcal{Y}}_b[g; t_*]$ is closed.*

Proof. Fix $g \in \mathfrak{M}_\psi$, $t_* \in T$, and $b \in [0, \infty[$. Since the above-mentioned topology τ_* of $C_n([t_*, \vartheta_0]) \times [t_*, \vartheta_0]$ is metrizable, it is sufficient to prove the sequential closure of $\tilde{\mathcal{Y}}_b[g; t_*]$. First of all, let $(z_k)_{k \in \mathbb{N}}$ be a sequence in $\tilde{\mathcal{Y}}_b[g; t_*]$ and assume $z^* \in C_n([t_*, \vartheta_0]) \times [t_*, \vartheta_0]$. Also, let

$$(z_k)_{k \in \mathbb{N}} \xrightarrow{\tau_*} z^*. \tag{7.6}$$

For the convergence of (7.6), the equivalent metric representation takes place. Namely, let

$$(x_k(\cdot) \triangleq pr_1(z_k)\ \forall k \in \mathbb{N})\ \&\ (\vartheta_k \triangleq pr_2(z_k)\ \forall k \in \mathbb{N})$$
$$\&\ (x^*(\cdot) \triangleq pr_1(z^*))\ \&\ (\vartheta^* \triangleq pr_2(z^*)),$$

where $pr_1(h)$ and $pr_2(h)$ are first and second elements of ordered pair h respectively. Then, from (7.6), we obtain

$$((x_k(\cdot))_{k \in \mathbb{N}} \rightrightarrows x^*(\cdot))\ \&\ ((\vartheta_k)_{k \in \mathbb{N}} \to \vartheta^*). \tag{7.7}$$

Of course, for $k \in \mathbb{N}$, we have the inequalities

$$(\sup_{\tau \in [t_*, \vartheta_k]} g(\tau, x_k(\tau)) \leq b)\ \&\ (\psi(\vartheta_k, x_k(\vartheta_k)) \leq b). \tag{7.8}$$

From (7.7), the next convergence property in $(\mathbb{R}^n, \|\cdot\|)$ follows:

$$(x_k(\vartheta_k))_{k \in \mathbb{N}} \to x^*(\vartheta^*).$$

As a result,
$$\rho((\vartheta_k, x_k(\vartheta_k)), (\vartheta^*, x^*(\vartheta^*)))_{k \in \mathbb{N}} \to 0. \tag{7.9}$$

Because of the continuity of ψ we obtain from (7.8) and (7.9),

$$\psi(\vartheta^*, x^*(\vartheta^*)) \leq b. \tag{7.10}$$

Now, we will show that
$$\mathfrak{H}[g; t_*](z^*) \leq b. \tag{7.11}$$

Indeed, by contradiction, let $b < \mathfrak{H}[g; t_*](z^*)$. Then, by (7.4) and (7.10) we obtain for some $t^* \in [t_*, \vartheta^*]$

$$b < g(t^*, x^*(t^*)). \tag{7.12}$$

Let $t_k^* \triangleq \inf(\{\vartheta_k; t^*\})\ \forall k \in \mathbb{N}$. Then, from (7.7), we have the convergence

$$(t_k^*)_{k \in \mathbb{N}} \to t^*. \tag{7.13}$$

Moreover, from (7.7) and (7.13), the convergence holds:

$$(\rho((t_k^*, x_k(t_k^*)), (t^*, x^*(t^*))))_{k \in \mathbb{N}} \to 0. \tag{7.14}$$

By (7.8), $g(t_k^*, x_k(t_k^*)) \leq b\ \ \forall k \in \mathbb{N}$. We obtain

$$(t_k^*, x_k(t_k^*)) \in g^{-1}([0, b])\ \forall k \in \mathbb{N}. \tag{7.15}$$

However, because of the selection of g, we have the property $g^{-1}([0,b]) \in \mathcal{F}$. Then, by (7.14) and (7.15): $(t^*, x^*(t^*)) \in g^{-1}([0,b])$ and $g(t^*, x^*(t^*)) \leq b$, which contradicts (7.12). Thus, (7.11) is correct. Then $z^* \in \mathfrak{H}[g;t_*]^{-1}([0,b])$ and, as a corollary, $z^* \in \tilde{\mathcal{Y}}_b[g;t_*]$. Finally, since the selection of $(z_k^*)_{k\in\mathbb{N}}$ and z^* was arbitrary, the required property is established. $\square$

For $g \in \mathfrak{M}_\psi$, $(t_*, x_*) \in T \times \mathbb{R}^n$, and $\nu \in \mathcal{E}_{t_*}$, we define the functional-constriction:

$$\mathbf{h}[g;t_*;x_*;\nu] \triangleq (\mathfrak{H}[g;t_*]|\mathcal{X}_\Pi(t_*,x_*,\nu)\times[t_*,\vartheta_0]) \in \mathcal{R}_+[\mathcal{X}_\Pi(t_*,x_*,\nu)\times[t_*,\vartheta_0]]; \quad (7.16)$$

by Proposition 7.1 while $b \in [0,\infty[$, the following set

$$\mathcal{Y}_b[g;t_*;x_*;\nu] \triangleq \mathbf{h}[g;t_*;x_*;\nu]^{-1}([0,b]) \in \mathcal{P}(\mathcal{X}_\Pi(t_*,x_*,\nu) \times [t_*,\vartheta_0])$$

is closed in the natural topology of $C_n([t_*,\vartheta_0]) \times [t_*,\vartheta_0]$, since (3.5) is compactum. In particular, this set is closed in $\mathcal{X}_\Pi(t_*,x_*,\nu) \times [t_*,\vartheta_0]$ with the topology of a subspace. By using (7.2) and (7.4), we show that the finite value

$$\begin{aligned}\tilde{\mathbf{a}}_* &\triangleq \inf_{(x(\cdot),\vartheta)\in\mathcal{X}_\Pi(t_*,x_*,\nu)\times[t_*,\vartheta_0]} \mathbf{h}[g;t_*;x_*;\nu](x(\cdot),\vartheta)\\ &= \inf_{(x(\cdot),\vartheta)\in\mathcal{X}_\Pi(t_*,x_*,\nu)\times[t_*,\vartheta_0]} \mathfrak{H}[g;t_*](x(\cdot),\vartheta) \in [0,\infty[\end{aligned}$$

is defined. For some $\bar{x}(\cdot) \in \mathcal{X}_\Pi(t_*,x_*,\nu)$ and $\bar{\vartheta} \in [t_*,\vartheta_0]$, $\mathbf{h}[g;t_*;x_*;\nu](\bar{x}(\cdot),\bar{\vartheta}) = \tilde{\mathbf{a}}_*$. Then we use the property of the lower semi-continuity of g. Therefore, for all $g \in \mathfrak{M}_\psi$, $(t_*,x_*) \in T \times \mathbb{R}^n$, and $\nu \in \mathcal{E}_{t_*}$, the following expression

$$\min_{(x(\cdot),\vartheta)\in\mathcal{X}_\Pi(t_*,x_*,\nu)\times[t_*,\vartheta_0]} \mathbf{h}[g;t_*;x_*;\nu](x(\cdot),\vartheta) \in [0,\infty[\quad (7.17)$$

is defined. Let us recall that by (5.25), $\forall (t_*,x_*) \in T \times \mathbb{R}^n$, $\exists c \in [0,\infty[$:

$$\psi(t,x(t)) \leq c \quad \forall \nu \in \mathcal{E}_{t_*} \quad \forall x(\cdot) \in \mathcal{X}_\Pi(t_*,x_*,\nu)\ \forall t \in [t_*,\vartheta_0]. \quad (7.18)$$

As a corollary, by (7.4) and (7.18), for $g \in \mathfrak{M}_\psi$ and $(t_*,x_*) \in T \times \mathbb{R}^n$

$$\exists c \in [0,\infty[: \min_{(x(\cdot),\vartheta)\in\mathcal{X}_\Pi(t_*,x_*,\nu)\times[t_*,\vartheta_0]} \mathbf{h}[g;t_*;x_*;\nu](x(\cdot),\vartheta) \leq c\ \forall \nu \in \mathcal{E}_{t_*}.$$

Therefore, we can define an operator

$$\Gamma : \mathfrak{M}_\psi \to \mathcal{R}_+[T \times \mathbb{R}^n] \quad (7.19)$$

by the following rules: for all $g \in \mathfrak{M}_\psi$ and $(t_*,x_*) \in T \times \mathbb{R}^n$

$$\Gamma(g)(t_*,x_*) \triangleq \sup_{\nu\in\mathcal{E}_{t_*}} \min_{(x(\cdot),\vartheta)\in\mathcal{X}_\Pi(t_*,x_*,\nu)\times[t_*,\vartheta_0]} \mathbf{h}[g;t_*;x_*;\nu](x(\cdot),\vartheta). \quad (7.20)$$

We consider (7.19) as a program operator. This operator plays important role in questions connected with representation of the sequence $(\varepsilon_0^{(k)})_{k\in\mathbb{N}_0}$. Now, we will consider one of the main results of this paper; namely by using program operator (7.19), we can recursively find values of the guaranteed result function ε_0.

Theorem 7.2. *The equality* $\varepsilon_0^{(k+1)} = \Gamma(\varepsilon_0^{(k)})$ *holds for any iteration number* $k \in \mathbb{N}_0$.

Proof. Fix $k \in \mathbb{N}_0$. Then, we have the functions $\varepsilon_0^{(k)} \in \mathfrak{M}_\psi$, $\varepsilon_0^{(k+1)} \in \mathfrak{M}_\psi$, and $\Gamma(\varepsilon_0^{(k)}) \in \mathcal{R}_+[T \times \mathbb{R}^n]$. Fix the position $(t_*, x_*) \in T \times \mathbb{R}^n$. We compare the numbers

$$(a_* \triangleq \varepsilon_0^{(k+1)}(t_*, x_*) \in [0, \infty[) \ \& \ (b_* \triangleq \Gamma(\varepsilon_0^{(k)})(t_*, x_*) \in [0, \infty[). \tag{7.21}$$

In addition, $a_* \in \Sigma_0^{(k+1)}(t_*, x_*)$. Then, $(t_*, x_*) \in \mathcal{W}_{k+1}(S_0(\mathbf{M}, a_*), S_0(\mathbf{N}, a_*))$. By using (5.2) and (5.4), we obtain $(t_*, x_*) \in \mathcal{W}_k(S_0(\mathbf{M}, a_*), S_0(\mathbf{N}, a_*))$ and, moreover, $\forall \nu \in \mathcal{E}_{t_*}$ $\exists (x(\cdot), \vartheta) \in \mathcal{X}_\Pi(t_*, x_*, \nu) \times [t_*, \vartheta_0]$:

$$((\vartheta, x(\vartheta)) \in S_0(\mathbf{M}, a_*)) \ \& \ ((t, x(t)) \in \mathcal{W}_k(S_0(\mathbf{M}, a_*), S_0(\mathbf{N}, a_*)) \ \forall t \in [t_*, \vartheta]). \tag{7.22}$$

From (5.9), (5.10), (5.25) and (7.22), we obtain that for all $\nu \in \mathcal{E}_{t_*}$ there exists some $(x(\cdot), \vartheta) \in \mathcal{X}_\Pi(t_*, x_*, \nu) \times [t_*, \vartheta_0]$ such that

$$(\psi(\vartheta, x(\vartheta)) \leq a_*) \ \& \ (\varepsilon_0^{(k)}(t, x(t)) \leq a_* \ \forall t \in [t_*, \vartheta]). \tag{7.23}$$

As a result, by (7.14) and (7.23) next inequalities are correct:

$$\min_{(x(\cdot),\vartheta) \in \mathcal{X}_\Pi(t_*, x_*, \nu) \times [t_*, \vartheta_0]} \mathbf{h}[\varepsilon_0^{(k)}; t_*; x_*; \nu](x(\cdot), \vartheta) \leq a_* \ \forall \nu \in \mathcal{E}_{t_*}.$$

From (7.20) and (7.21), the following inequality is valid:

$$b_* = \Gamma(\varepsilon_0^{(k)})(t_*, x_*) \leq a_*. \tag{7.24}$$

Thus, $b_* \leq a_*$. Now, we will establish the opposite inequality.

Fix $\hat{\nu} \in \mathcal{E}_{t_*}$. Then, by (7.21) we obtain

$$\min_{(x(\cdot),\vartheta) \in \mathcal{X}_\Pi(t_*, x_*, \hat{\nu}) \times [t_*, \vartheta_0]} \mathbf{h}[\varepsilon_0^{(k)}; t_*; x_*; \hat{\nu}](x(\cdot), \vartheta) \leq b_*.$$

For some $\hat{x}(\cdot) \in \mathcal{X}_\Pi(t_*, x_*, \hat{\nu})$ and $\hat{\vartheta} \in [t_*, \vartheta_0]$, we have $\mathbf{h}[\varepsilon_0^{(k)}; t_*; x_*; \hat{\nu}](\hat{x}(\cdot), \hat{\vartheta}) \leq b_*$. From (7.4) and (7.16), we obtain

$$(\psi(\hat{\vartheta}, \hat{x}(\hat{\vartheta})) \leq b_*) \ \& \ (\varepsilon_0^{(k)}(t, \hat{x}(t)) \leq b_* \ \forall t \in [t_*, \hat{\vartheta}]).$$

Then, $(\hat{\vartheta}, \hat{x}(\hat{\vartheta})) \in S_0(\mathbf{M}, b_*)$ and $b_* \in \Sigma_0^{(k)}(t, \hat{x}(t))$ $\forall t \in [t_*, \hat{\vartheta}]$ (see Proposition 5.3). Then, for $(\hat{x}(\cdot), \hat{\vartheta})$ we have the next properties for all $t \in [t_*, \hat{\vartheta}]$:

$$\big((\hat{\vartheta}, \hat{x}(\hat{\vartheta})) \in S_0(\mathbf{M}, b_*)\big) \ \& \ \big((t, \hat{x}(t)) \in \mathcal{W}_k(S_0(\mathbf{M}, b_*), S_0(\mathbf{N}, b_*))\big). \tag{7.25}$$

Since the selection of $\hat{\nu}$ was arbitrary, by (5.2), (5.4), and (7.25) we obtain:

$$(t_*, x_*) \in \mathcal{W}_{k+1}(S_0(\mathbf{M}, b_*), S_0(\mathbf{N}, b_*));$$

(we omit the simple verification of the property $(t_*, x_*) \in \mathcal{W}_k(S_0(\mathbf{M}, b_*), S_0(\mathbf{N}, b_*))$, it is established by relations similar to (7.25)). Thus, by (5.9) $b_* \in \Sigma_0^{(k+1)}(t_*, x_*)$ and, as a corollary (see (5.10)), $a_* \leq b_*$. Therefore, we finally have $a_* = b_*$ and by (7.21)

$$\varepsilon_0^{(k+1)}(t_*, x_*) = \Gamma(\varepsilon_0^{(k)})(t_*, x_*).$$

Since the selection of (t_*, x_*) was arbitrary, the theorem is fully proved. □

Now, we recall the property of $\varepsilon_0^{(0)}$, namely $\varepsilon_0^{(0)} = \rho(\cdot\,, \mathbf{N})$. The proof of this fact can be obtained from Proposition 5.6.

Proposition 7.3. *If $g \in \mathfrak{M}_\psi$, then $g \leqq \Gamma(g)$.*

Proof. Fix $g \in \mathfrak{M}_\psi$ and $(t_*, x_*) \in T \times \mathbb{R}^n$. Recall that $\mathcal{E}_{t_*} \neq \emptyset$. By using this property, we choose $\nu_0 \in \mathcal{E}_{t_*}$. For the sake of brevity, we assume that $\mathbf{h}_0 \triangleq \mathbf{h}[g; t_*; x_*; \nu_0]$; $\mathbf{h}_0 \in \mathcal{R}_+[\mathcal{X}_\Pi(t_*, x_*, \nu_0) \times [t_*, \vartheta_0]]$. Of course,

$$\min_{(x(\cdot),\vartheta) \in \mathcal{X}_\Pi(t_*,x_*,\nu_0) \times [t_*,\vartheta_0]} \mathbf{h}_0(x(\cdot\,), \vartheta) \leq \Gamma(g)(t_*, x_*), \tag{7.26}$$

where $g(t_*, \tilde{x}(t_*)) \leq \mathbf{h}_0(\tilde{x}(\cdot\,), \vartheta)$ $\forall \tilde{x}(\cdot\,) \in \mathcal{X}_\Pi(t_*, x_*, \nu_0)$ $\forall \vartheta \in [t_*, \vartheta_0]$. By using (7.26), we choose $\bar{x}(\cdot\,) \in \mathcal{X}_\Pi(t_*, x_*, \nu_0)$ and $\bar{\vartheta} \in [t_*, \vartheta_0]$ for which $\mathbf{h}_0(\bar{x}(\cdot\,), \bar{\vartheta}) \leq \Gamma(g)(t_*, x_*)$. Then, by (7.3), (7.16), and definition of $\mathbf{h}_0$ we obtain

$$\sup_{\tau \in [t_*, \vartheta_0]} g(\tau, \bar{x}(\tau)) \leq \Gamma(g)(t_*, x_*).$$

As a corollary, $g(t_*, \bar{x}(t_*)) \leq \Gamma(g)(t_*, x_*)$. However, $\bar{x}(t_*) = x_*$. In consequence we have $g(t_*, x_*) \leq \Gamma(g)(t_*, x_*)$. Since the selection of (t_*, x_*) was arbitrary, the required property is established. □

Proposition 7.4. *The operator Γ is isotonic: $\forall g_1 \in \mathfrak{M}_\psi$ $\forall g_2 \in \mathfrak{M}_\psi$*

$$(g_1 \leqq g_2) \Rightarrow (\Gamma(g_1) \leqq \Gamma(g_2)).$$

The corresponding proof can be obtained from definitions above (see ex. (7.20)).

8. The fixed point property

In this section, we consider one important property of the function of guaranteed result ε_0. Namely, this function is a fixed point of operator Γ. Indeed, the next statement takes place.

Theorem 8.1. *Function of guaranteed result ε_0 is the fixed point of Γ: $\varepsilon_0 = \Gamma(\varepsilon_0)$.*

Proof. By Proposition 7.3 the property $\varepsilon_0 \leqq \Gamma(\varepsilon_0)$ holds. We will show that $\varepsilon_0 = \Gamma(\varepsilon_0)$. For an arbitrary fixed position $(t_*, x_*) \in T \times \mathbb{R}^n$, let $a_* \triangleq \varepsilon_0(t_*, x_*)$ and $b_* \triangleq \Gamma(\varepsilon_0)(t_*, x_*)$, thus $a_* \in [0, \infty[$ and $b_* \in [0, \infty[$. We assume that

$$\mathbf{h}_\nu \triangleq \mathbf{h}[\varepsilon_0; t_*; x_*; \nu] \quad \forall \nu \in \mathcal{E}_{t_*}. \tag{8.1}$$

Then, we obtain the next equality

$$b_* = \sup_{\nu \in \mathcal{E}_{t_*}} \min_{(x(\cdot),\vartheta)\in \mathcal{X}_\Pi(t_*,x_*,\nu)\times[t_*,\vartheta_0]} \mathbf{h}_\nu(x(\cdot),\vartheta). \quad (8.2)$$

Of course, by (5.9) with (5.10) $a_* \in \Sigma_0(t_*, x_*)$ and

$$(t_*, x_*) \in \mathcal{W}(S_0(\mathbf{M}, a_*), S_0(\mathbf{N}, a_*)). \quad (8.3)$$

By (8.3) and the properties of $\mathcal{W}(M, N)$, $M \in \mathcal{F}, N \in \mathcal{F}$, we obtain that for all $\nu \in \mathcal{E}_{t_*}$ there exists $(x(\cdot), \vartheta) \in \mathcal{X}_\Pi(t_*, x_*, \nu) \times [t_*, \vartheta_0]$ such that

$$((\vartheta, x(\vartheta)) \in S_0(\mathbf{M}, a_*)) \;\&\; ((t, x(t)) \in \mathcal{W}(S_0(\mathbf{M}, a_*), S_0(\mathbf{N}, a_*)) \;\; \forall t \in [t_*, \vartheta]). \quad (8.4)$$

By using (5.9), (5.10), and (8.4), we obtain that for all $\nu \in \mathcal{E}_{t_*}$ there exists some $(x(\cdot), \vartheta) \in \mathcal{X}_\Pi(t_*, x_*, \nu) \times [t_*, \vartheta_0]$ such that

$$(\psi(\vartheta, x(\vartheta)) \leq a_*) \;\&\; (\varepsilon_0(t, x(t)) \leq a_* \;\; \forall t \in [t_*, \vartheta]). \quad (8.5)$$

As a corollary, from (8.5), the next property follows

$$\forall \nu \in \mathcal{E}_{t_*} \; \exists (x(\cdot), \vartheta) \in \mathcal{X}_\Pi(t_*, x_*, \nu) \times [t_*, \vartheta_0] : \mathbf{h}_\nu(x(\cdot), \vartheta) \leq a_*. \quad (8.6)$$

From (8.2) and (8.6), the inequality $b_* \leq a_*$ holds. However, from Proposition 7.3, we obtain inequality $a_* \leq b_*$. Therefore, $a_* = b_*$. Since the selection of (t_*, x_*) was arbitrary, theorem is fully proved. □

So, ε_0 is the fixed point of operator Γ. In a similar way with [10,11], we introduce the set

$$\tilde{\mathfrak{M}}_\psi^{(\Gamma)} \triangleq \{g \in \mathfrak{M}_\psi | (g = \Gamma(g)) \;\&\; (\rho(\cdot, \mathbf{N}) \leqq g)\}. \quad (8.7)$$

Of course, $\varepsilon_0 \in \tilde{\mathfrak{M}}_\psi^{(\Gamma)}$, thus, we obtain $\tilde{\mathfrak{M}}_\psi^{(\Gamma)} \neq \emptyset$.

Theorem 8.2. *The function of guaranteed result ε_0 is the $\leqq$-smallest element of $\tilde{\mathfrak{M}}_\psi^{(\Gamma)}$, namely:*

$$(\varepsilon_0 \in \tilde{\mathfrak{M}}_\psi^{(\Gamma)}) \;\&\; (\varepsilon_0 \leqq g \; \forall g \in \tilde{\mathfrak{M}}_\psi^{(\Gamma)}). \quad (8.8)$$

Proof. The inclusion $\varepsilon_0 \in \tilde{\mathfrak{M}}_\psi^{(\Gamma)}$ was shown previously. We shall consider the verification of the second statement in (8.8). Let $\mathbf{g} \in \tilde{\mathfrak{M}}_\psi^{(\Gamma)}$. Then, by (8.7),

$$(\mathbf{g} \in \mathfrak{M}_\psi) \;\&\; (\mathbf{g} = \Gamma(\mathbf{g})) \;\&\; (\rho(\cdot, \mathbf{N}) \leqq \mathbf{g}). \quad (8.9)$$

Thus, $\varepsilon_0^{(0)} \leqq \mathbf{g}$. We introduce the set $\mathfrak{N} \triangleq \{k \in \mathbb{N}_0 | \varepsilon_0^{(k)} \leqq \mathbf{g}\}$. Then $0 \in \mathfrak{N}$ and, as a corollary, $\mathfrak{N} \in \mathcal{P}'(\mathbb{N}_0)$. Let $r \in \mathfrak{N}$. With this, $r \in \mathbb{N}_0$ and $\varepsilon_0^{(r)} \leqq \mathbf{g}$. By Proposition 7.4 $\Gamma(\varepsilon_0^{(r)}) \leqq \Gamma(\mathbf{g})$, where $\varepsilon_0^{(r+1)} = \Gamma(\varepsilon_0^{(r)})$. Therefore, by (8.9), $\varepsilon_0^{(r+1)} \leqq \mathbf{g}$ and, as a corollary, $r + 1 \in \mathfrak{N}$. So,

$$(0 \in \mathfrak{N}) \;\&\; (k + 1 \in \mathfrak{N} \; \forall k \in \mathfrak{N}).$$

By induction, we obtain $\mathfrak{N} = \mathbb{N}_0$. As a result, by definition of $\mathfrak{N}$ we show that $\varepsilon_0^{(r)} \leqq \mathbf{g} \; \forall k \in \mathbb{N}_0$. As a corollary, by (5.21) the inequality $\varepsilon_0 \leqq \mathbf{g}$. Since the selection of $\mathbf{g}$ was arbitrary, the required property of ε_0 is finally established. □

9. One special application

In this final section, we consider a special setting of retention problem on the finite time interval T. Following this setting, we have the lacking target set and special kind of state constraints set $\tilde{\mathbf{N}} \in \mathcal{F}$, which t-cross-sections are defined as follows:

$$\tilde{\mathbf{N}}\langle t\rangle \triangleq \{x \in \mathbb{R}^n | (t,x) \in \tilde{\mathbf{N}}\}, \quad t \in T. \tag{9.1}$$

The retention problem for player I can be stated in a following way. At this moment, we implement informative notation. Assume that we have a starting position $(t_*, x_*) \in \tilde{\mathbf{N}}$. It is required to find $U \in \mathfrak{U}\langle t_*, x_*\rangle$, such that

$$x(t) \in \tilde{\mathbf{N}}\langle t\rangle \quad \forall x(\cdot) \in \mathcal{X}(t_*, x_*, U) \quad \forall t \in [t_*, \vartheta_0]; \tag{9.2}$$

in (9.2), we keep in mind the informative setting similar to one in Section 2. Of course, by (9.1), the relation (9.2) can be rewritten in the form

$$(t, x(t)) \in \tilde{\mathbf{N}} \quad \forall x(\cdot) \in \mathcal{X}(t_*, x_*, U) \quad \forall t \in [t_*, \vartheta_0].$$

However, in a similar way with (5.5), for $\varepsilon = 0$ we can consider more specific setting of the problem with non-anticipating strategies. This problem is a possible application for the relaxation provided in this research. Namely, for $(t_*, x_*) \in T \times \mathbb{R}^n$ we are trying to find the smallest number $\varepsilon_* \in [0, \infty[$ for which (see (4.4)) $\exists \alpha \in \tilde{A}_{t_*}\ \forall x(\cdot) \in \mathbb{X}[t_*; x_*; \alpha]$

$$(t, x(t)) \in S_0(\tilde{\mathbf{N}}, \varepsilon_*)\ \forall t \in [t_*, \vartheta_0 - \varepsilon_*], \tag{9.3}$$

(here and below we must require the additional agreement; we admit the application of expression $[a, b]$ for arbitrary $a \in \mathbb{R}$ and $b \in \mathbb{R}$: $[a, b] \triangleq \{\xi \in \mathbb{R} | (a \leq \xi)\ \&\ (\xi \leq b)\}$; of course, $[a, b] = \emptyset$ for $b < a$; similar expressions are used for $]a, b[$, $[a, b[$, and $]a, b]$). So, we employ order-wise intervals of different type. These agreements allows us to reduce the amount of possible cases. Of course, this initial setting is informative for $t_* < \vartheta_0$ and (9.3) is informative for $\varepsilon_* \leq \vartheta_0 - t_*$. However, we omit these requirements and investigate the question about guaranteed implementation of (9.3) is general case. Now, we consider the variant of transformation of this setting to general constructions provided in this paper. Furthermore, we assume that

$$\mathbf{M} \triangleq \{(\vartheta_0, x) : x \in \mathbb{R}^n\},\ \mathbf{N} \triangleq \tilde{\mathbf{N}} \cup \mathbf{M}. \tag{9.4}$$

In this case, $\mathbf{M} \in \mathcal{F}, \mathbf{N} \in \mathcal{F}$, and $\mathbf{M} \subset \mathbf{N}$. We obtain a variant of our general assumptions (see Section 2).

Proposition 9.1. *If* $(t_*, x_*) \in T \times \mathbb{R}^n$, $x(\cdot) \in C_n([t_*, \vartheta_0])$, *and* $\varepsilon \in [0, \vartheta_0 - t_*[$, *then*

$$\begin{aligned}(\exists \vartheta \in [t_*, \vartheta_0] : ((\vartheta, x(\vartheta)) \in S_0(\mathbf{M}, \varepsilon))\ \&\ ((t, x(t)) \in S_0(\mathbf{N}, \varepsilon)\ \forall t \in [t_*, \vartheta])) \\ \iff ((t, x(t)) \in S_0(\tilde{\mathbf{N}}, \varepsilon)\ \forall t \in [t_*, \vartheta_0 - \varepsilon]). \end{aligned} \tag{9.5}$$

Proof. We denote by (1') and (2') the expressions on the left-hand side and on the right-hand side of (9.5) respectively. Let (1') is true and $\vartheta^0 \in [t_*, \vartheta_0]$ implements the next expression

$$((\vartheta^0, x(\vartheta^0)) \in S_0(\mathbf{M}, \varepsilon)) \ \& \ ((t, x(t)) \in S_0(\mathbf{N}, \varepsilon) \ \forall t \in [t_*, \vartheta^0]). \tag{9.6}$$

Since (see (9.4)) $S_0(\mathbf{M}, \varepsilon) = [\vartheta_0 - \varepsilon, \vartheta_0] \times \mathbb{R}^n$, by (2.5) and (2.7), from the first expression in (9.6), we obtain $\vartheta^0 \in [\vartheta_0 - \varepsilon, \vartheta_0]$. So, from (9.6), we have the following:

$$(t, x(t)) \in S_0(\mathbf{N}, \varepsilon) \ \forall t \in [t_*, \vartheta_0 - \varepsilon]. \tag{9.7}$$

Let $\bar{t} \in [t_*, \vartheta_0 - \varepsilon[$. Then by (9.7), $(\bar{t}, x(\bar{t})) \in S_0(\mathbf{N}, \varepsilon)$ and by (2.7), $\rho((\bar{t}, x(\bar{t})), \mathbf{N}) \leq \varepsilon$. In addition, $\rho((\bar{t}, x(\bar{t})), \mathbf{M}) = |\bar{t} - \vartheta_0| = \vartheta_0 - \bar{t} > \varepsilon$. As a corollary, by (9.4) and (9.7)

$$\rho((\bar{t}, x(\bar{t})), \tilde{\mathbf{N}}) = \rho((\bar{t}, x(\bar{t})), \mathbf{N}) \leq \varepsilon.$$

Therefore, $(\bar{t}, x(\bar{t})) \in S_0(\tilde{\mathbf{N}}, \varepsilon)$. Since the selection of $\bar{t}$ was arbitrary, we obtain

$$\rho((t, x(t)), \tilde{\mathbf{N}}) \leq \varepsilon \ \forall t \in [t_*, \vartheta_0 - \varepsilon[, \tag{9.8}$$

where $\vartheta_0 - \varepsilon > t_*$ and $[t_*, \vartheta_0 - \varepsilon[\neq \emptyset$. From (9.8), by continuity of $x(\cdot)$ we obtain $\rho((\vartheta_0 - \varepsilon, x(\vartheta_0 - \varepsilon)), \tilde{\mathbf{N}}) \leq \varepsilon$ and, as a corollary, by (9.8)

$$\rho((t, x(t)), \tilde{\mathbf{N}}) \leq \varepsilon \ \forall t \in [t_*, \vartheta_0 - \varepsilon].$$

So, $(t, x(t)) \in S_0(\tilde{\mathbf{N}}, \varepsilon) \ \forall t \in [t_*, \vartheta_0 - \varepsilon]$. We obtain (2'). Thus, (1') $\Longrightarrow$ (2').

Let (2') is true; then $(t, x(t)) \in S_0(\tilde{\mathbf{N}}, \varepsilon) \ \forall t \in [t_*, \vartheta_0 - \varepsilon]$. In particular, $(\vartheta_0 - \varepsilon, x(\vartheta_0 - \varepsilon)) \in S_0(\tilde{\mathbf{N}}, \varepsilon)$ and, as a corollary,

$$\rho((\vartheta_0 - \varepsilon, x(\vartheta_0 - \varepsilon)), \mathbf{M}) = \vartheta_0 - (\vartheta_0 - \varepsilon) = \varepsilon. \tag{9.9}$$

Then $(\vartheta_0 - \varepsilon, x(\vartheta_0 - \varepsilon)) \in S_0(\mathbf{M}, \varepsilon)$. On the other hand, $\tilde{\mathbf{N}} \subset \mathbf{N}$ and with $t \in [t_*, \vartheta_0 - \varepsilon]$

$$\rho((t, x(t)), \mathbf{N}) \leq \rho((t, x(t)), \tilde{\mathbf{N}}) \leq \varepsilon.$$

We obtain $(t, x(t)) \in S_0(\mathbf{N}, \varepsilon) \ \forall t \in [t_*, \vartheta_0 - \varepsilon]$. In addition, $\vartheta_0 - \varepsilon \in]t_*, \vartheta_0]$. So, (1') holds. We have the implication (2') $\Longrightarrow$ (1'). So, (1') $\Longleftrightarrow$ (2'). □

Now, under **M** and **N** with the properties (9.4), we can implement the general procedure described in the previous sections. In addition, we obtain function ε_0 for representation (9.4). Now, we introduce the set (we keep in mind Proposition 9.1):

$$\mathbb{G} \triangleq \{(t, x) \in T \times \mathbb{R}^n | \varepsilon_0(t, x) < \vartheta_0 - t\}. \tag{9.10}$$

Proposition 9.2. *Let* $(t_*, x_*) \in \mathbb{G}$ *and* $\varepsilon_* \triangleq \varepsilon_0(t_*, x_*)$.
Then $(t_*, x_*) \in \mathcal{W}(S_0(\mathbf{M}, \varepsilon_*), S_0(\mathbf{N}, \varepsilon_*))$.

The proof follows from the definition of ε_0. By Proposition 9.2, we obtain

$$\pi^{(\mathcal{W})}_{t_*, x_*} \langle \cdot \, | S_0(\mathbf{M}, \varepsilon_0(t_*, x_*)), S_0(\mathbf{N}, \varepsilon_0(t_*, x_*)) \rangle \in \tilde{A}^{\Pi}_{t_*} \ \ \forall (t_*, x_*) \in \mathbb{G}. \tag{9.11}$$

The proof of the proposition above follows from (6.4).

Proposition 9.3. *Let* $(t_*, x_*) \in \mathbb{G}$ *and* $\alpha_* \triangleq \pi^{(\mathcal{W})}_{t_*,x_*}\langle\cdot\,|S_0(\mathbf{M},\varepsilon_*), S_0(\mathbf{N},\varepsilon_*)\rangle$, *where* $\varepsilon_* = \varepsilon_0(t_*, x_*)$. *Then*

$$(t, x(t)) \in S_0(\tilde{\mathbf{N}}, \varepsilon_*)\ \forall x(\cdot\,) \in \mathbb{X}[t_*; x_*; \alpha_*]\ \forall t \in [t_*, \vartheta_0 - \varepsilon_*]. \tag{9.12}$$

Proof. By the selection of (t_*, x_*) and α_* we obtain (see (6.2) and (6.5))

$$\forall x(\cdot\,) \in \mathbb{X}[t_*; x_*; \alpha_*] \exists \vartheta \in [t_*, \vartheta_0] :$$

$$((\vartheta, x(\vartheta)) \in S_0(\mathbf{M}, \varepsilon_*))\ \&\ ((t, x(t)) \in S_0(\mathbf{N}, \varepsilon_*) \forall t \in [t_*, \vartheta]).$$

In addition, by (9.10) $\varepsilon_* \in [0, \vartheta_0 - t_*[$. By Proposition 9.1 we show that (9.12) is valid. □

Proposition 9.4. *Let* $(t_*, x_*) \in \mathbb{G}$ *and* $\varepsilon \in [0, \varepsilon_0(t_*, x_*)[$. *Then*

$$\forall \alpha \in \tilde{A}_{t_*}\ \exists x(\cdot\,) \in \mathbb{X}[t_*; x_*; \alpha]\ \exists t \in [t_*, \vartheta_0 - \varepsilon] :\ (t, x(t)) \notin S_0(\tilde{\mathbf{N}}, \varepsilon_*).$$

Proof. By (5.9) and (5.10) we obtain $(t_*, x_*) \notin \mathcal{W}(S_0(\mathbf{M}, \varepsilon), S_0(\mathbf{N}, \varepsilon))$. Since $(t_*, x_*) \in T \times \mathbb{R}^n$, then by (6.4)

$$\tilde{\Pi}_{t_*}(\alpha)\ \setminus\ \mathcal{S}_{S_0(\mathbf{M},\varepsilon), S_0(\mathbf{N},\varepsilon)}(t_*, x_*) \neq \emptyset\ \forall \alpha \in \tilde{A}_{t_*}. \tag{9.13}$$

Let $\bar{\alpha} \in \tilde{A}_{t_*}$ and $\bar{\eta} \in \tilde{\Pi}_{t_*}(\bar{\alpha})\ \setminus\ \mathcal{S}_{S_0(\mathbf{M},\varepsilon), S_0(\mathbf{N},\varepsilon)}(t_*, x_*)$; see (9.13). Then for $\bar{x}(\cdot\,) \triangleq \varphi(\cdot\,, t_*, x_*, \bar{\eta}) \in C_n([t_*, \vartheta_0])$ we have the next property: $\forall \vartheta \in [t_*, \vartheta_0]$

$$((\vartheta, \bar{x}(\vartheta)) \in S_0(\mathbf{M}, \varepsilon)) \Rightarrow (\exists t \in [t_*, \vartheta] : (t, \bar{x}(t)) \notin S_0(\mathbf{N}, \varepsilon)). \tag{9.14}$$

We note that $\bar{x}(\cdot\,) \in \mathbb{X}[t_*; x_*; \bar{\alpha}]$ (see (4.4)). Then

$$\exists t \in [t_*, \vartheta_0 - \varepsilon] : (t, \bar{x}(t)) \notin S_0(\tilde{\mathbf{N}}, \varepsilon). \tag{9.15}$$

Indeed, assume the contrary:

$$(t, \bar{x}(t)) \in S_0(\tilde{\mathbf{N}}, \varepsilon)\ \forall t \in [t_*, \vartheta_0 - \varepsilon]. \tag{9.16}$$

By the selection of (t_*, x_*) we obtain $\varepsilon_0(t_*, x_*) < \vartheta_0 - t_*$. Since $\varepsilon < \varepsilon_0(t_*, x_*)$, we have the inequality $\varepsilon < \vartheta_0 - t_*$. Then by Proposition 9.1

$$(\exists \vartheta \in [t_*, \vartheta_0] : ((\vartheta, \bar{x}(\vartheta)) \in S_0(\mathbf{M}, \varepsilon))\ \&\ ((t, \bar{x}(t)) \in S_0(\mathbf{N}, \varepsilon)\ \forall t \in [t_*, \vartheta]))$$
$$\iff\ ((t, \bar{x}(t)) \in S_0(\tilde{\mathbf{N}}, \varepsilon)\ \forall t \in [t_*, \vartheta_0 - \varepsilon]). \tag{9.17}$$

From (9.16) and (9.17) we show that $\exists \vartheta \in [t_*, \vartheta_0]$:

$$((\vartheta, \bar{x}(\vartheta)) \in S_0(\mathbf{M}, \varepsilon))\ \&\ ((t, \bar{x}(t)) \in S_0(\mathbf{N}, \varepsilon)\ \forall t \in [t_*, \vartheta]).$$

The last property is impossible by (9.14). The obtained contradiction means that (9.16) is impossible. Thus, (9.15) takes place. Finally, due to selection of $\bar{\alpha}$ was arbitrary, the required property is established. □

From (9.11), Proposition 9.3 and Proposition 9.4, for $(t_*, x_*) \in \mathbb{G}$, in the form of $\varepsilon_* \triangleq \varepsilon_0(t_*, x_*) \in [0, \infty[$, we have the smallest number $\varepsilon \in [0, \infty[$ for which

$$\exists \alpha \in \tilde{A}_{t_*} : (t, x(t)) \in S_0(\tilde{\mathbf{N}}, \varepsilon)\ \forall x(\cdot\,) \in \mathbb{X}[t_*; x_*; \alpha]\ \forall t \in [t_*, \vartheta_0 - \varepsilon].$$

Therefore, we obtain a variant of the relaxation for the initial retention problem.

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