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Hypoelliptic Mean Field Games – a Case Study

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We study hypoelliptic mean-field games (MFG) that arise in stochastic control problems of degenerate diffusions. Here, we consider MFGs with quadratic Hamiltonians and prove the existence and uniqueness of solutions. Our main tool is the Hopf-Cole transform that converts the MFG into an eigenvalue problem. We prove the existence of a principal eigenvalue and a positive eigenfunction, which are then used to construct the unique solution to the original MFG.

Keywords: Mean-Field Games, stationary problems, hypoelliptic operator, eigenvalue problems.

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1. Introduction

Mean-field games (MFGs) are mathematical models to study the behavior of large populations of rational agents who seek to optimize a utility, identical for every agent. MFGs were introduced in the mathematical community in [24], [25], and [26], and, independently, in the engineering community in [22] and [23]. In this paper, we consider hypoelliptic MFGs; that is, MFGs that arise in stochastic control problems of degenerate diffusions.

Given a family of smooth vector fields on $\mathbb{T}^d$, we consider the following differential operators. First, each smooth vector field, $X = (X^1, ..., X^d)\colon \mathbb{T}^d \to \mathbb{R}^d$, is identified with the differential operator

$$Xf := \sum_{j=1}^{d} X^j \partial_{x_j} f$$

for $f \in C^1(\mathbb{T}^d)$. The gradient, $D_{\mathcal{X}} u$, the divergence, $\operatorname{div}_{\mathcal{X}^*}$, and the Laplacian, $\Delta_{\mathcal{X}} u$, are

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$$D_{\mathcal{X}}u = (X_1u, ..., X_lu)^T,\ \operatorname{div}_{\mathcal{X}^*}\boldsymbol{v} = -\sum_{i=1}^{l}(X_i + \operatorname{div} X_i)v_i,\ \Delta_{\mathcal{X}}u = \operatorname{div}_{\mathcal{X}^*}(D_{\mathcal{X}}u),$$

where $\operatorname{div} X_i$ is the *Euclidean divergence*, $\operatorname{div}_{\mathcal{X}^*}\boldsymbol{v}$, however, is the divergence with respect to the family $\boldsymbol{\mathcal{X}}^*$ of the dual differential operators, X_i^* . The operator $X_i^* := -X_i - \operatorname{div} X_i$ is the dual of X_i in the sense that $\int (X_iu)v = \int uX_i^*v$ for all $u, v \in C^1(\mathbb{T}^d)$.

Here, we study the following problem.

Problem 1.1. *Let $d \in \mathbb{N}$ with $d \geq 3$ and let $\mathcal{X} = \{X_1, ..., X_l\}$ be a family of smooth vector fields on $\mathbb{T}^d$ with $X_i\colon \mathbb{T}^d \to \mathbb{R}^d$ for $1 \leq i \leq l$. We consider, for $m \in L^1(\mathbb{T}^d)$, a potential $V[m](\cdot)\colon \mathbb{T}^d \to \mathbb{R}$. Find (λ, m, u) satisfying $m \geq 0$, $\int_{\mathbb{T}^d} m\,\mathrm{d}x = 1$, and*

$$\begin{cases} \lambda + \Delta_{\mathcal{X}}u - \frac{|D_{\mathcal{X}}u|^2}{2} = V[m](x) & \text{in } \mathbb{T}^d, \\ \Delta_{\mathcal{X}}m + \operatorname{div}_{\mathcal{X}^*}(mD_{\mathcal{X}}u) = 0 & \text{in } \mathbb{T}^d. \end{cases} \tag{1}$$

The prior problem arises in the analysis of a large population of agents. Each of these agents controls a degenerate diffusion,

$$\begin{cases} \mathrm{d}Y(t) = b(Y(t), \alpha(t))\mathrm{d}s + \sqrt{2}\,\sigma(Y(t)) \circ \mathrm{d}W(t), \\ Y(t_0) = x_0, \end{cases} \tag{2}$$

where $\alpha \in \mathcal{A}$, with $\mathcal{A}$ the set of all progressively measurable controls from $[0, T]$ to $\mathbb{R}^l$, $b(x, \alpha) = \sum_{i=1}^{l} X_i(x)\alpha_i$, σ is a $d \times l$ matrix, $\sigma(x) = (X_1(x), ..., X_l(x)) \in \mathbb{R}^{d\times l}$, and $\sigma(Y(t)) \circ \mathrm{d}W(t)$ denotes Stratonovich integration. For given $\alpha \in \mathcal{A}$, define

$$J(t, x; \alpha) := \mathbb{E}\left[\int_t^T \big(L(\alpha(s)) + V(Y(s))\big)\,\mathrm{d}s + g(Y(T))\right],$$

where $L(\alpha) = \frac{1}{2}|\alpha|^2$. In (1), the first equation is a Hamilton-Jacobi equation that gives the value function, u, for an agent as

$$u(x, t) = \inf_{\alpha\in\mathcal{A}} J(t, x; \alpha).$$

The Hamiltonian, H, is given by $H(x, p) = \sup_{\alpha\in\mathcal{A}}\{-p\cdot b(x, \alpha) - L(x, \alpha)\}$. Then, because $b(x, \alpha) = \sum_{i=1}^{l} X_i(x)\alpha_i$, $\sigma(x) = (X_1(x), ..., X_l(x))$, and $L(\alpha) = \frac{|\alpha|^2}{2}$, applying Itô's lemma to (2) with standard assumptions, u solves

$$\begin{cases} -u_t - \Delta_{\mathcal{X}}u + \frac{|D_{\mathcal{X}}u|^2}{2} = V(x), \\ u(T, x) = g(x). \end{cases}$$

Each agent seeks to minimize J. However, this cost depends on the distribution of the remaining agents. The second equation, a Fokker-Planck equation, describes the statistical distribution, m, of the agents. V encodes the spatial preferences of the agents and couples the Hamilton-Jacobi equation with the Fokker-Planck equation.

A general theory of existence of solutions to MFGs that satisfy certain monotonicity conditions was derived in [18] (and in [19] for MFGs with boundary conditions). However, for particular MFGs, more precise statements can be made concerning the regularity of solutions. In particular, MFGs with uniformly elliptic operators have been studied widely. Because of the well-known regularity properties of the Laplacian, that part of the theory is currently well-established. For example, the existence of classical solutions to MFGs was examined in [20], [21], and [29].

The Hopf-Cole transformation has been explored and used for ergodic MFGs extensively in [5], [6], [7], [9], and [14]. In particular, in the appendix of [5], the authors proved the existence and the uniqueness of classical solutions to time-dependent MFGs without any growth assumption but with the below bound on the coupling. In the proof of Lemma 3.2 at the appendix, this below bound and the first-order time derivative term were crucial to get the uniform bound of Du. Hence, the method in [5] may fail to apply to stationary cases. However, it may be possible to use this approach to time-dependent hypoelliptic MFGs associated with infinite-horizon control problems. Through the Hopf-Cole transformation, we obtain eigenvalue problems corresponding to appropriate ergodic problems. The existence of classical solutions to ergodic MFGs was addressed in [8].

The hypoelliptic case, which is an intermediate case between uniformly elliptic and general degenerate elliptic, is less well studied. This problem was addressed in [13] for Hamiltonian with linear growth. However, hypoelliptic operators enjoy more regularizing properties than general degenerate elliptic operators. Thus, in this paper, we investigate a model where these properties, combined with certain transformations, give the existence, regularity, and uniqueness of solutions. Because we use the Hopf-Cole transformation, our methods borrow techniques from the theory of eigenvalue problems for non-linear elliptic equations. For instance, in [28], the authors investigated eigenvalue problems with the standard Laplacian and homogeneous Dirichlet or Neumann boundary conditions. They assumed polynomial growth for the non-linearity, $V[m]$; that is, $V[m](x) = m(x)^q$ where $q \in (0, (2^* - 2)/2)$, which is similar to what is required here in Assumption A6. Finally, concerning degenerate elliptic equations, the degenerate time-dependent MFG was studied in [4]. The existence of solutions to Hamilton-Jacobi equations in Carnot groups was explained in [1]. The authors assumed only convexity and coercivity for the Hamiltonian. Homogenization problems in the Heisenberg group were examined in [11] and [12].

The following two theorems describe our main results. The first theorem concerns the local interaction case (when Assumption A4 holds, cf. Section 3) and the second the non-local case (when Assumptions A7–A8 hold, cf. Section 3).

Theorem 1.2. *Assume that Assumptions* A1–A6 *hold, i.e.,* $V[m](x) = V(m(x))$. *Then, there exists a unique solution* $(\lambda, m, u) \in \mathbb{R} \times C^\infty(\mathbb{T}^d) \times C^\infty(\mathbb{T}^d)$ *to Problem* 1.1.

Theorem 1.3. *Assume that Assumptions* A1–A3 *and* A7–A8 *hold; that is, V is non local. Then, there exists a unique solution $(\lambda, m, u) \in \mathbb{R} \times C^2_{\mathcal{X}}(\mathbb{T}^d) \times C^2_{\mathcal{X}}(\mathbb{T}^d)$ to Problem* 1.1.

The solution is unique here as well because V is monotone, see e.g. [13, Theorem 4.2].

The analysis of both cases relies on the Hopf-Cole transform, ($u = 2\log w$). Then, we have

$$\Delta_{\mathcal{X}} u = 2\frac{\Delta_{\mathcal{X}} w}{w} - \frac{2|D_{\mathcal{X}} w|}{w^2}, \quad \frac{|D_{\mathcal{X}} u|^2}{2} = \frac{2|D_{\mathcal{X}} w|}{w^2}.$$

Hence, applying the Hopf-Cole transform to the first equation in (1), we get the following eigenvalue problem:

$$\frac{1}{2}V[m]w - \Delta_{\mathcal{X}} w = \frac{1}{2}\lambda w,$$

which by redefining V, to simplify the notation, becomes

$$V[m]w - \Delta_{\mathcal{X}} w = \lambda w \ \text{ in } \mathbb{T}^d. \tag{3}$$

We split the analysis of this eigenvalue problem into three cases. First, in Section 4, we establish preliminary results by considering the $m-$independent case. In particular, we give conditions for the existence of a unique principal eigenvalue and corresponding positive eigenfunction. Next, in Section 5, we analyze the MFG case. In the local case, addressed in Section 5.1, we rely on variational methods. In the non-local case, addressed in Section 5.2, we rely on Schaefer's fixed-point theorem.

2. Preliminaries

In this section, we introduce some key definitions and explain the notation used throughout this paper. In particular, we define Hölder and Sobolev spaces with respect to a family of vector fields. Then, we examine various embedding results that extend those for standard Hölder and Sobolev spaces.

Given two vector fields, X and Y, their *commutator* (or, *Lie bracket*), $[\cdot,\cdot]$, is the vector field

$$[X,Y] := XY - YX = \sum_{i,j=1}^{d} \Big(X^j Y^i_{x_j} - Y^j X^i_{x_j} \Big) \partial_{x_i}. \tag{4}$$

Definition 2.1. The *Lie algebra*, $\mathcal{L}(X_1, ..., X_l)$, generated by a family, $X_1, ..., X_l$, of smooth vector fields endowed with the commutator, $[\cdot,\cdot]$, in (4) is the smallest subspace of $C^\infty(\mathbb{T}^d, \mathbb{R}^d)$ that is closed under commutation and contains $X_1, ..., X_l$.

Remark 2.2. From the preceding definition, we gather that the Lie algebra is the span of

$$X_1, ..., X_l, [X_i, X_j], [X_i, [X_j, X_k]], ... \ , \tag{5}$$

the *iterated commutators* of the family of vector fields $X_1, \ldots, X_l$.

The *length* of an iterated commutator is the minimal number of distinct vector fields needed for its definition/computation. In particular, vector fields are iterated commutators of length one, the "usual" commutator $[X_1, X_2]$ is an iterated commutator of length 2, $[[X_1, X_2], X_3]$ is an iterated commutator of length 3, $[[[X_1, X_2], X_3], X_4]$ are $[[X_1, X_2], [X_3, X_4]]$ are iterated commutators of length 4, and $[[X_1, [X_2, X_3]], [X_4, X_5]]$ is an iterated commutator of length 5, for example. This definition can be formalized rigorously (see [16], [17]) and corresponds to the intuitive notions of "iterated commutator" and its "length" outlined here.

The standard *Euclidean divergence* of a vector field, Y, is

$$\operatorname{div} Y = \sum_{j=1}^{d} \partial_{x_j} Y^j.$$

A family $\mathcal{X} = \{X_1, ..., X_l\}$ of smooth vector fields on $\mathbb{T}^d$ satisfies the *Hörmander condition* if

$$\mathcal{L}(X_1, ..., X_l)(x) = T_x\mathbb{T}^d = \mathbb{R}^d \quad \text{for } x \in \mathbb{T}^d,$$

where $T_x\mathbb{T}^d$ denotes the tangent space at $x \in \mathbb{T}^d$. The step of the Hörmander condition at $x \in \mathbb{T}^d$, $s(x)$, is

$$\begin{aligned} s(x) = \inf \Big\{ & r \in \mathbb{N} \colon \text{ there exist } B = \{B_i\}_{i=1}^n \text{ iterated Lie brackets } \textit{s.t.} \\ & \text{Length}(B) \le r \text{ and } \operatorname{Span}\big(B_1(\mathcal{X})(x), ..., B_n(\mathcal{X})(x)\big) = T_x\mathbb{T}^d = \mathbb{R}^d \Big\}. \end{aligned}$$

Since the dimension of the tangent space is d we may take $n = d$ above.

Remark 2.3. By Remark 2.2, for each $x \in \mathbb{T}^d$, if $(B_1(x), ..., B_n(x))$ spans $\mathbb{R}^d$, then any vector in $\mathbb{R}^d$ is a finite linear combination of (5). Therefore, $s(x)$ is finite.

Since $\mathcal{X}$ is a family of smooth vector fields, for each $x \in \mathbb{T}^d$, there exists a neighborhood $\mathcal{O}(x)$ such that Length(B) is constant in $\mathcal{O}(x)$. Thus, because $\mathbb{T}^d$ is compact and $s(x)$ is upper semi-continuous, the Hörmander step

$$s = \sup_{x \in \mathbb{T}^d} s(x) \tag{6}$$

is finite. Through this paper, we suppose that $s \ge 2$.

Example 2.4. Consider the local coordinates (x_1, x_2, x_3) in $\mathbb{T}^3$ and assume $X_1 = \cos(2\pi x_3)\partial_{x_1} + \sin(2\pi x_3)\partial_{x_2}$, $X_2 = \partial_{x_3}$. Then,

$$[X_1, X_2] = X_1X_2 - X_2X_1 = -2\pi \sin(2\pi x_3)\partial_{x_1} + 2\pi \cos(2\pi x_3)\partial_{x_2}.$$

Thus, $\mathcal{X} := \{X_1, X_2\}$ satisfies the Hörmander condition. Also, $s(x) = s = 2$ for any $\mathbf{x} \in \mathbb{T}^3$.

For given a family of smooth vector fields, $\mathcal{X} = \{X_1, ..., X_l\}$, on $\mathbb{T}^d$, an absolutely continuous curve $\gamma\colon [0,T] \to \mathbb{T}^d$ is *horizontal* if there exists a measurable function $\alpha\colon [0,T] \to \mathbb{R}^l$ such that

$$\dot\gamma(t) = \sum_{i=1}^{l} \alpha_i(t) X_i(\gamma(t)), \quad \text{a.e. } t \in (0,T).$$

The *length* of a horizontal curve is $\quad l(\gamma) = \displaystyle\int_0^T \sqrt{\sum_{i=1}^{l} \alpha_i^2(t)}\,\mathrm{d}t.$

The *Carnot-Carathéodory distance* associated to $\mathcal{X}$, $\mathcal{X}(\cdot,\cdot)$, between $x, y \in \mathbb{T}^d$ is

$$\mathcal{X}(x,y) = \inf \Big\{ l(\gamma)\colon \gamma \text{ is a horizontal curve connecting } x \text{ with } y \Big\}.$$

From Proposition 1.1 in [27], there exist constants, C_1 and C_2, such that

$$C_1|x-y| \le \mathcal{X}(x,y) \le C_2|x-y|^{\frac{1}{s}}, \tag{7}$$

where s is the Hörmander step in (6).

Next, we introduce the Hölder and Sobolev spaces with respect to $\mathcal{X}$ (see [13]). For $k \in \mathbb{Z}_+$ and $\alpha \in (0,1)$, let

$$C^{k,\alpha}_{\mathcal{X}}(\mathbb{T}^d) = \left\{ u \in L^\infty(\mathbb{T}^d)\colon \sup_{\substack{x,y\in\mathbb{T}^d \\ x\neq y}} \frac{|\mathcal{X}^J u(x) - \mathcal{X}^J u(y)|}{\mathcal{X}(x,y)^\alpha} < \infty, \forall J \in \mathbb{Z}^l_+, |J| \le k \right\}.$$

Because of (7), we have $\qquad C^{0,\alpha}(\mathbb{T}^d) \subset C^{0,\alpha}_{\mathcal{X}}(\mathbb{T}^d) \subset C^{0,\frac{\alpha}{s}}(\mathbb{T}^d). \qquad (8)$

Also, we have the following embedding theorem.

Proposition 2.5. *For any* $1 \le k < \infty$ *and* $0 < \alpha \le 1$, *we have*

$$C^{sk,\alpha}_{\mathcal{X}}(\mathbb{T}^d) \hookrightarrow C^{k,\alpha/s}(\mathbb{T}^d).$$

Proof. See Lemma 2.1 in [34]. □

For $k \in \mathbb{Z}_+$ and $1 \le p \le \infty$, let

$$W^{k,p}_{\mathcal{X}}(\mathbb{T}^d) = \big\{ u \in L^p(\mathbb{T}^d)\colon \mathcal{X}^J u \in L^p(\mathbb{T}^d), \forall J \in \mathbb{Z}^l_+, |J| \le k \big\},$$

and $\qquad H^k_{\mathcal{X}}(\mathbb{T}^d) = W^{k,2}_{\mathcal{X}}(\mathbb{T}^d).$

$W^{k,p}_{\mathcal{X}}(\mathbb{T}^d)$ is a Banach space with the norm

$$\|u\|_{W^{k,p}_{\mathcal{X}}(\mathbb{T}^d)} = (\sum_{|J|\le k} \int_{\mathbb{T}^d} |\mathcal{X}^J u|^p \,\mathrm{d}x)^{1/p}$$

(see Theorem 1 of Section 2 in [33]).

If $d > kp/s$, let
$$p^\sharp := \frac{dp}{d - kp/s}. \tag{9}$$

This exponent plays the role of the Sobolev conjugate exponent for the spaces $W^{k,p}_{\mathcal{X}}(\mathbb{T}^d)$. For $s \geq 2$, $p^\sharp$ is smaller than the Sobolev conjugate exponent, but it plays a similar role in the following embedding theorems. The first result is the analog of Sobolev's theorem and gives the continuous injection of $W^{k,p}_{\mathcal{X}}(\mathbb{T}^d)$ in $L^p(\mathbb{T}^d)$.

Proposition 2.6. *Fix $p \in (1, \infty)$ and let $p^\sharp$ be as in (9). If $d > kp/s$, then we have $W^{k,p}_{\mathcal{X}}(\mathbb{T}^d) \hookrightarrow L^{p^\sharp}(\mathbb{T}^d)$.*

Proof. See Theorem 3 of Section 2 in [33]. □

The second result gives the embedding of the Sobolev space associated with $\mathcal{X}$ in the standard Sobolev space.

Proposition 2.7. *Let s be as in (6). For any $p \in [1, \infty)$, we have*
$$W^{k,p}_{\mathcal{X}}(\mathbb{T}^d) \hookrightarrow W^{k/s,p}(\mathbb{T}^d).$$

Proof. See Theorem 13 in [31]. □

The third result is the analog of Rellich-Kondrachov compactness theorem.

Proposition 2.8. *Fix $p \in [1, \infty)$. If $d > kp/s$, then, for every $q \in [1, p^\sharp)$, we have*
$$W^{k,p}_{\mathcal{X}}(\mathbb{T}^d) \hookrightarrow\hookrightarrow L^q(\mathbb{T}^d).$$

Proof. From Proposition 2.7 and Corollary 7.2 in [10], for $q \in [1, p^\sharp)$, we obtain
$$W^{k/s,p}(\mathbb{T}^d) \hookrightarrow\hookrightarrow L^q(\mathbb{T}^d).$$ □

3. Assumptions

To prove our main result, we work under the following assumptions on $\mathcal{X}$ and V. The first two assumptions concern the Hörmander condition and impose a symmetry condition for $\mathcal{X}$.

Assumption A1. $\mathcal{X} = \{X_1, ..., X_l\}$ satisfies the Hörmander condition; that is,
$$\mathcal{L}(X_1(x), ..., X_l(x)) = T_x\mathbb{T}^d = \mathbb{R}^d, \forall x \in \mathbb{T}^d,$$
where $\mathcal{L}(X_1(x), ..., X_l(x))$ denotes the Lie algebra induced by the given vector fields and $T_x\mathbb{T}^d$ denotes the tangent space at $x \in \mathbb{T}^d$.

Assumption A2. $\mathcal{X} = \{X_1, ..., X_l\}$ is symmetric; that is, $\operatorname{div} X_i = 0$ for $i = 1, ..., l$. Therefore, $\operatorname{div}_{\mathcal{X}^*} = -\operatorname{div}_{\mathcal{X}}$.

Example 3.1. For $\mathcal{X}$ given by Example 2.4, since $\operatorname{div} X_1 = \operatorname{div} X_2 = 0$, Assumptions A1–A2 hold.

The next assumptions concern monotonicity, regularity, and growth of the potential, V.

Assumption A3. V is *monotone*, that is, for $m_1, m_2 \in L^1(\mathbb{T}^d)$ such that $m_1, m_2 \geq 0$,

$$\int_{\mathbb{T}^d} (V[m_1] - V[m_2])(m_1 - m_2)\,\mathrm{d}x \geq 0.$$

Assumption A4. V is local, i.e., $V[m](x) = V(m(x))$ and $V(\cdot) \in C^\infty((0, +\infty))$.

Assumption A5. For V as in Assumption A4, let $G(u) = \int_0^u tV(t^2)\,\mathrm{d}t$. There exits a constant, $\sigma \in (0, 2]$, such that

$$\liminf_{|w|\to\infty} \frac{G(w)}{|w|^\sigma} > -\infty.$$

Assumption A5 implies the following.

Lemma 3.2. *Let G be as in Assumption* A5. *Then, G is even.*

Proof. By the change of variables formula, we have

$$G(-u) = \int_0^{-u} tV(t^2)\,\mathrm{d}t = \int_0^u sV(s^2)\,\mathrm{d}s = G(u),$$

Thus, G is even. □

Remark 3.3. Assumption A5 is equivalent to this statement: there is a constant C such that, for every $w \in \mathbb{R}$, $G(w) \geq -C(|w|^\sigma + 1)$.

Assumption A6. If $d > 2/s$, let $2^\sharp := \dfrac{2d}{d - 2/s} > 2$. Then, V is local as in Assumption A4 and for some $\beta \in (0, \min\{(2^\sharp - 1)/2, 2^\sharp/sd\})$, V satisfies

$$|V(t)| \leq C(|t|^\beta + 1) \quad \forall t \in \mathbb{R}_0^+.$$

Remark 3.4. Let $w \in H^1_{\mathcal{X}}(\mathbb{T}^d)$ and suppose that Assumption A6 holds. Then, Proposition 2.6 gives that

$$wV[w^2] \in L^{\frac{2^\sharp}{(2\beta+1)}}(\mathbb{T}^d), \quad \text{where } 2^\sharp/(2\beta + 1) > 1.$$

Assumption A7. The map $u \mapsto V[u^2]$ from $H^1_{\mathcal{X}}(\mathbb{T}^d)$ to $L^\infty(\mathbb{T}^d)$ is bounded and continuous in the following sense: for any sequence $(u_n) \subset H^1_{\mathcal{X}}(\mathbb{T}^d)$ and any $u \in H^1_{\mathcal{X}}(\mathbb{T}^d)$, we have, for some $p > (2^\sharp/2)'$:

$$u_n \to u \text{ in } H^1_{\mathcal{X}}(\mathbb{T}^d) \quad \Longrightarrow \quad V[u_n^2] \rightharpoonup V[u^2] \text{ in } L^p(\mathbb{T}^d). \tag{10}$$

Assumption A8. Let V be as in Assumption A7. For all $u \in H^1_{\mathcal{X}}(\mathbb{T}^d)$, we have $V[u^2] \in C^\infty(\mathbb{T}^d)$.

The preceding assumptions are not the only possible ones for the existence of a solution. For example, in section 5.3, we show that we can replace Assumptions

A3–A6 by the following two assumptions and still establish the existence of a solution (Theorem 5.6).

Assumption A9. $V[\cdot]$ is given by $V[m](x) = W(x, m(x))$ for all $m \in W^{1,1}_{\mathcal{X}}(\mathbb{T}^d)$, $x \in \mathbb{T}^d$, where $W\colon \mathbb{T}^d \times \mathbb{R}^+_0 \to \mathbb{R}$ is continuous and satisfies the growth condition

$$-C(m^r + 1) \le W(x, m) \le M \qquad \forall x \in \mathbb{T}^d,\ m \ge 0$$

for some $M, C \ge 0$ and $0 \le r \le 1$. Also, for some $p > (2^\sharp/2)'$, V satisfies (10).

Example 3.5. Let $V_1[m]$ and $V_2[m]$ be such that $V_1[m](x) = 1 - \frac{1}{1+m(x)}$ and $V_2[m](x) = \zeta * \zeta * m(x)$, where $\zeta \in C^\infty_c(\mathbb{R}^d)$ with $\zeta \ge 0$ and ζ is even. Then, $V_1[m]$ satisfies Assumptions A3–A6 and A9, and $V_2[m]$ satisfies A7–A8.

4. Existence of solutions to the eigenvalue problem

Here, we investigate the eigenvalue problem corresponding to the Hamilton-Jacobi equation in (1). In this section, we consider the case where V is independent of m; that is, $V[m](x) \equiv f(x) \in C^{0,\alpha}_{\mathcal{X}}(\mathbb{T}^d)$. As we showed in (3), using the Hopf-Cole transform, $u = 2\log w$, i.e., $w = \exp(u/2)$, the first equation becomes

$$fw - \Delta_{\mathcal{X}} w = \lambda w. \tag{11}$$

If $m = w^2$, then, m solves the second equation in (1). Because (11) is linear, if w solves (11), then, for any constant c, cw is also a solution. Hence, we can assume that $\int_{\mathbb{T}^d} w^2\,\mathrm{d}x = 1$. Since we need to invert the Hopf-Cole transformation, we want to find $w > 0$ and a constant λ such that

$$\begin{cases} fw - \Delta_{\mathcal{X}} w = \lambda w \quad \text{in } \mathbb{T}^d, \\ \int_{\mathbb{T}^d} w^2\,\mathrm{d}x = 1. \end{cases} \tag{12}$$

Inspired by the discussion in Section 6.5.1 in [15], we prove the following proposition.

Proposition 4.1. *Suppose that Assumptions* A1–A2 *hold and that* $f \in C^{0,\alpha}_{\mathcal{X}}(\mathbb{T}^d)$ *with* $f(x) > 0$ *for all* $x \in \mathbb{T}^d$*. Then,* $S := (f - \Delta_{\mathcal{X}})^{-1}$ *is a bounded, linear, and compact operator from* $L^2(\mathbb{T}^d)$ *to* $L^2(\mathbb{T}^d)$*. Moreover,* S *is symmetric.*

Proof. Because $f \in C^{0,\alpha}_{\mathcal{X}}(\mathbb{T}^d)$, we can define a bilinear form

$$B[\cdot,\cdot]\colon H^1_{\mathcal{X}}(\mathbb{T}^d) \times H^1_{\mathcal{X}}(\mathbb{T}^d) \to \mathbb{R}$$

by
$$B[v_1, v_2] := \int_{\mathbb{T}^d} (f(x) v_1 v_2 + D_{\mathcal{X}} v_1 \cdot D_{\mathcal{X}} v_2)\,\mathrm{d}x.$$

Then, there exist constants $C_1, C_2 > 0$ such that, for all $v_1, v_2 \in H^1_{\mathcal{X}}(\mathbb{T}^d)$, we have $|B[v_1, v_2]| \le C_1 \|v_1\|_{H^1_{\mathcal{X}}(\mathbb{T}^d)} \|v_2\|_{H^1_{\mathcal{X}}(\mathbb{T}^d)}$ and furthermore, because $f > 0$ in $\mathbb{T}^d$, $B[v_1, v_1] \ge C_2 \|v_1\|^2_{H^1_{\mathcal{X}}(\mathbb{T}^d)}$.

Fix $h \in L^2(\mathbb{T}^d)$. Consider the functional, $\tilde{h}$, given by

$$\langle \tilde{h}, v \rangle := \int_{\mathbb{T}^d} h(x)v(x)\,\mathrm{d}x \quad \text{for} \quad v \in H^1_{\mathcal{X}}(\mathbb{T}^d).$$

Using Hölder's inequality, we obtain $|\langle \tilde{h}, v \rangle| \le \|h\|_{L^2(\mathbb{T}^d)} \|v\|_{L^2(\mathbb{T}^d)} \le C\|v\|_{H^1_{\mathcal{X}}(\mathbb{T}^d)}$. Therefore, by Lax-Milgram theorem, there exists $u \in H^1_{\mathcal{X}}(\mathbb{T}^d)$ such that

$$B[u,v] = \langle \tilde{h}, v \rangle \quad \text{for} \quad v \in H^1_{\mathcal{X}}(\mathbb{T}^d); \tag{13}$$

that is, u is a weak solution to $fu - \Delta_{\mathcal{X}} u = h$.

Next, we define a map $S\colon L^2(\mathbb{T}^d) \to H^1_{\mathcal{X}}(\mathbb{T}^d)$ as $Sh = (f - \Delta_{\mathcal{X}})^{-1}h = u$. For $h_1, h_2 \in L^2(\mathbb{T}^d)$, there exist $u_1, u_2 \in H^1_{\mathcal{X}}(\mathbb{T}^d)$ such that

$$fu_i - \Delta_{\mathcal{X}} u_i = h_i, \quad i = 1, 2.$$

Clearly S is linear. Moreover,

$$C_2\|u\|^2_{H^1_{\mathcal{X}}(\mathbb{T}^d)} \le |B[u,u]| = |\langle \tilde{h}, u \rangle| \le \|h\|_{L^2(\mathbb{T}^d)} \|u\|_{H^1_{\mathcal{X}}(\mathbb{T}^d)},$$

Therefore, $\|Sh\|_{H^1_{\mathcal{X}}(\mathbb{T}^d)} = \|u\|_{H^1_{\mathcal{X}}(\mathbb{T}^d)} \le \frac{1}{C_2}\|h\|_{L^2(\mathbb{T}^d)}$. Thus, S is bounded. By Proposition 2.8, $H^1_{\mathcal{X}}(\mathbb{T}^d)$ is compactly embedded in $L^2(\mathbb{T}^d)$. Therefore, S is a compact mapping from $L^2(\mathbb{T}^d)$ into itself.

Next, fix $h_1, h_2 \in L^2(\mathbb{T}^d)$ and let $u_1, u_2 \in H^1_{\mathcal{X}}(\mathbb{T}^d)$ be solutions to (13) corresponding to h_1 and h_2 respectively. Then, because of Assumption A2, we obtain $(\Delta_{\mathcal{X}})^* = \Delta_{\mathcal{X}}$ and $B[u,v] = B[v,u]$. Thus, we have

$$\langle h_1, Sh_2 \rangle = \langle h_1, u_2 \rangle = B[u_1, u_2], \ \langle Sh_1, h_2 \rangle = \langle h_2, u_1 \rangle = B[u_2, u_1],$$

from which we get the identity $\langle Sh_1, h_2 \rangle = \langle h_1, Sh_2 \rangle$. □

Remark 4.2. In Proposition 4.1, we can relax the condition of f such as only $f \in C^{0,\alpha}_{\mathcal{X}}(\mathbb{T}^d)$. In fact, there exists a constant, C, such that $f(x) + C > 0$ for all $x \in \mathbb{T}^d$. Thus, we can apply Proposition 4.1 to the following:

$$(f + C)w - \Delta_{\mathcal{X}} w = (\lambda + C)w \quad \text{in} \quad \mathbb{T}^d.$$

Proposition 4.3. *Assume that Assumptions* A1–A2 *hold and let* $L = (f - \Delta_{\mathcal{X}})$. *Then, each eigenvalue* λ *of* L *is real and has finite* geometric multiplicity*, that is, finite-dimensional eigenspace. Moreover, sorting these eigenvalues in increasing order, repeating each of them according to its finite geometric multiplicity, we obtain a sequence* $\{\lambda_i\}_{i=1}^{\infty}$ *such that*

$$-\infty < \lambda_1 \le \lambda_2 \le \lambda_3 \le \ldots,$$

with $\lambda_i \to \infty$ *as* $i \to \infty$. *Actually, there exists an orthonormal basis of* $L^2(\Omega)$ *consisting of eigenvectors of* L.

Proof. The statement follows from Theorem 1 in Section 6.5.1 of [15]. □

Definition 4.4. Let λ_1 be as in Proposition 4.3. Then, we call λ_1 the principal eigenvalue.

By a similar argument to the one in Section 6.5.1 in [15], we prove the following.

Proposition 4.5. *Assume that Assumptions* A1–A2 *hold and that* $f \in C^{0;\alpha}_{\mathcal{X}}(\mathbb{T}^d)$. *Then, the principal eigenvalue of* $-\Delta_{\mathcal{X}} + f$ *is simple and has a positive eigenfunction. In other words, there exists a unique pair* $(\lambda, w) \in \mathbb{R} \times H^1_{\mathcal{X}}(\mathbb{T}^d)$ *such that* $w > 0$ *and* (12) *holds (the PDE in* (12) *holds in weak sense).*

We recall that an eigenvalue of a linear operator is *simple* if the corresponding eigenspace is one-dimensional; that is, if any two corresponding eigenvectors are multiples of each other. Thus, the smallest eigenvalue of $-\Delta_{\mathcal{X}} + f$, that is, its *principal eigenvalue*, is simple and has a positive eigenfunction.

Proof. From Theorem 1 in Section 2 of [33], for $1 \le p < \infty$, $W^{k,p}_{\mathcal{X}}(\mathbb{T}^d)$ is separable. Furthermore, $L^2(\mathbb{T}^d)$ is separable. Therefore, by Proposition 4.1 and Theorem 7 in Appendix D of [15], there exists a countable orthonormal basis of $L^2(\mathbb{T}^d)$ consisting of eigenvectors of S. Since $\dim(L^2(\mathbb{T}^d)) = \infty$, the number of the eigenvalues $\{\mu_n\}$ is infinite. Fix $h \in L^2(\mathbb{T}^d)$, let u solve (13), and let $S = (f - \Delta_{\mathcal{X}})^{-1}$. From Remark 4.2, we can assume that f is positive in $\mathbb{T}^d$. Then, for some constant $C > 0$, we have

$$\langle Sh, h \rangle = \langle u, h \rangle = B[u, u] \ge C\|u\|^2_{H^1_{\mathcal{X}}(\mathbb{T}^d)} \ge 0,$$

from which we conclude that the real parts of $\{\mu_n\}$ are nonnegative. Then, applying Theorem 6 from Appendix D in [15], $\{\mu_n\}_{n=1}^{\infty}$ are real, positive, and converge to 0. Thus, we can assume that $\mu_i \ge \mu_{i+1}$ for all $i \in \mathbb{N}$. Therefore, the principal eigenvalue is $\lambda_1 = 1/\mu_1$. In weak sense, the corresponding eigenvector u_1 satisfies

$$fu_1 - \Delta_{\mathcal{X}} u_1 = \lambda_1 u_1.$$

Because $u_1 \not\equiv 0$, we can set $w_1 := u_1/\|u_1\|_{L^2(\mathbb{T}^d)}$. Then, w_1 satisfies

$$B[w_1, w_1] = \lambda_1 \|w_1\|^2_{L^2(\mathbb{T}^d)} = \lambda_1.$$

Finally, we want to show that w_1 is positive. Let c_1 and c_2 be

$$c_1 := \int_{\mathbb{T}^d} (w_1^+)^2 \,\mathrm{d}x, \text{ and } c_2 := \int_{\mathbb{T}^d} (w_1^-)^2 \,\mathrm{d}x.$$

Because $w_1^\pm \in H^1_{\mathcal{X}}(\mathbb{T}^d)$, we have $B[w_1^+, w_1^-] = 0$. Therefore, because $c_1 + c_2 = 1$, we obtain

$$\begin{aligned}\lambda_1 = B[w_1, w_1] &= B[w_1^+, w_1^+] + B[w_1^-, w_1^-] \\ &\ge \lambda_1 \|w_1^+\|^2_{L^2(\mathbb{T}^d)} + \lambda_1 \|w_1^-\|^2_{L^2(\mathbb{T}^d)} = (c_1 + c_2)\lambda_1 = \lambda_1.\end{aligned}$$

Consequently, the above inequality is an equality and, thus, we have

$$B[w_1^+, w_1^+] = \lambda_1 \|w_1^+\|^2_{L^2(\mathbb{T}^d)}, \quad B[w_1^-, w_1^-] = \lambda_1 \|w_1^-\|^2_{L^2(\mathbb{T}^d)}.$$

Since $f \in C^{0;\alpha}_{\mathcal{X}}(\mathbb{T}^d)$, by Proposition 1.1 in [2], the weak solution $w_1 \in H^1_{\mathcal{X}}(\mathbb{T}^d)$ is in $C^{2;\alpha}_{\mathcal{X}}(\mathbb{T}^d)$. Moreover, we have

$$f w_1^+ - \Delta_{\mathcal{X}} w_1^+ = \lambda_1 w_1^+ \geq 0.$$

Therefore, applying Bony's strong maximum principle (see Theorem 1.1 in [32]), we have either $w_1^+ > 0$ in $\mathbb{T}^d$ (and, hence, $w_1^- = 0$) or $-w_1^- < 0$ in $\mathbb{T}^d$ (and, hence, $w_1^+ = 0$). Let $w = w_1^+$ if $w_1^+ > 0$ or set $w = w_1^-$ if $w_1^- > 0$. Then, (λ_1, w) satisfies (12) and w is positive.

For uniqueness, assume that w and $\tilde{w}$ are two nontrivial positive weak solutions to (12). Then, since $\int_{\mathbb{T}^d} w \, \mathrm{d}x, \int_{\mathbb{T}^d} \tilde{w} \, \mathrm{d}x \neq 0$, there exists $C \in \mathbb{R}$ such that

$$\int_{\mathbb{T}^d} (w - C\tilde{w}) \, \mathrm{d}x = 0. \tag{14}$$

Because the first equation of (12) is linear, $w^* := (w - C\tilde{w})$ satisfies

$$f w^* - \Delta_{\mathcal{X}} w^* = \lambda_1 w^*.$$

Hence, by the same argument for w_1, w_1^+, and w_1^-, $(w - C\tilde{w})$ is positive, or negative, or vanishes identically. Consequently, due to (14), we have $w - C\tilde{w} \equiv 0$. Therefore, because $\|w\|^2_{L^2(\mathbb{T}^d)} = \|\tilde{w}\|^2_{L^2(\mathbb{T}^d)} = 1$ and both are positive, we have $w = \tilde{w}$, from which (λ_1, w) is unique. □

Corollary 4.6. *Suppose that Assumptions* A1–A2 *hold and that* $f \in C^{0;\alpha}_{\mathcal{X}}(\mathbb{T}^d)$. *Then, there exist a principal eigenvalue* $\lambda \in \mathbb{R}$ *and a positive eigenvector* $w \in H^1_{\mathcal{X}}(\mathbb{T}^d)$ *such that* (λ, w) *satisfies* (12) *in weak sense.*

Proof. Because $f \in C^{0;\alpha}_{\mathcal{X}}(\mathbb{T}^d)$, f is bounded from below. Thus, there exists a constant, $C > 0$, such that $f(x) + C > 0$. Then, we consider the following equation:

$$(f + C)w - \Delta_{\mathcal{X}} w = (\lambda + C)w, \tag{15}$$

which is equivalent to (11). Therefore, applying Proposition 4.5 to (15), we obtain the result. □

5. Mean-field games

Now, we revisit Problem 1.1. We consider two cases: V is a local operator, in Subsection 5.1 or V is a non-local operator, in Subsection 5.2. As before, we consider the eigenvalue problem with the non-linear term.

5.1. Local case

First, we consider the local case; that is, when Assumption A4 holds. To prove the existence and uniqueness of solutions to Problem 1.1, we use a variational method. By applying the Hopf-Cole transform, $u = 2 \log w$, to the first equation of (1) and letting $m = w^2$, we obtain the following equation:

$$wV(w^2) - \Delta_{\mathcal{X}} w = \lambda w. \tag{16}$$

Let
$$\mathcal{B} := \Big\{ v \in H^1_{\mathcal{X}}(\mathbb{T}^d)\colon \int_{\mathbb{T}^d} v^2 \,\mathrm{d}x = 1 \Big\},$$

and let G be as in Assumption A5. We consider the following variational problem:

$$\inf_{v\in\mathcal{B}} J(v) = \inf_{v\in\mathcal{B}} \int_{\mathbb{T}^d} \Big(\frac{1}{2}|D_{\mathcal{X}} v|^2 + G(v) \Big) \,\mathrm{d}x, \tag{17}$$

whose Euler-Lagrange equation is (16).

Proposition 5.1. *Suppose that Assumptions* A1–A2 *and* A4–A5 *hold. Then, there exists a minimizer,* $w \in \mathcal{B}$*, of* J*; that is,* $w \in \mathcal{B}$ *such that*

$$J(w) = \inf_{v\in\mathcal{B}} J(v).$$

Proof. First, we claim that, for any $v \in \mathcal{B}$, $J(v)$ is bounded from below. This claim follows from Assumption A5, because

$$\inf_{\mathcal{B}} J(v) \geq \int_{\mathbb{T}^d} G(v) \,\mathrm{d}x \geq -C \int_{\mathbb{T}^d} (|v|^2 + 1) \,\mathrm{d}x = -2C.$$

Let $\{v_n\}_{n=1}^\infty$ be a minimizing sequence in $\mathcal{B}$. Thus, $\{J(v_n)\}_{n=1}^\infty$ is bounded, and, therefore,

$$\int_{\mathbb{T}^d} \frac{1}{2}|D_{\mathcal{X}} v_n|^2 \,\mathrm{d}x - 2C \leq \int_{\mathbb{T}^d} \Big(\frac{1}{2}|D_{\mathcal{X}} v_n|^2 + G(v_n) \Big) \,\mathrm{d}x \leq \sup_n J(v_n) < \infty.$$

Because $\|v_n\|_{L^2(\mathbb{T}^d)} = 1$, $\{v_n\}_{n=1}^\infty$ is bounded in $H^1_{\mathcal{X}}(\mathbb{T}^d)$. Because $H^1_{\mathcal{X}}(\mathbb{T}^d)$ is reflexive, there exists a subsequence $\{v_{n_j}\}_{j=1}^\infty$ and $w \in H^1_{\mathcal{X}}(\mathbb{T}^d)$ such that

$$v_{n_j} \rightharpoonup w \ \text{ in } H^1_{\mathcal{X}}(\mathbb{T}^d).$$

By Proposition 2.8, extracting a subsequence if necessary, we have

$$v_{n_j} \to w \ \text{ in } L^2(\mathbb{T}^d) \ \text{ and } \int_{\mathbb{T}^d} w^2 \,\mathrm{d}x = 1, \tag{18}$$

from which we gather that $w \in \mathcal{B}$. In particular, extracting a subsequence again, $v_{n_j}(x)$ converges to $w(x)$ for a.e. $x \in \mathbb{T}^d$. Consequently,

$$\lim_{j\to\infty} G(v_{n_j}(x)) = G(w(x)) \ \text{ for a.e. } x \in \mathbb{T}^d.$$

Applying Proposition 2.8 again, $\big(G(v_n)+C(|v_n|^2+1)\big)$ is non-negative. Therefore, combining Fatou's lemma with (18), we have

$$\begin{aligned}\liminf_{j\to\infty} \int_{\mathbb{T}^d} \big(G(v_{n_j}) + C(|w|^2+1)\big) \,\mathrm{d}x &= \liminf_{j\to\infty} \int_{\mathbb{T}^d} \big(G(v_{n_j}) + C(|v_{n_j}|^2+1)\big) \,\mathrm{d}x \\ &\geq \int_{\mathbb{T}^d} \big(G(w) + C(|w|^2+1)\big) \,\mathrm{d}x.\end{aligned}$$

Consequently, $$\liminf_{j\to\infty} \int_{\mathbb{T}^d} G(v_{n_j})\,\mathrm{d}x \geq \int_{\mathbb{T}^d} G(w)\,\mathrm{d}x. \tag{19}$$

Therefore, because $v \mapsto \int_{\mathbb{T}^d} |D_{\mathcal{X}} v|^2\,\mathrm{d}x$ is lower semi-continuous, by (19), we have

$$J(w) \leq \liminf_{j\to\infty} J(v_{n_j}) = \inf_{v\in\mathcal{B}} J(v).$$ □

Corollary 5.2. *Suppose that Assumptions* A1–A2 *and* A4–A5 *hold and let w be a minimizer of J as in Proposition* 5.1. *Then, $|w|$ is also a minimizer of J.*

Proof. For $v \in H^1_{\mathcal{X}}(\mathbb{T}^d)$, we have $|v| \in H^1_{\mathcal{X}}(\mathbb{T}^d)$. If $v \in \mathcal{B}$, then $\||v|\|_{L^2(\mathbb{T}^d)} = \|v\|_{L^2(\mathbb{T}^d)} = 1$; that is, $|w| \in \mathcal{B}$. Moreover, by Lemma 3.2, we have

$$J(w) \geq J(|w|),$$

from which follows that $|w|$ is also a minimizer of (17). □

From Corollary 5.2, we can assume that the minimizer w of (17) is non-negative. Next, we verify that w is a weak solution to (16).

Theorem 5.3. *Assume that Assumptions* A1–A2 *and* A4–A5 *hold and let w be a non-negative minimizer of* (17). *Then, there exists $\lambda \in \mathbb{R}$ such that w satisfies* (16) *in weak sense.*

Proof. Fix $\varphi \in H^1_{\mathcal{X}}(\mathbb{T}^d)$ and let $\tilde{q} = 2^\sharp/(2\alpha+1)$. By Proposition 2.6, there exists $q < 2^\sharp$ such that $\frac{1}{q} + \frac{1}{\tilde{q}} = 1$. By Remark 3.4 and Hölder inequality, we have

$$\int_{\mathbb{T}^d} |wV[w^2]||\varphi|\,\mathrm{d}x \leq \|wV[w^2]\|_{L^{\tilde{q}}(\mathbb{T}^d)}\|\varphi\|_{L^q(\mathbb{T}^d)} < \infty.$$

Therefore, $wV[w^2]\varphi \in L^1(\mathbb{T}^d)$. Following the proof of Theorem 2 in Section 8.4.1 of [15], we verify that there exists $\lambda_1 \in \mathbb{R}$ such that, for all $\varphi \in H^1_{\mathcal{X}}(\mathbb{T}^d)$, we have

$$\int_{\mathbb{T}^d} (D_{\mathcal{X}} w \cdot D_{\mathcal{X}}\varphi + wV[w^2]\varphi)\,\mathrm{d}x = \lambda \int_{\mathbb{T}^d} w\varphi\,\mathrm{d}x.$$

Fix $\varphi \in H^1_{\mathcal{X}}(\mathbb{T}^d)$ and let w be a minimizer given by Proposition 5.1. Because $\|w\|_{L^2(\mathbb{T}^d)} = 1$, we have $w \not\equiv 0$ a.e. in $\mathbb{T}^d$. Then, we select $v \in H^1_{\mathcal{X}}(\mathbb{T}^d)$ with

$$\int_{\mathbb{T}^d} wv\,\mathrm{d}x \neq 0. \tag{20}$$

For $(\tau_1, \tau_2) \in \mathbb{R}^2$, let $\xi(\tau_1, \tau_2)$ be $\xi(\tau_1, \tau_2) := \|w + \tau_1\varphi + \tau_2 v\|^2_{L^2(\mathbb{T}^d)} - 1$.

Note that $\xi(0,0) = \|w\|^2_{L^2(\mathbb{T}^d)} - 1 = 0$. Since $\xi \in C^1(\mathbb{R}^d)$, we have

$$\frac{\partial\xi}{\partial\tau_1}(\tau_1, \tau_2) = \int_{\mathbb{T}^d} (w + \tau_1\varphi + \tau_2 v)\varphi\,\mathrm{d}x, \tag{21}$$

$$\frac{\partial\xi}{\partial\tau_2}(\tau_1, \tau_2) = \int_{\mathbb{T}^d} (w + \tau_1\varphi + \tau_2 v)v\,\mathrm{d}x. \tag{22}$$

According to (20), we have $\dfrac{\partial \xi}{\partial \tau_2}(0,0) \neq 0$.

Hence, by the implicit function theorem, there exist $\tau_0 > 0$ and a C^1 function $h\colon \mathbb{R} \to \mathbb{R}$ such that for all $|\tau| \leq \tau_0$,

$$h(0) = 0, \quad \xi(\tau, h(\tau)) = 0. \tag{23}$$

Hence,
$$\frac{\partial \xi}{\partial \tau_1}(\tau, h(\tau)) + \frac{\partial \xi}{\partial \tau_2}(\tau, h(\tau))h'(\tau) = 0.$$

Thus, from (21) and (22), we get

$$h'(0) = -\frac{\int_{\mathbb{T}^d} w\varphi \,\mathrm{d}x}{\int_{\mathbb{T}^d} wv \,\mathrm{d}x}. \tag{24}$$

Now, for $|\tau| \leq \tau_0$, we set $v(\tau) := \tau\varphi + h(\tau)v$ and define $i(\cdot) \in C^1([-\tau_0, \tau_0])$ as

$$i(\tau) := J[w + v(\tau)].$$

Because of (23), we obtain $\|w + v(\tau)\|^2_{L^2(\mathbb{T}^d)} = 1$ and $w + v(\tau) \in \mathcal{B}$. Therefore, $i(\cdot)$ has a minimum at 0; that is,

$$0 = i'(0) = \int\limits_{\mathbb{T}^d} \Big[D_{\mathcal{X}}w \cdot D_{\mathcal{X}}\varphi + wV[w^2]\varphi + h'(0)(D_{\mathcal{X}}w \cdot D_{\mathcal{X}}v + wV[w^2]v)\Big] \mathrm{d}x. \tag{25}$$

Let
$$\lambda = \frac{\int_{\mathbb{T}^d} D_{\mathcal{X}}w \cdot D_{\mathcal{X}}v + wV[w^2]v \,\mathrm{d}x}{\int_{\mathbb{T}^d} wv \,\mathrm{d}x}.$$

Applying (24) into (25), we have

$$\int_{\mathbb{T}^d} (D_{\mathcal{X}}w \cdot D_{\mathcal{X}}\varphi + wV[w^2]\varphi) \,\mathrm{d}x = \lambda \int_{\mathbb{T}^d} w\varphi \,\mathrm{d}x. \qquad \square$$

Proposition 5.4. *Assume that Assumptions* A1–A2 *and* A4–A6 *hold. Let $w \in \mathcal{B}$ be a weak solution to* (16). *Then, $w \in C^\infty(\mathbb{T}^d)$.*

Proof. From Remark 3.4, we have $wV[w^2] \in L^{2^\sharp/(2\beta+1)}(\mathbb{T}^d)$. Set $q_0 := 2^\sharp/(2\beta+1)$. In addition, w satisfies $-\Delta_{\mathcal{X}}w = \lambda w - wV(w^2) \in L^{q_0}(\mathbb{T}^d)$.

Thus, from Theorem 16 in [31], we have $w \in W^{2,q_0}_{\mathcal{X}}(\mathbb{T}^d)$. Applying Proposition 2.6, $w \in L^{q_0^\sharp}(\mathbb{T}^d)$, where $q_0^\sharp = \frac{dq_0}{d-2q_0/s}$. For $k \geq 0$, define q_k inductively by

$$q_{k+1} = \frac{q_k^\sharp}{2\beta+1} = \frac{dq_k}{(d-2q_k/s)(2\beta+1)}.$$

Then, because $\alpha < \frac{2^\sharp}{sd}$, we obtain $q_1 > q_0$, $wV[w^2] \in L^{q_1}(\mathbb{T}^d)$.

Also, w satisfies $-\Delta_{\mathcal{X}}w = \lambda w - wV(w^2) \in L^{q_1}(\mathbb{T}^d)$.

Because of Theorem 16 in [31] again, we get $w \in W^{2,q_1}_{\mathcal{X}}(\mathbb{T}^d)$. Using a bootstrapping argument for the regularity of w, there exists $q > sd/2$ such that $w \in W^{2,q}_{\mathcal{X}}(\mathbb{T}^d)$. In fact, when $\alpha < 2^\sharp/sd$, we have

$$q_1 = \frac{dq_0}{(d-2q_0/s)(2\beta+1)} > q_0; \quad \text{that is} \quad \frac{d}{(d-2q_0/s)(2\beta+1)} > 1.$$

Define $r(q) := d/((d-2q_0/s)(2\beta+1))$. Since $r(q)$ is increasing for $q < sd/2$, we get $r(q_1) > r(q_0) > 1$ and, iterating, q_k is increasing. Moreover, because $1 < r(q_0)$, there exits k^* such that

$$q_{k^*} = r(q_{k^*-1})q_{k^*-1} > r(q_0)q_{k^*-1} > r(q_0)^{k^*}q_0 > \frac{sd}{2}.$$

Therefore, $w \in W^{2,q_{k^*}}_{\mathcal{X}}(\mathbb{T}^d)$. By Proposition 2.7, we have $w \in W^{2/s,q_{k^*}}(\mathbb{T}^d)$. By the inclusion in (8) and according to Theorem 8.2 in [10], we get $w \in C^{0,\delta}(\mathbb{T}^d)$ for some $0 < \delta < 1$. When $w \in C^{0,\delta}(\mathbb{T}^d)$. Then, because w is bounded and V is smooth, $wV[w^2] \in C^{0,\delta}(\mathbb{T}^d)$. Hence, for $w \in C^{0,\delta}(\mathbb{T}^d)$, repeating Theorem 3.3 and 3.4 in [35], we have $w \in C^{2,\delta}_{\mathcal{X}}(\mathbb{T}^d)$. Then, we obtain $wV[w^2] \in C^{2,\delta}_{\mathcal{X}}(\mathbb{T}^d)$. According to Theorem 2.1 in [3], we get $w \in C^{4,\delta}_{\mathcal{X}}(\mathbb{T}^d)$. Consequently, applying the bootstrap argument again, we prove that, for any k, $w \in C^{k,\delta}_{\mathcal{X}}(\mathbb{T}^d)$. Thus, by Proposition 2.5, for any k, we have $w \in C^k(\mathbb{T}^d)$. □

Proof of Theorem 1.2. From Theorem 5.3, there exists $(\lambda, w) \in \mathbb{R} \times H^1_{\mathcal{X}}(\mathbb{T}^d)$ such that w is non-negative and (λ, w) is a weak solution to (16). By Proposition 5.4, $w \in C^\infty(\mathbb{T}^d)$. Therefore, applying the strong maximum principle, w is positive. Hence, if we set $u = 2\log w$ and $m = w^2$, then, $(\lambda, m, u) \in \mathbb{R} \times C^\infty(\mathbb{T}^d) \times C^\infty(\mathbb{T}^d)$ solves Problem 1.1. From the monotonicity in Assumption A3, (λ, m, u) is unique. □

5.2. Non-local case

Next, we consider the non-local case. In this case, we use the Schaefer's fixed-point theorem to obtain the existence and uniqueness of a solution to Problem 1.1.

Theorem 5.5. *Assume that the Assumptions* A1–A2 *and* A7 *hold. Then, the problem*

$$\begin{cases} -\Delta_{\mathcal{X}} w + \lambda w = wV[w^2] & \text{in } \mathbb{T}^d \\ \displaystyle\int_{\mathbb{T}^d} w^2\, dx = 1, \quad w \geq 0 \end{cases} \tag{26}$$

has a solution $(\lambda, w) \in \mathbb{R} \times H^2_{\mathcal{X}}(\mathbb{T}^d)$.

Proof. The proof relies on a fixed-point argument. We introduce the map

$$\Phi\colon H^1_{\mathcal{X}}(\mathbb{T}^d) \ni u \mapsto \Phi[u] := w \in H^1_{\mathcal{X}}(\mathbb{T}^d),$$

where w is the solution of the linear eigenvalue problem

$$\begin{cases} -\Delta_{\mathcal{X}} w + \lambda w = wV[u^2] \\ \displaystyle\int_{\mathbb{T}^d} w^2\, dx = 1, \quad w \geq 0 \end{cases}$$

for the principal (that is, the smallest) eigenvalue $-\lambda$ of the symmetric linear operator with compact resolvent $-\Delta_{\mathcal{X}} - V[u^2]$. From Section 4, the principal eigenvalue of such an operator is real, simple; that is, it has a one-dimensional eigenspace generated by an eigenfunction $w \geq 0$. In particular, that eigenfunction, w, is unique under the additional normalization conditions $\int_{\mathbb{T}^d} w^2\, dx = 1$. Thus, the map Φ is well-defined.

Next, we show that Φ is continuous and compact. Let $u_n \xrightarrow{n} u$ in $H^1_{\mathcal{X}}(\mathbb{T}^d)$. Let $w_n = \Phi[u_n]$. Let $(-\lambda_n, w_n)$ and $(-\lambda, w)$ be the principal eigenvalue and the eigenfunction of the operators $-\Delta_{\mathcal{X}} - V[u_n^2]$ and $-\Delta_{\mathcal{X}} - V[u^2]$, respectively.

Since, by Assumption A7, V is bounded, let $M \geq 0$ be such that

$$\|V[u^2]\|_\infty \leq M \tag{27}$$

for all $u \in H^1_{\mathcal{X}}(\mathbb{T}^d)$. For the test function $w_n \in H^1_{\mathcal{X}}(\mathbb{T}^d)$,

$$\int_{\mathbb{T}^d} |D_{\mathcal{X}} w_n|^2\, dx + \lambda_n \int_{\mathbb{T}^d} w_n^2\, dx = \int_{\mathbb{T}^d} w_n^2 V[u_n^2]\, dx\,. \tag{28}$$

To find a lower bound for λ_n, we use the characterization of the principal eigenvalue of a symmetric operator with compact resolvent as the minimizer of Rayleigh-Ritz quotient; that is,

$$-\lambda_n = \min_{\phi \in H^1_{\mathcal{X}}(\mathbb{T}^d), \phi \not\equiv 0} \frac{B[\phi, \phi]}{\|\phi\|^2_{L^2(\mathbb{T}^d)}}, \tag{29}$$

where $$B[\phi, \phi] = \int_{\mathbb{T}^d} \left(|D_{\mathcal{X}} \phi|^2 - V[u_n^2]\phi^2 \right) \mathrm{d}x.$$

Using the test function $\phi \equiv 1$ in (29), we conclude that

$$-\lambda_n \leq M \qquad \forall n \in \mathbb{N}.$$

About the above bound, by (27), and recalling that by the definition of Φ, $\int_{\mathbb{T}^d} w_n^2\, dx = 1$, we have $\lambda_n \leq M$. Thus, $\{\lambda_n\}_n \subset [-M, M]$.

By Theorem 16 in [31], we infer that $w_n \in H^2_{\mathcal{X}}(\mathbb{T}^d)$ for all $n \in \mathbb{N}$ and that there is some constant C depending only on the vector fields $\mathcal{X}$ such that

$$\|w_n\|_{H^2_{\mathcal{X}}(\mathbb{T}^d)} \leq C \left\| w_n V[u_n^2] - \lambda_n u_n \right\|_{L^2(\mathbb{T}^d)} \leq C \left(M + \sup_n \|u_n\|_{L^2(\mathbb{T}^d)} \right).$$

Thus, $\{w_n\}_n$ is bounded in $H^2_{\mathcal{X}}(\mathbb{T}^d)$. Rellich-Kondrachov's theorem (see [31] and [10]) implies that $H^2_{\mathcal{X}}(\mathbb{T}^d)$ is compactly embedded into $H^1_{\mathcal{X}}(\mathbb{T}^d)$. Additionally, the

Bolzano-Weierstrass theorem gives that $[-M, M]$ is compactly embedded into $\mathbb{R}$. Combining these two results, we infer that up to a subsequence, we have

$$\lambda_n \xrightarrow{n} \lambda \text{ in } \mathbb{R} \quad \text{ and } \quad w_n \xrightarrow{n} w \text{ in } H^1_{\mathcal{X}}(\mathbb{T}^d)$$

for some $\lambda \in \mathbb{R}$ and $w \in H^1_{\mathcal{X}}(\mathbb{T}^d)$. Taking also into account (10) in Assumption A7, for any test function φ, we can pass to the limit in

$$\int_{\mathbb{T}^d} D_{\mathcal{X}} w_n \cdot D_{\mathcal{X}} \varphi \, dx + \lambda_n \int_{\mathbb{T}^d} w_n \varphi \, dx = \int_{\mathbb{T}^d} w_n V[u_n^2] \varphi \, dx$$

as $n \to \infty$ and, therefore, find that

$$\int_{\mathbb{T}^d} D_{\mathcal{X}} w \cdot D_{\mathcal{X}} \varphi \, dx + \lambda \int_{\mathbb{T}^d} w \varphi \, dx = \int_{\mathbb{T}^d} w V[u^2] \varphi \, dx \, .$$

Because the prior identity holds for any arbitrary test function $\varphi \in H^1_{\mathcal{X}}(\mathbb{T}^d)$, this means that $(-\lambda, w)$ is an eigenvalue-eigenfunction pair of $-\Delta_{\mathcal{X}} - V[u^2]$. The eigenvalue $-\lambda$ is the principal eigenvalue because in the limit $w \geq 0$, while as the eigenfunctions corresponding to other eigenvalues must change sign due to orthogonality. Therefore, $\Phi[u] = w$. Consequently, Φ is continuous. Since $\Phi(H^1_{\mathcal{X}}(\mathbb{T}^d)) \subset H^2_{\mathcal{X}}(\mathbb{T}^d)$ and $H^2_{\mathcal{X}}(\mathbb{T}^d)$ is compactly embedded into $H^1_{\mathcal{X}}(\mathbb{T}^d)$, we conclude that Φ is also compact.

Lastly, we show that the set of fixed points of the maps $\{s\Phi\}_{0 \leq s \leq 1}$ is bounded in $H^1_{\mathcal{X}}(\mathbb{T}^d)$. For that, let $u \in H^1_{\mathcal{X}}(\mathbb{T}^d)$ be such that $u = s\Phi[u]$ for some $0 \leq s \leq 1$. This means that

$$\begin{cases} -\Delta_{\mathcal{X}} u + \lambda u = u V[u^2] \\ \displaystyle\int_{\mathbb{T}^d} u^2 \, dx = s^2, \qquad u \geq 0 \end{cases}$$

in weak sense for a suitable $\lambda \in \mathbb{R}$. If $s = 0$, $u = 0$. So, let $s > 0$. Since $u \geq 0$, $u \neq 0$, $-\lambda$ is actually the principal eigenvalue of $-\Delta_{\mathcal{X}} - V[u^2]$. Therefore, λ is estimated in the same way as above. More precisely, from (28) with $\|u\|^2_{L^2(\mathbb{T}^d)} = s^2$, we have

$$s^2 \lambda \leq \lambda \int_{\mathbb{T}^d} u^2 \, \mathrm{d}x \leq \int_{\mathbb{T}^d} u^2 V[u^2] \, \mathrm{d}x \leq s^2 M.$$

Thus, $\lambda \leq M$. For a bound from below, using the Rayleigh-Ritz quotient, we obtain $-\lambda \leq M$.

Hence, we find that $\lambda \in [-M, M]$. Next, by taking $u \in H^1_{\mathcal{X}}(\mathbb{T}^d)$ as test function we have the estimate

$$\int_{\mathbb{T}^d} |D_{\mathcal{X}} u|^2 \, dx \leq |\lambda| \int_{\mathbb{T}^d} u^2 \, dx + \int_{\mathbb{T}^d} u^2 |V[u^2]| \, dx \leq s^2(|\lambda| + M) \leq 2M \, .$$

Using $\int_{\mathbb{T}^d} u^2 \, dx = s^2 \leq 1$ again, we find that $\|u\|_{H^1_{\mathcal{X}}(\mathbb{T}^d)} \leq \sqrt{2M + 1}$.

As we see, this constant is independent of $u \in H^1_{\mathcal{X}}(\mathbb{T}^d)$ and s. Therefore, applying Schaefer's fixed-point theorem (see Theorem 4 in Section 9.2.2 in [15]), we conclude that Φ has a fixed point, w; that is, there is a solution $(\lambda, w) \in \mathbb{R} \times H^1_{\mathcal{X}}(\mathbb{T}^d)$ of (26) with $w \geq 0$. Because $wV[w^2] \in L^2(\mathbb{T}^d)$, by Theorem 16 in [31], we have $w \in H^2_{\mathcal{X}}(\mathbb{T}^d)$. □

Proof of Theorem 1.3. From Theorem 5.5, there exists $(\lambda, w) \in \mathbb{R} \times H^2_{\mathcal{X}}(\mathbb{T}^d)$ such that w is a weak solution to (26). Because of Assumption A8, for any $i = 1, ..., d$, we have

$$\frac{\partial}{\partial x_i}(wV[w^2]) = w_{x_i}V[w^2] + w(V[w^2])_{x_i} \in L^2(\mathbb{T}^d).$$

Consequently, $wV[w^2] \in H^1_{\mathcal{X}}(\mathbb{T}^d)$. Thus, by Theorem 16 in [31] again, we have $w \in H^3_{\mathcal{X}}(\mathbb{T}^d)$. Using a bootstrapping argument similar to the one in the proof of Proposition 5.4 and applying Theorem 8.2 in [10], we obtain $w \in C^2_{\mathcal{X}}(\mathbb{T}^d)$. Hence, by the strong maximum principle in [32], we conclude that w is positive. If we set $u = 2\log w \in C^2_{\mathcal{X}}(\mathbb{T}^d)$ and $m = w^2 \in C^2_{\mathcal{X}}(\mathbb{T}^d)$, then, $(\lambda, m, u) \in \mathbb{R} \times C^2_{\mathcal{X}}(\mathbb{T}^d) \times C^2_{\mathcal{X}}(\mathbb{T}^d)$ is a weak solution to Problem 1.1. By Assumption A3, (λ, m, u) is unique. □

5.3. Final remarks

Finally, applying Theorem 5.5, we show a different approach to prove the existence of weak solutions to (26) in the case of a local potential.

Theorem 5.6. *Assume that Assumptions* A1–A2 *and* A9 *hold. Then* (26) *has a solution* $(\lambda, w) \in \mathbb{R}\times]H^2_{\mathcal{X}}(\mathbb{T}^d)$. *Moreover,* $w \geq 0$.

Proof. We consider the following truncations W_N of W:

$$W_N := \max\{W, -N\} \qquad \forall N \in \mathbb{N},$$

and set $V_N[u^2](x) = W_N(x, u^2(x))$.

Clearly $-N \leq V_N \leq M$. V_N satisfy all the assumptions of Theorem 5.5, therefore for each $N \in \mathbb{N}$, there is a solution $(\lambda_N, w_N) \in \mathbb{R} \times H^2_{\mathcal{X}}(\mathbb{T}^d)$ with $w_N \geq 0$. Clearly $\lambda_N \leq M$ for all N. On the other hand, if using the test function $\phi = 1$ in (29) with λ_n replaced by λ_N, we find that

$$-\lambda_N \leq \int_{\mathbb{T}^d} -V_N[w_N^2]\, dx \leq -\int_{\mathbb{T}^d} V[w_N^2] \leq C \int_{\mathbb{T}^d} (|w_N|^{2r} + 1)\, dx.$$

From Assumption A9, $2r \leq 2$, applying Young's inequality and $\|w_N\|_{L^2(\mathbb{T}^d)} = 1$, we have

$$\int_{\mathbb{T}^d} |w_N|^{2r}\, dx \leq \int_{\mathbb{T}^d} |w_N|^2\, dx + C = 1 + C.$$

Using w_N as a test function in the definition of weak solutions with $\|w_N\|_{L^2(\mathbb{T}^d)} = 1$, we find

$$\int_{\mathbb{T}^d} |D_{\mathcal{X}} w_N|^2\, dx + \lambda_N = \int_{\mathbb{T}^d} w_N^2 V[w_N^2]\, ds \leq M.$$

Therefore, we obtain $$\int_{\mathbb{T}^d} |D_{\mathcal{X}} w_N|^2 \, dx \le -\lambda_N + M \le C,$$

from which we have $\|w_N\|_{H^1_{\mathcal{X}}(\mathbb{T}^d)} \le C$.

This in turn, by the L^2-regularity theorem in [31], implies that for some $C \ge 0$ depending only on $\mathcal{X}$ and M, we have $\|w_N\|_{H^2_{\mathcal{X}}(\mathbb{T}^d)} \le C$.

Let $q < 2^\sharp$ to be fixed in the sequel. Since $\{\lambda_N\}$ is bounded in $\mathbb{R}$ and $\{w_N\}$ is bounded in $H^2_{\mathcal{X}}(\mathbb{T}^d)$, it follows, by Rellich-Kondrachov theorem on the compact embedding of $H^2_{\mathcal{X}}(\mathbb{T}^d)$ into $L^q(\mathbb{T}^d)$ and into $H^1_{\mathcal{X}}(\mathbb{T}^d)$, that, up to a subsequence,

$$\lambda_N \to \lambda \in \mathbb{R} \tag{30}$$
$$w_N \rightharpoonup w \text{ (that is, weakly) in } H^2_{\mathcal{X}}(\mathbb{T}^d) \tag{31}$$
$$w_N \to w \text{ in } H^1_{\mathcal{X}}(\mathbb{T}^d) \tag{32}$$
$$w_N \to w \text{ in } L^q(\mathbb{T}^d) \tag{33}$$
$$\text{and} \quad w_N \to w \text{ a.e. in } \mathbb{T}^d \tag{34}$$

for some $\lambda \in \mathbb{R}$, $w \in H^1_{\mathcal{X}}(\mathbb{T}^d)$. Hence, (λ, w) solves (26). Clearly $\int_{\mathbb{T}^d} w_N^2 \, dx = 1$ and $w_N \ge 0$ are preserved in the limit: that is, $\int_{\mathbb{T}^d} w^2 \, dx = 1$ and $w \ge 0$. It is seen easily that

$$W_N(x, w_N^2(x)) = \max\{W(x, w_N^2(x)), -N\} \to W(x, w^2(x)) \quad \text{a.e. in } \mathbb{T}^d. \tag{35}$$

Since $(2r+1)2^\sharp/(2^\sharp - 1) < 2^\sharp$, it follows that there is $p > 2^\sharp/(2^\sharp - 1)$ such that we have $q := (2r+1)p < 2^\sharp$. In particular, we have:

(1.) $|w_N W_N(x, w_N(x)) - wW(x, w(x))|^p \le C\,(|w_N(x)|^q + |w(x)|^q) + C$ in $\mathbb{T}^d$,

(2.) $|w_N W_N(x, w_N(x)) - wW(x, w(x))|^p \to 0$ a.e. by (35),

(3.) (34), (4.) (33), and (5.) $w \in L^q(\mathbb{T}^d)$:

under these conditions it follows from a slight extension of Lebesgue's dominated convergence theorem: let $\{f_n\}$, $\{g_n\}$ be sequences of measurable functions on a measure space such that $f_n \to f$ and $g_n \to g$ a.e., for suitable (measurable) functions f, g; $|f_n| \le g_n$ a.e. for all n; and $\int g_n \to \int g < \infty$; then $\int |f_n - f| \to 0$. Therefore, we have

$$w_N W_N(\cdot, w_N) \to wW(\cdot, w) \text{ in } \mathrm{L}^p(\mathbb{T}^d).$$

Hence, since $p > (2^\sharp)' = 2^\sharp/(2^\sharp - 1)$ and thus $p' < 2^\sharp$ and, by the continuous embedding of $H^2_{\mathcal{X}}(\mathbb{T}^d)$ into $L^{p'}(\mathbb{T}^d)$, passing to the duals, $L^p(\mathbb{T}^d) \equiv \left(L^{p'}(\mathbb{T}^d)\right)^*$ is continuously embedded into $\left(H^2_{\mathcal{X}}(\mathbb{T}^d)\right)^*$, it follows that

$$w_N W_N(\cdot, w_N) \to wW(\cdot, w) \text{ in } \left(H^2_{\mathcal{X}}(\mathbb{T}^d)\right)^*. \tag{36}$$

Thus, for any test function $\varphi \in H^2_{\mathcal{X}}(\mathbb{T}^d)$, we can pass to the limit in the identity

$$\int_{\mathbb{T}^d} D_{\mathcal{X}} w_N \cdot D_{\mathcal{X}} \varphi \, dx + \lambda_N \int_{\mathbb{T}^d} w_N \varphi \, dx = \int_{\mathbb{T}^d} w_N W_N(x, w_N) \varphi \, dx$$

as $N \to \infty$ in view of (31), (32), (36), (30) and (33) and conclude that

$$\int_{\mathbb{T}^d} D_{\mathcal{X}} w \cdot D_{\mathcal{X}} \varphi \, dx + \lambda \int_{\mathbb{T}^d} w \varphi \, dx = \int_{\mathbb{T}^d} wW(x, w) \varphi \, dx.$$

Since this is true for any test function $\varphi \in H^2_{\mathcal{X}}(\mathbb{T}^d)$, we conclude that w is a weak solution of (26). □

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