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Approximation of the Integral Funnel of a Nonlinear Control System with Limited Control Resources

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Approximation of the integral funnel of the control system described by Urysohn type integral equation is studied. It is assumed that the control resource of the system is limited. The closed ball of the space L_p, $p > 1$, with radius r and centered at the origin is chosen as the set of admissible control functions. The integral funnel of the system is defined as the set of graphs of the system's trajectories generated by all admissible control functions. The approximation of the integral funnel by the set consisting of a finite number of points is presented and convergence of the approximation procedure is proved.

Keywords: Control system, integral constraint, Urysohn integral equation, integral funnel, approximation.

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1. Introduction

Integral funnel is one of the most important notions of the control systems and differential inclusions theories and is a generalization of the integral curve concept widely used in ordinary differential equations theory. The integral funnel is defined as a set of graphs of trajectories of the system generated by all admissible control functions. It can be redefined as the graph of the set valued map the values of which are the attainable sets of given dynamical system (see, e.g. [9], [11], [31] and references therein). Note that construction of the integral funnel

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allows to forecast the system's behaviour and design the trajectories with prescribed properties (see, e.g. [28]). It is known that integral funnel of a control system described by ordinary differential equation is a level set of the minimax (viscosity) solution of the appropriate Hamilton-Jacobi-Bellman-Isaacs equation, where the Hamiltonian of the system is convex (or concave) function with respect to the impulse variable (see, [11], [25], [34]).

Approximation of the integral funnel of the control system described by Urysohn type integral equation is important for various types of problems arising in theory and applications. Pointing out the importance of the integral equations, XX century scientist W. Heisenberg in his well known "Physics and Philosophy" writes: "The final equation of motion for matter will probably be some quantized nonlinear wave equation... This wave equation will probably be equivalent to rather complicated sets of integral equations..." (see, [13], p. 68). Nowadays the theory of integral equations is considered one of the origins of the contemporary functional analysis (see, [14], Chapter 1, p.2). It is known that the concept of the solution for initial and boundary value problems of different types differential equations can be reformulated via appropriate solutions notions for adequate integral equations (see, e.g. [7], [23]).

It should be noted that the integral models have some advantages over differential ones. For example the trajectories for such systems can be defined as continuous, even as p-integrable functions (see, e.g. [7], [16], [33]). The Urysohn-Hummerstein type equations are frequently studied among the nonlinear integral equations. Sometimes the processes described by integral equations have exterior influences called control actions which appear in the system's models as control functions. The control systems can be characterized by the types of constraints that the control functions must satisfy: control systems with geometric constraints on the control functions; control systems with integral constraints on the control functions; control systems with mixed type constraints on the control functions which include both the integral constraints and geometric constraints.

Geometric constraints on the control functions require that the values of the control functions belong to the given set and as it is seen, this kind of control efforts are not exhausted by consumption (see, e.g. [25], [26], [32], [36] and references therein). These types of control systems are also studied on the framework of the differential and integral inclusions theories (see, e.g. [2], [8], [30], [31]). But integral constraints on the control functions have different character than geometric constraints, since integral constraint does not guarantee the geometric boundedness. In general they describe the control efforts which are exhausted by consumption such as energy, fuel, finance, etc. (see, e.g. [6], [24], [29], [35]). For example, the mathematical model of the flying object with rapidly changing mass is described as control system with integral constraint on the control functions (see, e.g. [6], [24]). Note that by extending the system dimension, it is possible to rewrite the control system with integral constraint on the controls in the form of differential or integral inclusion with unbounded right hand side and with phase state constraint. Thus, the new system turns out to be more complex than original one. Therefore studying the considered system in its original form is more preferable.

The control systems described by ordinary differential equations with integral constraints on the control functions are considered in [12], [21], [22], [24], [35] (see also the references therein). In [19], [20] approximation of the set of trajectories of the control system described by Urysohn type integral equation is discussed, where the system is affine with respect to the control vector. Approximation of the sections of the set of trajectories is discussed in [17]. Controllability, existence of the optimal controls and dependence of the set of trajectories of the system's parameters of the control systems given by the Urysohn type integral equation are studied in [1], [4], [5], [15], [16], [18].

In this paper approximation of the integral funnel of the control systems described by the Urysohn type integral equation with integral constraint on the control functions are considered. The closed ball of the space L_p, $p > 1$, with radius r and centered at the origin is chosen as the set of admissible control functions. The trajectory of the system is defined as a continuous function which satisfies the system's equation everywhere. Approximation of the integral funnel of the system is given and convergence of the approximation procedure is proved. Note that in comparison with a section of the set of trajectories, the integral funnel includes more completed information about the system. The paper is organized as follows. In Section 2 the basic conditions and auxiliary propositions which are used in following arguments, are presented. In Section 3 the main result of the paper is proved (Theorem 3.1). The integral funnel is replaced by the set which consists of a finite number of sections of a finite number of trajectories. It is shown that in the appropriate specification of the discretization parameters, the Hausdorff distance between the system's integral funnel and a set consisting of a finite number of sections of a finite number of trajectories stands sufficiently small.

2. Preliminaries

Consider the control system described by a Urysohn type integral equation

$$x(\xi) = f(\xi, x(\xi)) + \lambda \int_a^b K(\xi, s, x(s), u(s))ds, \tag{1}$$

where $x(s) \in \mathbf{R}^n$ is the state vector of the system, $u(s) \in \mathbf{R}^m$ is the control vector, $\xi \in [a, b]$, $\lambda \geq 0$ is a real number. For given $p > 1$ and $r > 0$ we define

$$U_{p,r} = \left\{ u(\cdot) \in L_p([a, b]; \mathbf{R}^m) : \|u(\cdot)\|_p \leq r \right\},$$

where $\|u(\cdot)\|_p = \left(\int_a^b \|u(s)\|^p ds \right)^{\frac{1}{p}}$, $L_p([a, b]; \mathbf{R}^m)$ is the space of Lebesgue measurable functions $u(\cdot) : [a, b] \to \mathbf{R}^m$ such that $\|u(\cdot)\|_p < \infty$, $\|\cdot\|$ denotes the

Euclidean norm. The set $U_{p,r} \subset L_p\left([a,b];\mathbf{R}^m\right)$ is called the set of admissible control functions and a function $u(\cdot) \in U_{p,r}$ is said to be an admissible control function.

It is assumed that the functions

$$f(\cdot):[a,b]\times\mathbf{R}^n \to \mathbf{R}^n, \quad K(\cdot):[a,b]\times[a,b]\times\mathbf{R}^n\times\mathbf{R}^m \to \mathbf{R}^n$$

and the number $\lambda \in [0,\infty)$ given in equation (1) satisfy the following conditions:

(A1) The functions $f\colon [a,b]\times\mathbf{R}^n \to \mathbf{R}^n$ and $K\colon [a,b]\times[a,b]\times\mathbf{R}^n\times\mathbf{R}^m \to \mathbf{R}^n$ are continuous;

(A2) There exist $l_0 \in [0,1)$, $l_1 \geq 0$, $\alpha_1 \geq 0$, $l_2 \geq 0$, $\alpha_2 \geq 0$, $l_3 \geq 0$ and $\alpha_3 \geq 0$ such that $\|f(\xi,x_1)-f(\xi,x_2)\| \leq l_0\|x_1-x_2\|$ for every $(\xi,x_1) \in [a,b]\times\mathbf{R}^n$, $(\xi,x_2)\in[a,b]\times\mathbf{R}^n$ and

$$\left\|K(\xi_1,s,x_1,u_1)-K(\xi_2,s,x_2,u_2)\right\| \leq \left[l_1+\alpha_1\left(\|u_1\|+\|u_2\|\right)\right]|\xi_1-\xi_2| + \\ +\left[l_2+\alpha_2\left(\|u_1\|+\|u_2\|\right)\right]\|x_1-x_2\| + \left[l_3+\alpha_3\left(\|x_1\|+\|x_2\|\right)\right]\|u_1-u_2\|$$

for every (ξ_1,s,x_1,u_1) and $(\xi_2,s,x_2,u_2) \in [a,b]\times[a,b]\times\mathbf{R}^n\times\mathbf{R}^m$.

(A3) The inequality $0 \leq \lambda\left(l_2(b-a)+2\alpha_*(b-a)^{\frac{p-1}{p}}r\right) < 1-l_0$ is satisfied, where $\alpha_* = \max\{\alpha_1,\alpha_2,\alpha_3\}$.

Let $u(\cdot)\in U_{p,r}$. A continuous function $x(\cdot):[a,b]\to\mathbf{R}^n$ satisfying the equation (1) for every $\xi\in[a,b]$ is said to be a trajectory of the system (1) generated by the admissible control function $u(\cdot)\in U_{p,r}$.

The following propositions characterise the properties of the trajectories of the system (1).

Proposition 2.1. [15] *Every admissible control function $u(\cdot)\in U_{p,r}$ generates a unique trajectory of the system (1).*

We denote by $\mathbf{X}_{p,r}$ the set of all trajectories of the system (1) generated by all admissible control functions $u(\cdot)\in U_{p,r}$. The set $\mathbf{X}_{p,r}$ is called the set of trajectories of the system (1).

Proposition 2.2. [15] *The set of trajectories $\mathbf{X}_{p,r}$ is a bounded subset of the space $C([a,b];\mathbf{R}^n)$, i.e. there exists $\gamma_*>0$ such that*

$$\|x(\cdot)\|_C \leq \gamma_* \tag{2}$$

for every $x(\cdot)\in\mathbf{X}_{p,r}$, where $C([a,b];\mathbf{R}^n)$ is the space of continuous functions $x(\cdot):[a,b]\to\mathbf{R}^n$ with norm $\|x(\cdot)\|_C = \max\left\{\|x(\xi)\| : \xi\in[a,b]\right\}$.

For each fixed $\xi\in[a,b]$ we set

$$\mathbf{X}_{p,r}(\xi) = \{x(\xi)\in\mathbf{R}^n : x(\cdot)\in\mathbf{X}_{p,r}\} \tag{3}$$

and let

$$\mathcal{F}_{p,r} = \{(\xi,x(\xi))\in[a,b]\times\mathbf{R}^n : x(\cdot)\in\mathbf{X}_{p,r}\} \tag{4}$$

The set $\mathbf{X}_{p,r}(\xi)$ is called a *section* of the set of trajectories at the instant of ξ and the set $\mathcal{F}_{p,r}\subset[a,b]\times\mathbf{R}^n$ is called *integral funnel* of the system (1). It is obvious that the integral funnel $\mathcal{F}_{p,r}$ consists of graphs of all trajectories of the system (1). From Proposition 2.2 it follows that the integral funnel $\mathcal{F}_{p,r}$ is bounded.

Define
$$L(\lambda) = l_0 + \lambda\left(l_2\,(b-a) + 2\alpha_*(b-a)^{\frac{p-1}{p}} r\right),$$
and
$$g_* = \frac{\lambda\,(l_3 + 2\gamma_*\alpha_3)}{1 - L(\lambda)}\,, \tag{5}$$

where γ_* is defined by (2).

The validity of the following propositions follows from the conditions (A1)–(A3).

Proposition 2.3. *Let $x_1(\cdot) \in \mathbf{X}_{p,r}$ and $x_2(\cdot) \in \mathbf{X}_{p,r}$ be arbitrary trajectories of the system (1) generated by the admissible control functions $u_1(\cdot) \in U_{p,r}$ and $u_2(\cdot) \in U_{p,r}$ respectively. Then, for every $\xi \in [a,b]$,*

$$\|x_1(\xi) - x_2(\xi)\| \le g_* \int_a^b \|u_1(s) - u_2(s)\|\,ds.$$

We set $B_n(\gamma_*) = \{x \in \mathbf{R}^n : \|x\| \le \gamma_*\}\,, \quad B_n = \{x \in \mathbf{R}^n : \|x\| \le 1\}\,,$

$$B_C(1) = \{x(\cdot) \in C\left([a,b];\mathbf{R}^n\right) : \|x(\cdot)\|_C \le 1\}\,, \quad D_1 = [a,b] \times B_n(\gamma_*),$$
$$\omega_0(\Delta) = \max\left\{\,\|f(\xi_2,x) - f(\xi_1,x)\| : |\xi_2 - \xi_1| \le \Delta,\ (\xi_1,x) \in D_1,\ (\xi_2,x) \in D_1\right\},$$
$$\varphi\,(\Delta) = \frac{1}{1-L_0}\left\{\omega_0\,(\Delta) + \lambda\left[l_1(b-a) + 2\alpha_1\,(b-a)^{\frac{p-1}{p}}\,r\right]\Delta\right\}. \tag{6}$$

From (2) it follows that $\mathcal{F}_{p,r} \subset D_1$ and by virtue of condition (A1) we have that $\omega_0(\Delta) \to 0^+$, $\varphi(\Delta) \to 0^+$ as $\Delta \to 0^+$.

The Hausdorff distance between the sets $D \subset \mathbf{R}^n$ and $E \subset \mathbf{R}^n$ is denoted by $h_n(D,E)$, and the Hausdorff distance between the sets $G \subset C\left([a,b];\mathbf{R}^n\right)$ and $W \subset C\left([a,b];\mathbf{R}^n\right)$ is denoted by $h_C(G,W)$ (see, [3]).

Proposition 2.4. [15] *For every $x(\cdot) \in \mathbf{X}_{p,r}$, $\xi_1 \in [a,b]$, $\xi_2 \in [a,b]$ the inequality*

$$\|x(\xi_2) - x(\xi_1)\| \le \varphi\left(|\xi_2 - \xi_1|\right)$$

holds, and therefore $h_n\left(\mathbf{X}_{p,r}(\xi_2), \mathbf{X}_{p,r}(\xi_1)\right) \le \varphi\left(|\xi_2 - \xi_1|\right)$, where $\mathbf{X}_{p,r}(\xi_1)$ and $\mathbf{X}_{p,r}(\xi_2)$ are defined by (3), *$\varphi(\cdot)$ is defined by* (6).

Proposition 2.4 yields that the set valued map $\xi \to \mathbf{X}_{p,r}(\xi)$, $\xi \in [a,b]$, is a continuous one. Moreover, since the set of trajectories is a bounded subset of the space $C\left([a,b];\mathbf{R}^n\right)$, then Proposition 2.4 implies the validity of following proposition.

Proposition 2.5. [15] *The set of trajectories $\mathbf{X}_{p,r}$ is a precompact subset of the space $C\left([a,b];\mathbf{R}^n\right)$.*

3. Main Result: Approximation

Let $\beta \in (0,+\infty)$, $\Gamma = \{a = \xi_0, \xi_1, \dots, \xi_N = b\}$ be a uniform partition of the closed interval $[a,b]$, $\Lambda = \{0 = w_0, w_1, \dots, w_q = \beta\}$ be a uniform partition of the closed interval $[0,\beta]$, $\Delta = \xi_{i+1} - \xi_i = \frac{b-a}{N}$, $i = 0,1,\dots,N-1$ be the diameter of the partition Γ, $\delta = w_{j+1} - w_j = \frac{\beta}{q}$, $j = 0,1,\dots,q-1$ be the diameter of the partition Λ, $S = \{u \in \mathbf{R}^m : \|u\| = 1\}$ and for a given $\sigma > 0$ let $S_\sigma = \{s_l \in S : l = 1,2,\dots c_*\}$ be a finite σ-net of the compact set $S \subset \mathbf{R}^m$.

Now we define a new set of control functions by setting

$$U_{p,r}^{\beta,\Delta,\delta,\sigma} = \Big\{u(\cdot) \in L_p\left([a,b]\,;\mathbf{R}^m\right) : u(\xi) = w_{j_i} s_{l_i},\ \xi \in [\xi_i, \xi_{i+1}), \\ w_{j_i} \in \Lambda,\ \ s_{l_i} \in S_\sigma,\ i = 0,1,\dots,N-1,\ \Delta \cdot \sum_{i=0}^{N-1} w_{j_i}^p \le r^p \Big\}. \tag{7}$$

It is obvious that the set $U_{p,r}^{\beta,\Delta,\delta,\sigma}$ consists of a finite number of control functions. By $\mathbf{X}_{p,r}^{\beta,\Delta,\delta,\sigma}$ we denote the set of trajectories of the system (1) generated by all control functions $u(\cdot) \in U_{p,r}^{\beta,\Delta,\delta,\sigma}$ and let

$$\mathbf{X}_{p,r}^{\beta,\Delta,\delta,\sigma}(\xi) = \left\{x(\xi) \in \mathbf{R}^n : x(\cdot) \in \mathbf{X}_{p,r}^{\beta,\Delta,\delta,\sigma}\right\}. \tag{8}$$

Since the set $U_{p,r}^{\beta,\Delta,\delta,\sigma}$ consists of a finite number of control functions, then the set $\mathbf{X}_{p,r}^{\beta,\Delta,\delta,\sigma}$ consists of a finite number of trajectories and the set $\mathbf{X}_{p,r}^{\beta,\Delta,\delta,\sigma}(\xi)$ consists of a finite number of points of the space $\mathbf{R}^n$. Now we denote

$$\Phi_{p,r,\Gamma}^{\beta,\Delta,\delta,\sigma} = \bigcup_{i=0}^{N} \left(\xi_i, \mathbf{X}_{p,r}^{\beta,\Delta,\delta,\sigma}(\xi_i)\right), \quad \xi_i \in \Gamma. \tag{9}$$

Let us formulate the main result.

Theorem 3.1. *For each $\varepsilon > 0$ there exist $\beta(\varepsilon) > 0$, $\Delta_*(\varepsilon, \beta(\varepsilon)) > 0$, $\delta_*(\varepsilon) > 0$ and $\sigma_*(\varepsilon, \beta(\varepsilon)) > 0$ such that for every uniform partition Γ of the closed interval $[a,b]$, uniform partition Λ of the closed interval $[0, \beta(\varepsilon)]$ and σ-net S_σ, where $\Delta \le \Delta_*(\varepsilon, \beta(\varepsilon))$, $\delta \le \delta_*(\varepsilon)$, $\sigma \le \sigma_*(\varepsilon, \beta(\varepsilon))$, the inequality*

$$h_{n+1}\big(\mathcal{F}_{p,r}, \Phi_{p,r,\Gamma}^{\beta(\varepsilon),\Delta,\delta,\sigma}\big) \le \varepsilon$$

is satisfied. Here Δ is the diameter of the partition Γ, δ is the diameter of the partition Λ.

Proof. Let $\varepsilon > 0$ be fixed. The proof will be completed in 9 steps.

Step 1. For given $\beta > 0$, we set

$$U_{p,r}^{\beta} = \big\{u(\cdot) \in U_{p,r} : \|u(s)\| \le \beta \text{ for every } s \in [a,b]\big\},$$

and let $\mathbf{X}_{p,r}^{\beta}$ be the set of trajectories of the system (1) generated by the control functions $u(\cdot) \in U_{p,r}^{\beta}$. Let

$$\kappa_* = 2r^p g_*, \tag{10}$$

where g_* is defined by (5). We will prove that the following inequality holds:

$$h_C(\mathbf{X}_{p,r}, \mathbf{X}_{p,r}^{\beta}) \leq \frac{\kappa_*}{\beta^{p-1}}. \tag{11}$$

Let $x(\cdot) \in \mathbf{X}_{p,r}$ be an arbitrarily chosen trajectory generated by the admissible control function $u(\cdot) \in U_{p,r}$ and let the control function $u_*(\cdot) : [a,b] \to \mathbf{R}^m$ be defined as

$$u_*(s) = \begin{cases} u(s), & \text{if} \quad \|u(s)\| \leq \beta, \\ \beta \dfrac{u(s)}{\|u(s)\|}, & \text{if} \quad \|u(s)\| > \beta \end{cases} \tag{12}$$

where $s \in [a,b]$. It is obvious that $u_*(\cdot) \in U_{p,r}^{\beta}$. Let $x_*(\cdot)$ be the trajectory of the system (1) generated by the control function $u_*(\cdot)$. Setting

$$\Omega = \{s \in [a,b] : \|u(s)\| > \beta\}$$

we obtain from (12) and Proposition 2.3 that

$$\|x(\xi) - x_*(\xi)\| \leq g_* \int_{\Omega} \|u(s) - u_*(s)\| \, ds \tag{13}$$

for every $\xi \in [a,b]$, where $x_*(\cdot) \in \mathbf{X}_{p,r}^{\beta}$ and g_* is defined by (5).

Since $u(\cdot) \in U_{p,r}$, then Tchebyshev's inequality (see, [37], p.82) yields

$$\mu(\Omega) \leq \frac{r^p}{\beta^p} \ , \tag{14}$$

where $\mu(\Omega)$ denotes the Lebesgue measure of the set Ω. The inclusions $u(\cdot) \in U_{p,r}$, $u_*(\cdot) \in U_{p,r}^{\beta}$, Hölder's inequality and the relations (10), (13) and (14) imply that

$$\|x(\xi) - x_*(\xi)\| \leq 2r \left[\mu(\Omega)\right]^{\frac{p-1}{p}} g_* \leq \frac{\kappa_*}{\beta^{p-1}}$$

for every $\xi \in [a,b]$, and hence $\|x(\cdot) - x_*(\cdot)\|_C \leq \dfrac{\kappa_*}{\beta^{p-1}}$.

Taking into consideration that $x(\cdot) \in \mathbf{X}_{p,r}$ is arbitrarily chosen, we conclude that

$$\mathbf{X}_{p,r} \subset \mathbf{X}_{p,r}^{\beta} + \frac{\kappa_*}{\beta^{p-1}} B_C(1). \tag{15}$$

The inclusion $\mathbf{X}_{p,r}^{\beta} \subset \mathbf{X}_{p,r}$ and (15) complete the validity of the inequality (11).

Setting
$$\beta(\varepsilon) = \left[\frac{8k_*}{\varepsilon}\right]^{\frac{1}{p-1}} \tag{16}$$

we obtain from (11) that $h_C(\mathbf{X}_{p,r}, \mathbf{X}_{p,r}^{\beta(\varepsilon)}) \le \frac{\varepsilon}{8}$, and hence

$$h_n(\mathbf{X}_{p,r}(\xi), \mathbf{X}_{p,r}^{\beta(\varepsilon)}(\xi)) \le \frac{\varepsilon}{8} \tag{17}$$

for every $\xi \in [a, b]$.

Step 2. Let us introduce new set of control functions, setting

$$U_{p,r}^{\beta(\varepsilon),lip} = \left\{u(\cdot) \in U_{p,r}^{\beta(\varepsilon)} : u(\cdot) : [a, b] \to \mathbf{R}^m \text{ is Lipschitz continuous}\right\}$$

where $\beta(\varepsilon)$ is defined by (16).

By $\mathbf{X}_{p,r}^{\beta(\varepsilon),lip}$ we denote the set of trajectories of the system (1) generated by the control functions $u(\cdot) \in U_{p,r}^{\beta(\varepsilon),lip}$ and let

$$\mathbf{X}_{p,r}^{\beta(\varepsilon),lip}(\xi) = \left\{x(\xi) \in \mathbf{R}^n : x(\cdot) \in \mathbf{X}_{p,r}^{\beta(\varepsilon),lip}\right\}, \ \xi \in [a, b]. \tag{18}$$

In this step it will be proved that we have

$$h_C\left(\mathbf{X}_{p,r}^{\beta(\varepsilon)}, \mathbf{X}_{p,r}^{\beta(\varepsilon),lip}\right) = 0. \tag{19}$$

Let $\nu > 0$ be fixed. Now let $x(\cdot) \in \mathbf{X}_{p,r}^{\beta(\varepsilon)}$ be an arbitrarily chosen trajectory generated by the control function $u(\cdot) \in U_{p,r}^{\beta(\varepsilon)}$ and $\eta \in (0, 1)$. By $u_\eta(\cdot)$ we denote the Steklov function of $u(\cdot) \in U_{p,r}^{\beta(\varepsilon)}$, i.e.

$$u_\eta(s) = \frac{1}{2\eta}\int_{s-\eta}^{s+\eta} \widetilde{u}(\tau)\, d\tau\ , \quad s \in [a, b],$$

where
$$\widetilde{u}(\tau) = \begin{cases} u(\tau), & \tau \in [a, b] \\ 0, & \tau \in [a-1, a)\bigcup(b, b+1]. \end{cases}$$

The inclusion $u(\cdot) \in U_{p,r}^{\beta(\varepsilon)}$ yields that for each fixed $\eta \in (0, 1)$ the inequalities $\|u_\eta(\cdot)\|_p \le r$ and $\|u_\eta(s)\| \le \beta(\varepsilon)$ for every $s \in [a, b]$ are verified, the function $u_\eta(\cdot) : [a, b] \to \mathbf{R}^m$ is Lipschitz continuous with Lipschitz constant $\beta(\varepsilon)/\eta$ (see, e.g. [27]). Thus, we obtain that $u_\eta(\cdot) \in U_{p,r}^{\beta(\varepsilon),lip}$ for each fixed $\eta \in (0, 1)$. Moreover (see, e.g. [27]), $\lim_{\eta\to 0^+} \|u_\eta(\cdot) - u(\cdot)\|_p = 0$, and consequently for given $\nu/l_* > 0$ there exists $\eta_* \in (0, 1)$ such that

$$\|u_{\eta_*}(\cdot) - u(\cdot)\|_p \le \frac{\nu}{l_*}\ , \tag{20}$$

where $l_* = g_* \, (b-a)^{\frac{p-1}{p}}$, g_* is defined by (5). Let $u_*(\cdot) = u_{\eta_*}(\cdot)$ and let $x_*(\cdot)$ be the trajectory of the system (1) generated by the control function $u_*(\cdot)$. Thus, $x_*(\cdot) \in \mathbf{X}_{p,r}^{\beta(\varepsilon),lip}$ and Proposition 2.3, Hölder's inequality and (20) imply that

$$\begin{aligned} \|x(\xi) - x_*(\xi)\| &\leq g_* \int_a^b \|u(s) - u_*(s)\| \, ds \\ &\leq g_*(b-a)^{\frac{p-1}{p}} \|u(\cdot) - u_*(\cdot)\|_p \leq \nu \end{aligned}$$

for every $\xi \in [a,b]$, and consequently

$$\|x(\cdot) - x_*(\cdot)\|_C \leq \nu. \tag{21}$$

Since $x(\cdot) \in \mathbf{X}_{p,r}^{\beta(\varepsilon)}$ is an arbitrarily chosen trajectory, $x_*(\cdot) \in \mathbf{X}_{p,r}^{\beta(\varepsilon),lip}$, then inequality (21) implies that

$$\mathbf{X}_{p,r}^{\beta(\varepsilon)} \subset \mathbf{X}_{p,r}^{\beta(\varepsilon),lip} + \nu B_C(1). \tag{22}$$

Taking into consideration that $\mathbf{X}_{p,r}^{\beta(\varepsilon),lip} \subset \mathbf{X}_{p,r}^{\beta(\varepsilon)}$, we obtain from (22) that

$$h_C\left(\mathbf{X}_{p,r}^{\beta(\varepsilon)}, \mathbf{X}_{p,r}^{\beta(\varepsilon),lip}\right) \leq \nu. \tag{23}$$

Since $\nu > 0$ is an arbitrarily fixed, then from (23) we obtain the proof of the equality (19), and hence

$$h_n\left(\mathbf{X}_{p,r}^{\beta(\varepsilon)}(\xi), \mathbf{X}_{p,r}^{\beta(\varepsilon),lip}(\xi)\right) = 0 \tag{24}$$

for every $\xi \in [a,b]$.

Step 3. Now, we will define compact set of control functions and the set of trajectories generated by this set of control functions.

For given integer $M > 0$ we denote

$$\begin{aligned} U_{p,r}^{\beta(\varepsilon),lip,M} &= \{u(\cdot) \in U_{p,r}^{\beta(\varepsilon),lip} : u(\cdot) : [a,b] \to \mathbf{R}^m \text{ is Lipschitz continuous} \\ &\qquad \text{and Lipschitz constant is not greater than } M\}, \end{aligned}$$

and let $\mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M}$ be the set of trajectories of the system (1) generated by the control functions $u(\cdot) \in U_{p,r}^{\beta(\varepsilon),lip,M}$. For given $\xi \in [a,b]$ we set

$$\mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M}(\xi) = \left\{x(\xi) \in \mathbf{R}^n : x(\cdot) \in \mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M}\right\}. \tag{25}$$

It is not difficult to verify that for each fixed $M > 0$ the set of control functions $U_{p,r}^{\beta(\varepsilon),lip,M}$ and the set of trajectories $\mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M}$ are compact subsets of the space $C\big([a,b];\mathbf{R}^m\big)$ and $C\big([a,b];\mathbf{R}^n\big)$ respectively and

$$\mathbf{X}_{p,r}^{\beta(\varepsilon),lip} = \bigcup_{M=1}^{+\infty} \mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M}, \quad \mathbf{X}_{p,r}^{\beta(\varepsilon),lip}(\xi) = \bigcup_{M=1}^{+\infty} \mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M}(\xi) \tag{26}$$

for every $\xi \in [a,b]$. Proposition 2.2 together with the validity of the inclusion $\mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M}(\xi) \subset \mathbf{X}_{p,r}(\xi)$ for every $\xi \in [a,b]$ and $M = 1,2,\ldots$ yield that

$$\mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M}(\xi) \subset B_n(\gamma_*) \tag{27}$$

for every $\xi \in [a,b]$ and $M = 1,2,\ldots$. Since $\mathbf{X}_{p,r}^{\beta(\varepsilon),lip,\tilde{M}_1}(\xi) \subset \mathbf{X}_{p,r}^{\beta(\varepsilon),lip,\tilde{M}_2}(\xi)$ for every $\xi \in [a,b]$ and $\tilde{M}_1 < \tilde{M}_2$, then (26) and (27) implies that for each $\xi \in [a,b]$ the equality

$$\lim_{M\to\infty} \mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M}(\xi) = cl\Big(\mathbf{X}_{p,r}^{\beta(\varepsilon),lip}(\xi)\Big) \tag{28}$$

holds where cl denotes the closure of a set, the sets $\mathbf{X}_{p,r}^{\beta(\varepsilon),lip}(\xi)$ and $\mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M}(\xi)$ are defined by (18) and (25) respectively, $\lim_{M\to\infty} \mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M}(\xi)$ is the Kuratowski limit of the set sequence $\Big\{\mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M}(\xi)\Big\}_{M=1}^{\infty}$ (see, [3]). Taking into consideration (28) we have that for fixed $\xi \in [a,b]$ and given $\varepsilon > 0$ there exists $M_1(\varepsilon,\beta(\varepsilon),\xi) > 0$ such that for every $M \geq M_1(\varepsilon,\beta(\varepsilon),\xi)$ the following inequality is satisfied:

$$h_n\big(\mathbf{X}_{p,r}^{\beta(\varepsilon),lip}(\xi), \mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M}(\xi)\big) < \frac{\varepsilon}{8}. \tag{29}$$

Now let us show that the number $M_1(\varepsilon,\beta(\varepsilon),\xi)$ in inequality (29) can be chosen not depending on ξ. Assume contrary. Let there exists $\varepsilon_* > 0$, $M_i > 0$ and $\xi_i \in [a,b]$ such that $M_i \to +\infty$ as $i \to +\infty$ and the inequality

$$h_n\big(\mathbf{X}_{p,r}^{\beta(\varepsilon),lip}(\xi_i), \mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M_i}(\xi_i)\big) \geq \varepsilon_* \tag{30}$$

holds for every $i = 1,2,\ldots$

Since $\xi_i \in [a,b]$ for every $i = 1,2,\ldots$, then without loss of generality it is possible to assume that $\xi_i \to \xi_*$ as $i \to +\infty$ where $\xi_* \in [a,b]$. By virtue of (29) we have that for $\varepsilon_*/4$ there exists $N_1^* > 0$ such that

$$h_n\big(\mathbf{X}_{p,r}^{\beta(\varepsilon),lip}(\xi_*), \mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M_i}(\xi_*)\big) < \frac{\varepsilon_*}{4} \tag{31}$$

for every $i > N_1^*$. Since $\mathbf{X}_{p,r}^{\beta(\varepsilon),lip} \subset \mathbf{X}_{p,r}$ and $\mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M_i} \subset \mathbf{X}_{p,r}$ for every $i = 1,2,\ldots$, then analogously to the Proposition 2.4 it is not difficult to verify that for every $i = 1,2,\ldots$ the inequalities

$$h_n\left(\mathbf{X}_{p,r}^{\beta(\varepsilon),lip}(\xi_i), \mathbf{X}_{p,r}^{\beta(\varepsilon),lip}(\xi_*)\right) \leq \varphi\left(|\xi_i - \xi_*|\right) \tag{32}$$

and
$$h_n\left(\mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M_i}(\xi_i), \mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M_i}(\xi_*)\right) \leq \varphi\left(|\xi_i - \xi_*|\right) \tag{33}$$

are verified for every $i = 1,2,\ldots$ where $\varphi(\cdot)$ is defined by (6). Since $\xi_i \to \xi_*$ as $i \to +\infty$ and $\varphi(\Delta) \to 0^+$ as $\Delta \to 0^+$, then by virtue of (32) and (33) we obtain that for given $\varepsilon_*/4 > 0$ there exists $N_2^* > 0$ such that for every $i > N_2^*$ the inequalities

$$h_n\left(\mathbf{X}_{p,r}^{\beta(\varepsilon),lip}(\xi_i), \mathbf{X}_{p,r}^{\beta(\varepsilon),lip}(\xi_*)\right) \leq \frac{\varepsilon_*}{4} \tag{34}$$

and
$$h_n\left(\mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M_i}(\xi_i),\mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M_i}(\xi_*)\right)\leq\frac{\varepsilon_*}{4} \tag{35}$$

hold. Define $N_*=\max\left\{N_1^*,N_2^*\right\}$. Thus, (31), (34) and (35) imply that

$$\begin{aligned}&h_n\big(\mathbf{X}_{p,r}^{\beta(\varepsilon),lip}(\xi_i),\mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M_i}(\xi_i)\big)\leq h_n\big(\mathbf{X}_{p,r}^{\beta(\varepsilon),lip}(\xi_i),\mathbf{X}_{p,r}^{\beta(\varepsilon),lip}(\xi_*)\big)\\&+h_n\big(\mathbf{X}_{p,r}^{\beta(\varepsilon),lip}(\xi_*),\mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M_i}(\xi_*)\big)+h_n\big(\mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M_i}(\xi_*),\mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M_i}(\xi_i)\big)\\&<\frac{3\varepsilon_*}{4}<\varepsilon_*\end{aligned} \tag{36}$$

holds for every $i\geq N_*$. The inequalities (30) and (36) contradict which means that our assumption is not true, and hence for every $\varepsilon>0$ there exists $M_*(\varepsilon,\beta(\varepsilon))>0$ (not depending on ξ) such that for every $M\geq M_*(\varepsilon,\beta(\varepsilon))$ the inequality

$$h_n\big(\mathbf{X}_{p,r}^{\beta(\varepsilon),lip}(\xi),\mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M}(\xi)\big)<\frac{\varepsilon}{8}$$

is satisfied for every $\xi\in[a,b]$, and hence

$$h_n\big(\mathbf{X}_{p,r}^{\beta(\varepsilon),lip}(\xi),\mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M_*(\varepsilon,\beta(\varepsilon))}(\xi)\big)<\frac{\varepsilon}{8} \tag{37}$$

for every $\xi\in[a,b]$. Finally, (17), (24) and (37) yield that the inequality

$$h_n\big(\mathbf{X}_{p,r}(\xi),\mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M_*(\varepsilon,\beta(\varepsilon))}(\xi)\big)<\frac{\varepsilon}{4} \tag{38}$$

is verified for every $\xi\in[a,b]$ where the sets $\mathbf{X}_{p,r}(\xi)$ and $\mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M_*(\varepsilon,\beta(\varepsilon))}(\xi)$ are defined by (3) and (25) respectively.

Step 4. In this step we introduce the sets of control functions which consist of piecewise-constant control functions. Let

$$\begin{aligned}U_{p,r}^{\beta(\varepsilon),\Delta}&=\big\{u(\cdot)\in U_{p,r}^{\beta(\varepsilon)}:u(\xi)=u_i,\ \xi\in[\xi_i,\xi_{i+1}),\\&\qquad i=0,1,\ldots,N-1,\ u(b)=u_{N-1}\big\},\\V_{p,r}^{\beta(\varepsilon),\Delta,M}&=\big\{u(\cdot)\in U_{p,r}^{\beta(\varepsilon)}:u(\xi)=u_i,\ \xi\in[\xi_i,\xi_{i+1}),\ i=0,1,\ldots,N-1,\\&\qquad \|u_{i+1}-u_i\|\leq M\Delta,\ i=0,1,\ldots,N-2,\ u(b)=u_{N-1}\big\}\end{aligned}$$

where $\beta(\varepsilon)$ is defined by (16) and Δ is the diameter of the uniform partition $\Gamma=\{a=\xi_0,\xi_1,\ldots,\xi_N=b\}$ of the closed interval $[a,b]$.

By $\mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta}$ and $\mathbf{Z}_{p,r}^{\beta(\varepsilon),\Delta,M}$ we denote the set of trajectories of the system (1) generated by the control functions $u(\cdot)\in U_{p,r}^{\beta(\varepsilon),\Delta}$ and $v(\cdot)\in V_{p,r}^{\beta(\varepsilon),\Delta,M}$, respectively. For given $\xi\in[a,b]$ we define

$$\mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta}(\xi)=\left\{x(\xi)\in\mathbf{R}^n:x(\cdot)\in\mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta}\right\}, \tag{39}$$

$$\mathbf{Z}_{p,r}^{\beta(\varepsilon),\Delta,M}(\xi)=\left\{x(\xi)\in\mathbf{R}^n:x(\cdot)\in\mathbf{Z}_{p,r}^{\beta(\varepsilon),\Delta,M}\right\}. \tag{40}$$

It will be proved in the following that for every $M > 0$ and every uniform partition $\Gamma = \left\{a = \xi_0, \xi_1, \ldots, \xi_N = b\right\}$ of the interval $[a, b]$ the inclusion

$$\mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M} \subset \mathbf{Z}_{p,r}^{\beta(\varepsilon),\Delta,M} + g_* M(b-a)\Delta B_C(1) \tag{41}$$

holds. Choose an arbitrary $x(\cdot) \in \mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M}$ generated by the control function $u(\cdot) \in U_{p,r}^{\beta(\varepsilon),lip,M}$ and define new control function $v(\cdot) : [a, b] \to \mathbf{R}^m$, setting

$$v(\xi) = \frac{1}{\Delta} \int\limits_{\xi_i}^{\xi_{i+1}} u(s)ds, \quad \xi \in [\xi_i, \xi_{i+1}), \; i = 0, 1, \ldots, N-1, \quad v(\xi_N) = v(\xi_{N-1}). \tag{42}$$

It is obvious, that $\|v(\xi)\| \le \beta(\varepsilon)$ for every $\xi \in [a, b]$. From (42) and Hölder's inequality it follows that

$$\int_{\xi_i}^{\xi_{i+1}} \|v(s)\|^p \, ds \le \int_{\xi_i}^{\xi_{i+1}} \|u(s)\|^p \, ds,$$

for every $i = 0, 1, \ldots, N-1$, and hence

$$\int_a^b \|v(s)\|^p \, ds \le \int_a^b \|u(s)\|^p \, ds \le r^p$$

which means that $v(\cdot) \in U_{p,r}$. Since $\|v(\xi)\| \le \beta(\varepsilon)$ for every $\xi \in [a, b]$, then we have that $v(\cdot) \in U_{p,r}^{\beta(\varepsilon)}$.

Define $v(\xi_i) = v_i$ for $i = 0, 1, \ldots, N$. Then $v(\xi) = v_i$ for every $\xi \in [\xi_i, \xi_{i+1})$, $i = 0, 1, \ldots, N-1$, and $v(\xi_N) = v_N = v_{N-1}$. The inclusion $u(\cdot) \in U_{p,r}^{\beta(\varepsilon),lip,M}$ implies that $\|u(\xi_*) - u(\xi^*)\| \le M |\xi_* - \xi^*|$ for every $\xi_* \in [a, b]$ and $\xi^* \in [a, b]$. From (42) we have that the inequality

$$\begin{aligned} \|v_{i+1} - v_i\| &= \|v(\xi_{i+1}) - v(\xi_i)\| = \left\| \frac{1}{\Delta} \int_{\xi_{i+1}}^{\xi_{i+2}} u(s)ds - \frac{1}{\Delta} \int_{\xi_i}^{\xi_{i+1}} u(s)ds \right\| \\ &\le \frac{1}{\Delta} \int_{\xi_i}^{\xi_{i+1}} \|u(s+\Delta) - u(s)\| \, ds \le M\Delta \end{aligned} \tag{43}$$

is satisfied for each $i < N-1$. The inequality $\|v_N - v_{N-1}\| \le M\Delta$ is satisfied trivially. Since $v(\cdot) \in U_{p,r}^{\beta(\varepsilon)}$, then from (42) and (43) we conclude that $v(\cdot) \in V_{p,r}^{\beta(\varepsilon),\Delta,M}$. By $z(\cdot)$ we denote the trajectory of the system (1) generated by the control function $v(\cdot) \in V_{p,r}^{\beta(\varepsilon),\Delta,M}$. Then, $z(\cdot) \in \mathbf{Z}_{p,r}^{\beta(\varepsilon),\Delta,M} \subset \mathbf{X}_{p,r}$ and by virtue of Proposition 2.3 we obtain that

$$\|x(\xi) - z(\xi)\| \le g_* \int_a^b \|u(s) - v(s)\| \, ds \tag{44}$$

for every $\xi \in [a, b]$.

Let us choose an arbitrary $s \in [a,b)$. Then there exists $i = 0,1,\ldots,N-1$ such that $s \in [\xi_i, \xi_{i+1})$. Since the control function $u(\cdot)$ is Lipschitz continuous with Lipschitz constant M, then (42) yields that

$$\|u(s) - v(s)\| \leq \frac{1}{\Delta}\int_{\xi_i}^{\xi_{i+1}} \|u(s) - u(\tau)\|\, d\tau \leq \frac{1}{\Delta} M \int_{\xi_i}^{\xi_{i+1}} |s-\tau|\, d\tau \leq M\Delta. \tag{45}$$

Since $s \in [a,b)$ is arbitrarily chosen, (44) and (45) imply that

$$\|x(\xi) - z(\xi)\| \leq g_* M(b-a)\Delta \tag{46}$$

for every $\xi \in [a,b]$. From (46) it follows that for arbitrarily chosen $x(\cdot) \in \mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M}$ there exists $z(\cdot) \in \mathbf{Z}_{p,r}^{\beta(\varepsilon),\Delta,M}$ such that the inequality

$$\|x(\cdot) - z(\cdot)\|_C \leq g_* M(b-a)\Delta$$

holds, which completes the proof of validity of the inclusion (41). From inclusion (41) we obtain that

$$\mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M}(\xi) \subset \mathbf{Z}_{p,r}^{\beta(\varepsilon),\Delta,M}(\xi) + g_* M(b-a)\Delta B_n \tag{47}$$

for every for $M > 0$, uniform partition $\Gamma = \{a = \xi_0, \xi_1, \ldots, \xi_N = b\}$ of the interval $[a,b]$ and $\xi \in [a,b]$, where g_* is defined by (5), $\mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M}(\xi)$ and $\mathbf{Z}_{p,r}^{\beta(\varepsilon),\Delta,M}(\xi)$ are defined by (25) and (40) respectively, $\Delta = \xi_{i+1} - \xi_i$, $i = 0,1,\ldots,N-1$. In particular, from (47) it follows that

$$\mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M_*(\varepsilon,\beta(\varepsilon))}(\xi) \subset \mathbf{Z}_{p,r}^{\beta(\varepsilon),\Delta,M_*(\varepsilon,\beta(\varepsilon))}(\xi) + g_* M_*(\varepsilon,\beta(\varepsilon))(b-a)\Delta B_n, \tag{48}$$

where $M_*(\varepsilon,\beta(\varepsilon))$ is defined in (38).

Step 5. Now we will evaluate the Hausdorff distance between the sections of the sets of trajectories $\mathbf{X}_{p,r}$ and $\mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta}$. It is known that $\varphi(\delta) \to 0^+$ as $\Delta \to 0^+$ where $\varphi(\cdot)$ is defined by (6). Then for $\varepsilon > 0$ there exists $\Delta_1(\varepsilon) > 0$ such that for every $\Delta \in (0, \Delta_1(\varepsilon))$ the inequality

$$\varphi(\Delta) \leq \frac{\varepsilon}{10} \tag{49}$$

holds. Define

$$\Delta_*(\varepsilon,\beta(\varepsilon)) = \min\left\{\frac{\varepsilon}{8 g_* M_*(\varepsilon,\beta(\varepsilon))(b-a)},\ \Delta_1(\varepsilon),\ \frac{\varepsilon}{10}\right\}. \tag{50}$$

(48) implies that for every $\Delta > 0$ such that $\Delta \leq \Delta_*(\varepsilon,\beta(\varepsilon))$, the inclusion

$$\mathbf{X}_{p,r}^{\beta(\varepsilon),lip,M_*(\varepsilon,\beta(\varepsilon))}(\xi) \subset \mathbf{Z}_{p,r}^{\beta(\varepsilon),\Delta,M_*(\varepsilon,\beta(\varepsilon))}(\xi) + \frac{\varepsilon}{8} B_n \tag{51}$$

holds for every $\xi \in [a,b]$.

Since
$$\mathbf{Z}_{p,r}^{\beta(\varepsilon),\Delta,M_*(\varepsilon,\beta(\varepsilon))}(\xi) \subset \mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta}(\xi) \tag{52}$$
for every $\xi \in [a,b]$, then (38), (51) and (52) imply that if $\Delta \le \Delta_*(\varepsilon,\beta(\varepsilon))$, then
$$\mathbf{X}_{p,r}(\xi) \subset \mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta}(\xi) + \frac{3\varepsilon}{8} B_n \tag{53}$$
for every $\xi \in [a,b]$ where the set $\mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta}(\xi)$ is defined by (39). The inclusion $\mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta}(\xi) \subset \mathbf{X}_{p,r}(\xi)$, $\xi \in [a,b]$, and (53) imply that for every uniform partition $\Gamma = \{a = \xi_0, \xi_1, \ldots, \xi_N = b\}$ of the closed interval $[a,b]$ such that $\Delta \le \Delta_*(\varepsilon,\beta(\varepsilon))$, the inequality
$$h_n\big(\mathbf{X}_{p,r}(\xi), \mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta}(\xi)\big) \le \frac{3\varepsilon}{8} \tag{54}$$
is verified for every $\xi \in [a,b]$, where $\Delta = \xi_{i+1} - \xi_i$, $i = 0,1,\ldots,N-1$, is the diameter of the partition Γ, $\Delta_*(\varepsilon,\beta(\varepsilon))$ is defined by (50).

Step 6. In this step we reduce the set of control functions $U_{p,r}^{\beta(\varepsilon),\Delta}$ and introduce a new set of control functions.
For a given uniform partition $\Gamma = \{a = \xi_0, \xi_1, \ldots, \xi_N = b\}$ of the closed interval $[a,b]$ and uniform partition $\Lambda = \{0 = w_0, w_1, \ldots, w_q = \beta(\varepsilon)\}$ of the closed interval $[0,\beta(\varepsilon)]$ we define
$$U_{p,r}^{\beta(\varepsilon),\Delta,\delta} = \big\{u(\cdot) \in U_{p,r}^{\beta(\varepsilon)} : u(\xi) = u_i,\ \xi \in [\xi_i, \xi_{i+1}), \\ \|u_i\| \in \Lambda,\ i = 0,1,\ldots,N-1,\ u(b) = u(\xi_{N-1})\big\}.$$
where $\Delta = \xi_{i+1} - \xi_i$, $i = 0,1,\ldots,N-1$, is diameter of partition Γ, $\delta = w_{j+1} - w_j$, $j = 0,1,\ldots,q-1$, is diameter of the partition Λ.
By $\mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta,\delta}$ we denote the set of trajectories of the system (1), generated by the control functions $u(\cdot) \in U_{p,r}^{\beta(\varepsilon),\Delta,\delta}$ and for given $\xi \in [a,b]$ we set
$$\mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta,\delta}(\xi) = \big\{x(\xi) \in \mathbf{R}^n : x(\cdot) \in \mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta,\delta}\big\}. \tag{55}$$
Let us show that the inequality
$$h_C\left(\mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta}, \mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta,\delta}\right) \le g_*(b-a)\delta \tag{56}$$
is satisfied where g_* is defined by (5).

Let $x(\cdot) \in \mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta}$ be an arbitrarily chosen trajectory generated by the control function $u(\cdot) \in U_{p,r}^{\beta(\varepsilon),\Delta}$. The inclusion $u(\cdot) \in U_{p,r}^{\beta(\varepsilon),\Delta}$ implies that $u(\xi) = u_i$ for every $\xi \in [\xi_i, \xi_{i+1})$, $i = 0,1,\ldots,N-1$, $u(b) = u_{N-1}$, and moreover
$$\Delta \cdot \sum_{i=0}^{N-1} \|u_i\|^p \le r^p, \quad \|u_i\| \le \beta(\varepsilon) \ \text{ for every } i = 0,1,\ldots,N-1. \tag{57}$$

If $\|u_i\| < \beta(\varepsilon)$ $(i = 0, 1, \ldots, N-1)$, then there exists $w_{j_i} \in \Lambda$ such that

$$\|u_i\| \in [w_{j_i}, w_{j_i+1})\,. \tag{58}$$

Define a function $u_0(\cdot) : [a, b] \to \mathbf{R}^m$, setting

$$u_0(\xi) = \begin{cases} \dfrac{u_i}{\|u_i\|} w_{j_i}\ , & \text{if} \quad 0 < \|u_i\| < \beta(\varepsilon), \\ u_i\ , & \text{if} \quad \|u_i\| = 0 \text{ or } \|u_i\| = \beta(\varepsilon) \end{cases} \tag{59}$$

where $\xi \in [\xi_i, \xi_{i+1})$ for $i = 0, 1, \ldots, N-1$, $w_{j_i} \in \Lambda$ is defined by (58), and $u_0(b) = u_0(\xi_{N-1})$. (58) and (59) imply that $\|u_0(\xi)\| \le \|u(\xi)\|$ for every $\xi \in [a, b]$, and hence by virtue of (57) we have $u_0(\cdot) \in U_{p,r}^{\beta(\varepsilon),\Delta,\delta}$. Taking into consideration (58) and (59) we obtain that

$$\|u(\xi) - u_0(\xi)\| \le \delta \tag{60}$$

for every $\xi \in [a, b]$, where δ is the diameter of the partition Λ.

Now let us denote by $x_0(\cdot)$ the trajectory of the system (1) generated by the control function $u_0(\cdot)$. Then $x_0(\cdot) \in \mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta,\delta}$ and by virtue of (60) and Proposition 2.3 we have $\|x(\xi) - x_0(\xi)\| \le g_*(b-a)\delta$ for every $\xi \in [a, b]$, and consequently

$$\|x(\cdot) - x_0(\cdot)\|_C \le g_*(b-a)\delta. \tag{61}$$

The inequality (61) implies that

$$\mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta} \subset \mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta,\delta} + g_*(b-a)\delta B_C(1). \tag{62}$$

Since $\mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta,\Lambda} \subset \mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta}$, then from (62) we obtain the proof of the inequality (56), and hence

$$h_n\left(\mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta}(\xi), \mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta,\delta}(\xi)\right) \le g_*(b-a)\delta \tag{63}$$

for every $\xi \in [a, b]$. Let us set

$$\delta_*(\varepsilon) = \frac{\varepsilon}{8g_*(b-a)}. \tag{64}$$

(63) and (64) implies that for every $\Delta > 0$ and for every uniform partition Λ such that $\delta \le \delta_*(\varepsilon)$ the inequality

$$h_n\left(\mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta}(\xi), \mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta,\delta}(\xi)\right) \le \frac{\varepsilon}{8} \tag{65}$$

holds for every $\xi \in [a, b]$.

Step 7. According to (7), the set of control functions $U_{p,r}^{\beta(\varepsilon),\Delta,\delta,\sigma}$ can be redefined as

$$U_{p,r}^{\beta(\varepsilon),\Delta,\delta,\sigma} = \big\{u(\cdot) \in U_{p,r}^{\beta(\varepsilon),\Delta,\delta} : u(\xi) = w_{j_i} s_{l_i},\ \xi \in [\xi_i, \xi_{i+1}),$$
$$w_{j_i} \in \Lambda,\ s_{l_i} \in S_\sigma,\ i = 0, 1, \ldots, N-1,\ u(b) = u(\xi_{N-1})\big\}$$

where $\beta(\varepsilon)$ is defined by (16). Now we will show that for every uniform partition $\Gamma = \{a = \xi_0, \xi_1, \ldots, \xi_N = b\}$ of the interval $[a,b]$, uniform partition $\Lambda = \{0 = w_0, w_1, \ldots, w_q = \beta(\varepsilon)\}$ of the interval $[0, \beta(\varepsilon)]$ and σ-net S_σ, the inequality

$$h_C\left(\mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta,\delta}, \mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta,\delta,\sigma}\right) \leq g_* \beta(\varepsilon)(b-a)\sigma \tag{66}$$

holds where $\mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta,\delta,\sigma}$ is the set of trajectories of the system (1) generated by the control functions $u(\cdot) \in U_{p,r}^{\beta(\varepsilon),\Delta,\delta,\sigma}$.

Choose an arbitrary $x(\cdot) \in \mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta,\delta}$ generated by the control function $u(\cdot) \in U_{p,r}^{\beta(\varepsilon),\Delta,\delta}$. From the inclusion $u(\cdot) \in U_{p,r}^{\beta(\varepsilon),\Delta,\delta}$ it follows that $\|u(\xi)\| = w_{j_i}$ for every $\xi \in [\xi_i, \xi_{i+1})$, $i = 0, 1, \ldots, N-1$, $u(b) = u(\xi_{N-1})$ and $\Delta \cdot \sum_{i=0}^{N-1} w_{j_i}^p \leq r^p$, where $w_{j_i} \in \Lambda$, $\Delta = \xi_{i+1} - \xi_i$, $i = 0, 1, \ldots, N-1$, is the diameter of the partition Γ.

Since $\|u(\xi)\| = w_{j_i}$ for every $\xi \in [\xi_i, \xi_{i+1})$ and $u(\cdot)$ is constant on $[\xi_i, \xi_{i+1})$, then there exists $b_i \in S$ such that $u(\xi) = w_{j_i} b_i$ for every $\xi \in [\xi_i, \xi_{i+1})$, $i = 0, 1, \ldots, N-1$, where $u(b) = u(\xi_{N-1})$. The inclusion $b_i \in S = \{u \in \mathbf{R}^m : \|u\| = 1\}$ implies that there exists $s_{l_i} \in S_\sigma$ such that $\|b_i - s_{l_i}\| \leq \sigma$, $i = 0, 1, \ldots, N-1$. Define new control function $u_0(\cdot) : [a,b) \to \mathbf{R}^m$ setting

$$u_0(\xi) = w_{j_i} s_{l_i} \text{ for every } \xi \in [\xi_i, \xi_{i+1}),\ i = 0, 1, \ldots, N-1,\ u_0(b) = u_0(\xi_{N-1}).$$

It is not difficult to verify that $u_0(\cdot) \in U_{p,r}^{\beta(\varepsilon),\Delta,\delta,\sigma}$ and

$$\|u(\xi) - u_0(\xi)\| \leq \beta(\varepsilon)\sigma \tag{67}$$

hold for every $\xi \in [a,b]$. Let $x_0(\cdot)$ be the trajectory of the system (1) generated by the control function $u_0(\cdot)$. Then $x_0(\cdot) \in \mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta,\delta,\sigma}$ and according to the Proposition 2.3 and (67) we obtain

$$\|x(\xi) - x_0(\xi)\| \leq g_* \beta(\varepsilon)(b-a)\sigma$$

for every $\xi \in [a,b]$, and hence

$$\|x(\cdot) - x_0(\cdot)\|_C \leq g_* \beta(\varepsilon)(b-a)\sigma. \tag{68}$$

(68) implies that

$$\mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta,\delta} \subset \mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta,\delta,\sigma} + g_*(b-a)\beta(\varepsilon)\sigma B_C(1). \tag{69}$$

Since $\mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta,\delta,\sigma} \subset \mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta,\delta}$, then from (69) we obtain the proof of the validity of the inequality (66), and consequently

$$h_n\left(\mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta,\delta}(\xi), \mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta,\delta,\sigma}(\xi)\right) \leq g_* \beta(\varepsilon)(b-a)\sigma \tag{70}$$

for every $\xi \in [a,b]$, where g_* is defined by (5), $\mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta,\delta,\sigma}(\xi)$ and $\mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta,\delta}(\xi)$ are defined by (8) and (55) respectively.

Define
$$\sigma(\varepsilon,\beta(\varepsilon))=\frac{\varepsilon}{8g_*\beta(\varepsilon)(b-a)}. \tag{71}$$

In consequence of the formulas (70) and (71) it follows that for every uniform partition $\Gamma=\{a=\xi_0,\xi_1,\dots,\xi_N=b\}$ of the interval $[a,b]$, every uniform partition $\Lambda=\{0=w_0,w_1,\dots,w_q=\beta(\varepsilon)\}$ of the interval $[0,\beta(\varepsilon)]$, and every σ-net S_σ such that $\sigma\le\sigma_*(\varepsilon,\beta(\varepsilon))$ the inequality

$$h_n\left(\mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta,\delta}(\xi),\mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta,\delta,\sigma}(\xi)\right)\le\frac{\varepsilon}{8} \tag{72}$$

is satisfied for every $\xi\in[a,b]$ where Δ is diameter of the partition Γ, δ is diameter of the partition Λ.

Step 8. Let for given $\varepsilon>0$ the quantities $\beta(\varepsilon)>0$, $\Delta_*(\varepsilon,\beta(\varepsilon))>0$, $\delta_*(\varepsilon)>0$ and $\sigma_*(\varepsilon,\beta(\varepsilon))>0$ are defined by (16), (50), (64) and (71) respectively. If for uniform partition Γ of the closed interval $[a,b]$, for uniform partition Λ of the closed interval $[0,\beta(\varepsilon)]$ and for σ-net S_σ the inequalities $\Delta\le\Delta_*(\varepsilon,\beta(\varepsilon))$, $\delta\le\delta_*(\varepsilon)$, $\sigma\le\sigma_*(\varepsilon,\beta(\varepsilon))$ are satisfied, then from (54), (65) and (72) it follows that the inequality

$$h_n\big(\mathbf{X}_{p,r}(\xi),\mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta,\delta,\sigma}(\xi)\big)\le\frac{5\varepsilon}{8} \tag{73}$$

holds for every $\xi\in[a,b]$ where Δ is the diameter of the partition Γ, δ is the diameter of the partition Λ.

Step 9. Since $U_{p,r}^{\beta(\varepsilon),\Delta,\delta,\sigma}\subset U_{p,r}$, then we have that

$$\Phi_{p,r,\Gamma}^{\beta(\varepsilon),\Delta,\delta,\sigma}\subset\mathcal{F}_{p,r} \tag{74}$$

where the sets $\mathcal{F}_{p,r}$ and $\Phi_{p,r,\Gamma}^{\beta(\varepsilon),\Delta,\delta,\sigma}$ are defined by (4) and (9) respectively.

Now let $\Delta\le\Delta_*(\varepsilon,\beta(\varepsilon))$, $\delta\le\delta_*(\varepsilon)$, $\sigma\le\sigma_*(\varepsilon,\beta(\varepsilon))$ and let us choose an arbitrary $(\xi_*,x_*)\in\mathcal{F}_{p,r}$.

Let $\xi_*=\xi_i$ for some $i=0,1,\dots,N$. Then $x_*\in\mathbf{X}_{p,r}(\xi_*)=\mathbf{X}_{p,r}(\xi_i)$.

According to (73), there exists $y_*\in\mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta,\delta,\sigma}(\xi_i)$ such that $\|x_*-y_*\|\le 3\varepsilon/4$. Thus, $(\xi_*,y_*)=(\xi_i,y_*)\in\Phi_{p,r,\Gamma}^{\beta(\varepsilon),\Delta,\delta,\sigma}$, $\|(\xi_*,x_*)-(\xi_i,y_*)\|=\|x_*-y_*\|\le 3\varepsilon/4$ and hence

$$(\xi_*,x_*)\in\Phi_{p,r,\Gamma}^{\beta(\varepsilon),\Delta,\delta,\sigma}+\frac{3\varepsilon}{4}B_{n+1}. \tag{75}$$

Let $\xi_*\ne\xi_i$ for every $i=0,1,\dots,N$. Then there exists i_* such that $\xi_*\in(\xi_{i_*},\xi_{i_*+1})$ and hence $|\xi_*-\xi_{i_*}|<\Delta_*(\varepsilon,\beta(\varepsilon))$. Since $\Delta\le\Delta_*(\varepsilon,\beta(\varepsilon))$, then according to (49) and (50) we have

$$|\xi_*-\xi_{i_*}|\le\frac{\varepsilon}{10},\quad \varphi(\Delta)\le\frac{\varepsilon}{10}. \tag{76}$$

The inclusion $(\xi_*, x_*) \in \mathcal{F}_{p,r}$ yields that $x_* \in \mathbf{X}_{p,r}(\xi_*)$. According to the (49) and Proposition 2.4 we have that there exists $z_* \in \mathbf{X}_{p,r}(\xi_{i_*})$ such that

$$\|x_* - z_*\| \leq \frac{\varepsilon}{8}. \tag{77}$$

Since $z_* \in \mathbf{X}_{p,r}(\xi_{i_*})$, by virtue of (73) there exists $v_* \in \mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta,\delta,\sigma}(\xi_{i_*})$ such that

$$\|z_* - v_*\| \leq \frac{3\varepsilon}{4}. \tag{78}$$

The inclusion $v_* \in \mathbf{X}_{p,r}^{\beta(\varepsilon),\Delta,\delta,\sigma}(\xi_{i_*})$ implies that $(\xi_{i_*}, v_*) \in \Phi_{p,r,\Gamma}^{\beta(\varepsilon),\Delta,\delta,\sigma}$. From (76), (77) and (78) we conclude that

$$\|(\xi_*, x_*) - (\xi_{i_*}, v_*)\| \leq |\xi_* - \xi_{i_*}| + \|x_* - z_*\| + \|z_* - v_*| \leq \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{3\varepsilon}{4} = \varepsilon$$

and hence
$$(\xi_*, x_*) \in \Phi_{p,r,\Gamma}^{\beta(\varepsilon),\Delta,\delta,\sigma} + \varepsilon B_{n+1}. \tag{79}$$

Since $(\xi_*, x_*) \in \mathcal{F}_{p,r}$ is arbitrarily chosen, then from (75) and (79) we obtain that

$$\mathcal{F}_{p,r} \subset \Phi_{p,r,\Gamma}^{\beta(\varepsilon),\Delta,\delta,\sigma} + \varepsilon B_{n+1}. \tag{80}$$

Finally, (74) and (80) complete the proof of the theorem. □

Conclusion

In this paper an approximation method for construction of the integral funnel of the control system described by Urysohn type integral equation and with integral constraint on the control functions is presented. The set of control functions $U_{p,r}$ which is closed ball of the space $L_p([a,b];\mathbb{R}^m)$ centered at the origin with radius r is replaced by the set $U_{p,r}^{\beta,\Delta,\delta,\sigma}$ which consists of a finite number of piecewise constant control functions satisfying special type algebraic inequality. This inequality is an algebraic form of the integral constraint, the piecewise constant control function have to satisfy. Specifying a finite σ-net on the unit sphere $S = \{x \in \mathbb{R}^m : \|x\| = 1\}$ and aligning the control functions from the set $U_{p,r}^{\beta,\Delta,\delta,\sigma}$ (see, [10]) and using a numerical method for calculation of the appropriate Urysohn type integral equation, it is possible to construct the integral funnel $\Phi_{p,r,\Gamma}^{\beta,\Delta,\delta,\sigma}$ which consists of a finite number of points. An approximate construction of the integral funnel allows to solve various types of optimal control problems arising in different fields of theory and applications.

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