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On Problems of Minmax-Maxmin Type under Vector-Valued Criteria

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This paper is devoted to methods of solving problems of dynamic optimization under multivalued criteria. Such problems require a full description of the related Pareto boundary for the set of all values of the vector criteria and also an investigation of the dynamics of such set. Of special interest are problems for systems that also include a bounded disturbance in the system equation. Hence it appears useful to develop methods of calculating guaranteed estimates for possible realizations of related solution dynamics. Such estimates are included in this paper. Introduced here are the notions of vector values for minmax and maxmin with basic properties of such items. In the second part of this work there given are some sufficient conditions for the fulfilment of an analogy of classical scalar inequalities that involve relations between minmax and maxmin. An illustrative example is worked out for a linear-quadratic type of vector-valued optimization with bounded disturbance in the system equations.

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1. Essential definitions

1.1. Pareto order

Consider a mapping $F(x)\colon \mathbb{R}^n \to \mathbb{R}^p$.

Definition 1.1. A vector $x \in \mathbb{R}^p$ is said to be *dominated* by the vector $y \in \mathbb{R}^p$ in the sense of Pareto if $x \neq y$ and $y_i \leqslant x_i$ for $i = 1, \dots, p$. We will denote this relation as $y \leqslant x$.

Definition 1.2. The vector-valued minimum for the set of values of this mapping is defined as $\operatorname{Min} F(X) = \{f_* \in F(X) \mid \text{not } \exists\, x \in X\colon F(x) \leqslant f_*\}$.

Definition 1.3. The vector-valued maximum for this set is defined in the form $\operatorname{Max} F(X) = \{f^* \in F(X) \mid \text{not } \exists\, x \in X\colon f^* \leqslant F(x)\}$.

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1.2. The notion of vector-valued MinMax and MaxMin

Consider the mapping $F(u,v)\colon U\times V\to\mathbb{R}^p$.

We further assume that the set $F(U,V)$ satisfies certain conditions that ensure for this set the existence of both Pareto boundaries, namely, for minimum and for maximum. An example of such conditions are given for lower and upper bounds of $F(U,V)$ under the given ordering as

$$\exists M_*, M^*\colon\; M_*\leqslant F(u,v)\leqslant M^* \quad \forall u\in U, v\in V.$$

Besides this the set is supposed to be closed.

Definition 1.4. *The vector-valued minmax* for the elements of $F(u,v)$ over the set $U\times V$ will be taken as

$$\mathrm{Min}_u\mathrm{Max}_v F(u,v)=\mathrm{Min}\Big\{\bigcup_{u\in U}\mathrm{Max}\,F(u,V)\Big\}.$$

Each maximum inside the brackets is taken over values $v\in V$ with the value u fixed.

We need a similar definition for *vector-valued maxmin* as follows.

Definition 1.5. *The vector-valued maxmin* for the elements of $F(u,v)$ over $U\times V$ will be taken as the mapping

$$\mathrm{Max}_v\mathrm{Min}_u F(u,v)=\mathrm{Max}\Big\{\bigcup_{v\in V}\mathrm{Min}\,F(U,v)\Big\}.$$

Definition 1.6. We assume that arbitrary sets $A,B\subset\mathbb{R}^p$ satisfy the condition

$$A\leqslant B, \tag{1}$$

if the following property is true: $\forall\, b\in B\setminus A\Rightarrow\exists\, a\in A\colon a\leqslant b$.

Some trivial relations for the mappings above are as follows:

Proposition 1.7. *The next relations are true :*
(i) $\mathrm{Min}\,F(U,V)\leqslant\mathrm{Min}_u\mathrm{Max}_v F(u,v)$;
(ii) $\mathrm{Max}\,F(U,V)\geqslant\mathrm{Max}_v\mathrm{Min}_u F(u,v)$.

In addition to that, for validating the proofs of further propositions, we also indicate the correctness of the following relations:

Proposition 1.8. *The Pareto boundary satifies the next properties:*
(i) $\forall A\subset\mathbb{R}^p\Rightarrow\mathrm{Max}\,\{\mathrm{Min}\,A\}=\mathrm{Min}\,A,\quad \mathrm{Min}\,\{\mathrm{Max}\,A\}=\mathrm{Max}\,A$;
(ii) $\forall A\subseteq B\subset\mathbb{R}^p\Rightarrow\mathrm{Min}\,B\leqslant\mathrm{Min}\,A\leqslant\mathrm{Max}\,A\leqslant\mathrm{Max}\,B$;
(ii) *if* $\exists\,\mathrm{Min}\,(A+B)$, *then* $\mathrm{Min}\,(A+B)=\mathrm{Min}\,(A+\mathrm{Min}\,B)$.

Remark 1.9. The last equation is described in details in [1].

2. The linear-quadratic control problem under a vector-valued criterion

Consider a dynamic control system under disturbance of the type

$$\dot{x} = B(t)u + C(t)v, \quad t \in [t_0, \vartheta], \quad x_0 = x^0, \quad u(t) \in \mathcal{P}(t), \quad v(t) \in \mathcal{R}(t),$$

with the vector criterion

$$\vec{\mathcal{J}}(\vartheta, x, u, v) = \begin{bmatrix} \mathcal{J}_1(\vartheta, x, u, v) \\ \vdots \\ \mathcal{J}_p(\vartheta, x, u, v) \end{bmatrix} = \begin{bmatrix} \langle x, N_1 x\rangle + \langle u, M_1 u\rangle - \langle v, P_1 v\rangle \\ \vdots \\ \langle x, N_r x\rangle + \langle u, M_r u\rangle - \langle v, P_r v\rangle \end{bmatrix},$$

where $\langle a, b\rangle$ denotes as usual the scalar product of the elements a and b, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $v(t) \in \mathbb{R}^p$, $N_i \in \mathbb{R}^{n\times n}$, $M_i \in \mathbb{R}^{m\times m}$, and $P_i \in \mathbb{R}^{p\times p}$.

We will now investigate the properties of minmax and maxmin solutions of this problem. Using the Cauchy formula for the given equation of our system dynamics, we get:

$$x(t) = x^0 + \int_{t_0}^{t} B(\tau)u(\tau)d\tau + \int_{t_0}^{t} C(\tau)v(\tau)d\tau.$$

The components of the vector functional $\mathcal{J}[\vartheta]$ are

$$\begin{aligned} \mathcal{J}_i(\vartheta, x, u, v) &= \langle x, N_i x\rangle + \langle u, M_i u\rangle - \langle v, P_i v\rangle = \\ &= \left\langle \int_{t_0}^{\vartheta} B(\tau)u(\tau)d\tau, (N_i + N_i') \int_{t_0}^{\vartheta} C(\tau)v(\tau)d\tau \right\rangle + \\ &+ \left\langle x^0, (N_i + N_i') \int_{t_0}^{\vartheta} B(\tau)u(\tau)d\tau \right\rangle + \left\langle x^0, (N_i + N_i') \int_{t_0}^{\vartheta} C(\tau)v(\tau)d\tau \right\rangle + \\ &+ \left\langle \int_{t_0}^{\vartheta} B(\tau)u(\tau)d\tau, (N_i + N_i') \int_{t_0}^{\vartheta} B(\tau)u(\tau)d\tau \right\rangle + \\ &+ \left\langle \int_{t_0}^{\vartheta} C(\tau)v(\tau)d\tau, (N_i + N_i') \int_{t_0}^{\vartheta} C(\tau)v(\tau)d\tau \right\rangle + \\ &+ \langle u, M_i u\rangle - \langle v, P_i v\rangle + \left\langle x^0, (N_i + N_i')x^0 \right\rangle. \end{aligned}$$

We further regroup these components in such a way that each component of the vector functional can be represented in the form of a sum of the type

$$\mathcal{J}_i = F_i(u, v) + \varphi_i(u) + \psi_i(v),$$

where $F_i(\tilde{u}, \tilde{v}) = \langle \tilde{u}, A_i \tilde{v}\rangle$, and

$$\tilde{u} \in \left\{ \int_{t_0}^{\vartheta} B(\tau)u(\tau)d\tau \mid u(\tau) \in \mathcal{P}(\tau) \right\}, \quad \tilde{v} \in \left\{ \int_{t_0}^{\vartheta} C(\tau)v(\tau)d\tau \mid v(\tau) \in \mathcal{R}(\tau) \right\}.$$

In vector form the last relation reads

$$\vec{\mathcal{J}}[\vartheta] = \vec{F}(u, v) + \vec{\Phi}(u) + \vec{\Psi}(v).$$

We now give a separate description for the behaviour of the minmax and maxmin for each term of the last relation.

3. The functional of type $\vec{\Phi}(u) + \vec{\Psi}(v)$

Consider functional $\vec{S}(u,v) = \vec{\Phi}(u) + \vec{\Psi}(v)$.

Moving along a chain of inequalities, we have for the minmax:

$$\operatorname{Min}_u \operatorname{Max}_v \vec{S}(u,v) = \operatorname{Min}_u \operatorname{Max}_v \{\vec{\Phi}(u) + \vec{\Psi}(v)\} =$$
$$= \operatorname{Min} \left\{ \operatorname{Max} \left\{ \vec{\Phi}(\tilde{u}) + \vec{\Psi}(v) | v \in V \right\} | \tilde{u} \in U \right\} =$$
$$= \operatorname{Min} \left\{ \vec{\Phi}(\tilde{u}) + \operatorname{Max} \vec{\Psi}(V) | \tilde{u} \in U \right\} = \operatorname{Min} \left\{ \operatorname{Min} \vec{\Phi}(U) + \operatorname{Max} \vec{\Psi}(V) \right\}.$$

And similarly for the maxmin:

$$\operatorname{Max}_v \operatorname{Min}_u \vec{S}(u,v) = \operatorname{Max}_v \operatorname{Min}_u \{\vec{\Phi}(u) + \vec{\Psi}(v)\} =$$
$$= \operatorname{Max} \left\{ \operatorname{Min} \left\{ \vec{\Phi}(u) + \vec{\Psi}(\tilde{v}) | u \in U \right\} | \tilde{v} \in V \right\} =$$
$$= \operatorname{Max} \left\{ \operatorname{Min} \vec{\Phi}(U) + \vec{\Psi}(\tilde{v}) | \tilde{v} \in V \right\} = \operatorname{Max} \left\{ \operatorname{Min} \vec{\Phi}(U) + \operatorname{Max} \vec{\Psi}(V) \right\}.$$

Since a randomly selected nonempty set A satisfies relation

$$\operatorname{Min} A \leqslant \operatorname{Max} A,$$

if both $\operatorname{Min} A$ and $\operatorname{Max} A$ exist, we apply this relation to $\vec{S}(u,v) = \vec{\Phi}(u) + \vec{\Psi}(v)$, which yields the inequality

$$\operatorname{Min}_u \operatorname{Max}_v \vec{S}(u,v) \leqslant \operatorname{Max}_v \operatorname{Min}_u \vec{S}(u,v). \tag{2}$$

4. The functional of type $F(u,v) = [\langle u, A_1 v \rangle, \dots, \langle u, A_r v \rangle]'$

4.1. Case $v \in \mathbb{R}^1$

Consider a functional of the type

$$F(u,v) = \begin{bmatrix} \langle u, Av \rangle \\ \langle u, Bv \rangle \end{bmatrix} = \begin{bmatrix} \langle A'u, v \rangle \\ \langle B'u, v \rangle \end{bmatrix},$$

where $v \in V \subseteq \mathbb{R}^1, u \in U \subseteq \mathbb{R}^m$, $\langle a, b \rangle$ is the scalar product of a and b, and $A, B \in \mathbb{R}^m$. Given a scalar v, we introduce an alternative representation for this functional of the form

$$F(u,v) = v \begin{bmatrix} A'u \\ B'u \end{bmatrix}.$$

Then $F(U,v) = v \begin{bmatrix} A' \\ B' \end{bmatrix} U = v\tilde{U}$, where $\tilde{U} = \begin{bmatrix} A' \\ B' \end{bmatrix} U.$

Assume $v_{max} = \max v, v_{min} = \min(v)$, then

$$\operatorname{Min}_u \operatorname{Max}_v v\tilde{U} = \operatorname{Min}_u \left\{ v_{max} \cdot u | u \in \tilde{U} \right\} = v_{max} \cdot \operatorname{Min} \tilde{U},$$

and $$\operatorname{Max}_v \operatorname{Min}_u v\tilde{U} = \operatorname{Max}_v \left\{ v \cdot \operatorname{Min} \tilde{U} \right\} = v_{max} \cdot \operatorname{Min} \tilde{U},$$

which means $$\operatorname{Min}_u \operatorname{Max}_v v\tilde{U} = \operatorname{Max}_v \operatorname{Min}_u v\tilde{U}.$$

For describing the cases of higher dimensions we will need some additional propositions.

4.2. A necessary condition for the violation (2) of the minmax inequality

Consider functional $F(u,v)\colon U\times V\to\mathbb{R}^r$, where $U\subseteq\mathbb{R}^m$, $V\subseteq\mathbb{R}^p$.

The main minmax inequality for the Pareto ordering will be presented as

$$\mathrm{Max}_v\mathrm{Min}_u F(u,v)\leqslant \mathrm{Min}_u\mathrm{Max}_v F(u,v), \tag{3}$$

where the order $\leqslant$ is understood to be in the sense of (1).

Proposition 4.1. *Suppose that there exist $f^*=F(u^*,v^*)\in\mathrm{Min}_u\mathrm{Max}_v F$ and $f_*=F(u_*,v_*)\in\mathrm{Max}_v\mathrm{Min}_u F$ such that $f^*\leqslant f_*$. Then $\hat f=F(u^*,v_*)$ will be such that $\hat f\ngtrless f^*$ and $\hat f\ngtrless f_*$.*

Proof. The proof works by indicating that other cases are impossible.

Since $f^*\in\mathrm{Min}_u\mathrm{Max}_v F$, we have $f^*\in\mathrm{Max}\,F(u^*,V)$, which means $f^*\nleqslant\hat f$.

Similarly $f_*\in\mathrm{Min}\,F(U,v_*)$, which means $\hat f\nleqslant f_*$.

(a) Suppose $\hat f\leqslant f^*$. Then $\hat f\leqslant f^*\leqslant f_*$, and $\hat f\nleqslant f_*$, which leads to a contradiction.

(b) A similar reasoning is true for cases $f_*=\hat f$, $f_*\leqslant\hat f$ and $f_*=\hat f$.

(c) Now suppose $\hat f\ngtrless f^*$. Then if $f_*\leqslant\hat f$ and $f_*=\hat f$, we come to a contradiction.

(d) Case $\hat f\ngtrless f_*$ is treated similarly to the previous one. Hence condition $f^*\leqslant f_*$ automatically yields $\hat f\ngtrless f^*$ and $\hat f\ngtrless f_*$. □

Corollary 4.2. *Suppose $\vec F=(F_1,\dots,F_r)'$ and $\exists f^*=F(u^*,v*)\in\mathrm{Min}_u\mathrm{Max}_v F$, $f_*=F(u_*,v_*)\in\mathrm{Max}_u\mathrm{Min}_v F\colon f^*\leqslant f_*$. Then $\exists\, i\neq j,\ k\neq l;\ i,j,k,l=1,\dots,r:$*

$$\begin{cases}(F_i(u^*,v^*)-F_i(u^*,v_*))\,(F_j(u^*,v^*)-F_j(u^*,v_*))<0,\\ (F_k(u_*,v_*)-F_k(u^*,v_*))\,(F_l(u_*,v_*)-F_l(u^*,v_*))<0.\end{cases}$$

4.3. Case $\min\{\dim(u),\dim(v)\}\geqslant 2$

Suppose the criterion $\vec F=(F_1,\dots,F_r)'$ is given by the relations $F_i=\langle u,A_iv\rangle$, where $u\in U\subseteq\mathbb{R}^m$, $v\in V\subseteq\mathbb{R}^p$ and $A_i\in\mathbb{R}^{m\times p}$, $i=1,\dots,r$.

We now look for sufficient conditions for the validity of the minmax inequality for the Pareto ordering.

Suppose the last criterion is not true. Then for some $i\neq j$, $k\neq l$ it follows that:

$$\begin{cases}(\langle u^*,A_iv^*\rangle-\langle u^*,A_iv_*\rangle)\,(\langle u^*,A_jv^*\rangle-\langle u^*,A_jv_*\rangle)<0,\\ (\langle u_*,A_kv_*\rangle-\langle u^*,A_kv_*\rangle)\,(\langle u_*,A_lv^*\rangle-\langle u_*,A_lv_*\rangle)<0,\end{cases}$$

or
$$\begin{cases} \langle u^*, A_i(v^* - v_*)\rangle \langle u^*, A_j(v^* - v_*)\rangle < 0, \\ \langle (u^* - u_*), A_k v_*\rangle \langle (u^* - u_*), A_l v_*\rangle < 0. \end{cases}$$

Now we assume to find conditions which for all nonzero $u \in \mathbb{R}^n, v \in R^m$ yield

$$\langle u, A_i v\rangle \langle u, A_j v\rangle \geqslant 0,$$

then we come to a contradiction with the earlier supposition and these conditions will hence be sufficient for the validity of the minmax inequality (3).

We rewrite: $$\langle u, A_i v\rangle \langle u, A_j v\rangle = u' A_i v v' A_j' u \geqslant 0.$$

We will now look for the restrictions which ensure that matrix $Q = A_i v v' A_j'$ would be positive semi-definite simultaneously for all $v \in \mathbb{R}^m$.

For this situation we further use the next propositions..

Theorem 4.3. (The criterion of Silvester)*A symmetric matrix* $A = A' \in \mathbb{R}^{m\times m}$ *is positively semi-definite if and only if all its main minor matrices are nonnegative.*

Since in general the considered matrix Q, is not symmetrical, the application of Silvester criterion requires the next additional proposition.

Lemma 4.4. *For an arbitrary matrix* $A \in \mathbb{R}^{m\times m}$ *the relation* $\forall\, v \neq 0 \Rightarrow \langle v, Av\rangle \geqslant 0$ *is true if and only if the next inequality is true:* $(A + A')/2 \geqslant 0$.

Since for two arbitrary matrices the next relations are true,

$$\operatorname{rank} AB \leqslant \min\{\operatorname{rank} A, \operatorname{rank} B\}, \quad \operatorname{rank}(A + B) \leqslant \operatorname{rank} A + \operatorname{rank} B, \tag{4}$$

then, applying the equality $\operatorname{rank}[vv'] = 1$ we obtain

$$\operatorname{rank}(\frac{Q + Q'}{2}) \leqslant 2 \operatorname{rank} Q \leqslant 2 \min\left\{\operatorname{rank} A_i, \operatorname{rank}(vv'), \operatorname{rank} A_j'\right\} \leqslant 2.$$

Hence, all the minors of matrix $S = (Q + Q')$ of order 3 are equal to zero.

Proposition 4.5. *Suppose that for all* $i, j = 1, \ldots, r,\ i \neq j$ *all the angular minors* $\mathcal{M}_k[S]$ *of the matrix* $S = A_i v v' A_j' + A_j v v' A_i'$ *of order* $k \leqslant 2$ *are nonnegative simultaneously for all* $v \in (V - V)$. *Then the inequality* (3) *is true.*

Corollary 4.6. *Suppose that the conditions of Proposition* 4.5 *are true. Then*

$$\forall i \neq j = 1, \ldots, r \Rightarrow [A_i]_{kl}[A_j]_{kl} \geqslant 0, \ \ \forall k = 1, \ldots, n, \ l = 1, \ldots, m.$$

Proof. We now find the explicit relation for the diagonal elements of the matrix $S = Q + Q'$. For getting more suitable notations we rename some of the system variables, namely,

$$A_i = A = [a_{ij}], \quad A_j = B = [b_{ij}], \quad vv' = X = [x_{ij}].$$

Then, due to the definition of matrix products, we find

$$Q_{ii} = [AXB']_{ii} = \sum_k [AX]_{ik}[B']_{ki} = \sum_k \left(\sum_r a_{ir} x_{rk} \right) b_{ik} = \sum_k \sum_r a_{ir}(v_r v_k) b_{ik}.$$

Denote $\alpha_i = (a_{i1}, a_{i2}, \dots, a_{im})$, $\beta_i = (b_{i1}, b_{i2}, \dots, b_{im})$ to be the vectors of matrix rows for A and B respectively. Then

$$S_{ii} = 2Q_{ii} = 2 \sum_k b_{ik} v_k \sum_r a_{ir} v_r = 2 \langle \alpha_i', v \rangle \langle \beta_i', v \rangle = 2v' \alpha_i' \beta_i v \geqslant 0.$$

Thus, for all $k = 1, \dots, m$ we have $([A_i]_k)'[A_j]_k \geqslant 0$.

Note that for rank $((A_k)'B_k) = 1$, this relation allows to get a simpler formula of the last condition, having

$$A_{kl}B_{kl} \geqslant 0, \quad \forall k = 1, \dots, n, \; l = 1, \dots, m. \qquad \square$$

Remark 4.7. Note that the given reasoning is true not only for the case of square matrices A_i.

5. Examples

Example 5.1. Let $U = V = [-1; 1]$, $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$, $A_2 = -\begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$.

After a numerical calculation of maxmin and minmax for the given conditions we obtain the graph shown in Figure 5.1.

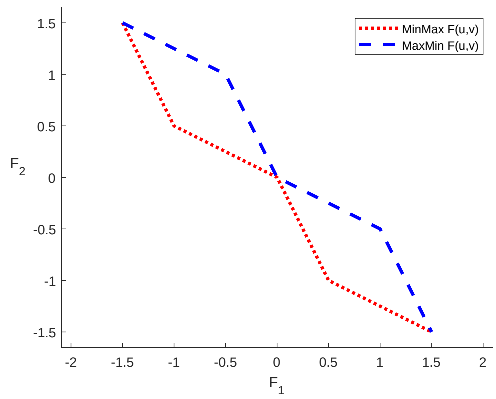

Figure 5.1: The result of numerical calculations for the boundaries in Example 1.

Under given values of matrices A_i we have have an inverse relation

$$\text{Min}_u \text{Max}_v F(u,v) \leqslant \text{Max}_v \text{Min}_u F(u,v).$$

Example 5.2. We first indicate that Corollary 4.6 is not a sufficient condition for the validity of the main inequality (3).

Let $U = V = [-1; 1]$, $A_1 = \begin{bmatrix} 0.5 & 0.2 \\ 0.3 & 0.5 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0.6 & 0 \\ -0.1 & -0.8 \end{bmatrix}$.

After a numerical calculation we get the graph of Figure 5.2.

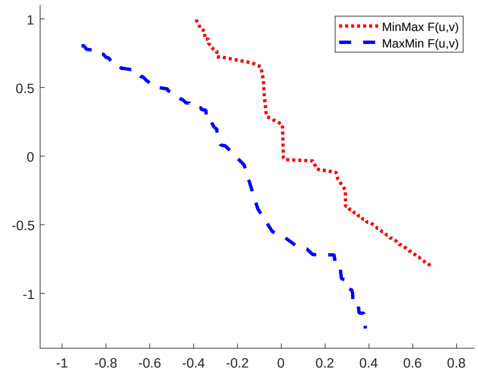

Figure 5.2: The result of numerical calculation of the boundaries in Example 5.2.

It thus occurs that Corollary 4.6 is not a sufficient condition for fulfilling inequality (3), hence an additional analysis of second order minors for matrix $S = Q + Q'$ cannot be omitted.

6. Conclusion

In the course of this paper we formulated a vector-valued analogy of minmax under Pareto ordering and a similar analogy for maxmin. It was determined that an analogy of the classical scalar inequality relations between minmax and maxmin may not always be true for the vector-valued case. It thus became necessary to investigate conditions to ensure the correctness of such an analogy through an example for a linear-quadratic vector-valued problem under bounded disturbance.

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