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Hamilton-Jacobi Equations for Neutral-Type Systems: Inequalities for Directional Derivatives of Minimax Solutions*

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The paper deals with the Hamilton-Jacobi equation in coinvariant derivatives arising in optimal control problems and differential games for dynamical systems described by differential equations of neutral type. On the basis of a suitable definition of directional derivatives, an infinitesimal criterion of the minimax (generalized) solution of this equation is given.

Keywords: Optimal control, Hamilton-Jacobi equations, neutral-type systems, minimax solutions.

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1. Introduction

The paper continues investigations [8, 12, 14, 15, 16] of Hamilton-Jacobi equations in coinvariant derivatives [10] for functional differential systems and aims to develop the theory of minimax (generalized) solutions [23, 24, 25] for Hamilton-Jacobi equations arising in optimal control problems and differential games [11, 13, 19] for dynamical systems described by differential equations of neutral type.

Let $\mathbb{R}^n$ be the n-dimensional Euclidian space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Denote by $\mathrm{Lip}([a,b], \mathbb{R}^n)$ the linear spaces of Lipschitz continuous functions from $[a,b]$ to $\mathbb{R}^n$.

The Hamilton-Jacobi equation considered in this paper arises, for example, in the following optimal control problem. Let a motion of a dynamical system be described by the neutral type differential equation in Hale's form [9]

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$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(x(t) - g(t, x(t-h))\Big) = f(t, x(t), x(t-h), u(t)), \tag{1}$$

$x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{U}$, $t \in [\tau, \vartheta]$, under the initial condition

$$x(t) = w(t-\tau), \quad t \in [\tau - h, \tau]. \tag{2}$$

Here τ and ϑ are initial and terminal times of the motion; $h > 0$ is the delay constant; the function $w(\cdot) \in \mathrm{Lip}([-h, 0], \mathbb{R}^n)$ determines the initial history of the motion; t is the current time; $x(t)$ is the current state vector; $u(t)$ is the current control action; $\mathbb{U} \subset \mathbb{R}^m$ is a compact set. Denote by $\mathcal{U}_\tau$ the set of Lebesgue measurable functions $u(\cdot)\colon [\tau, \vartheta] \mapsto \mathbb{U}$. The control problem is to minimize the value of the cost functional

$$J(\tau, w(\cdot), u(\cdot)) = \sigma(x(\vartheta)), \tag{3}$$

over all $u(\cdot) \in \mathcal{U}_\tau$. Let $t_0 < \vartheta$ be fixed. Define

$$\mathrm{Lip} = \mathrm{Lip}([-h, 0], \mathbb{R}^n), \quad \mathbb{G} = [t_0, \vartheta] \times \mathrm{Lip}.$$

The linear space Lip is endowed with the norm

$$\|w(\cdot)\|_\infty = \max_{\xi \in [-h, 0]} \|w(\xi)\|, \quad w(\cdot) \in \mathrm{Lip}.$$

Consider the value functional in optimal control problem (1)–(3):

$$\varphi(\tau, w(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}_\tau} J(\tau, w(\cdot), u(\cdot)), \quad (\tau, w(\cdot)) \in \mathbb{G}. \tag{4}$$

Following [12, 14, 15, 10], in order to obtain a Hamilton-Jacobi equation for this functional, we use the notion of coinvariant differentiability. Put

$$\Lambda(\tau, w(\cdot)) = \big\{x(\cdot) \in \mathrm{Lip}([\tau - h, \vartheta], \mathbb{R}^n)\colon x(t) = w(t - \tau),\ t \in [\tau - h, \tau]\big\}.$$

Definition 1.1. A functional $\varphi\colon \mathbb{G} \mapsto \mathbb{R}$ is called *coinvariantly differentiable (ci-differentiable)* at a point $(\tau, w(\cdot)) \in \mathbb{G}$, $\tau < \vartheta$, if there exist $\partial_\tau \varphi(\tau, w(\cdot)) \in \mathbb{R}$ and $\nabla \varphi(\tau, w(\cdot)) \in \mathbb{R}^n$ such that, for every $x(\cdot) \in \Lambda(\tau, w(\cdot))$ and $t \in [\tau, \vartheta]$, the following relation holds:

$$\begin{aligned} &\varphi(t, x_t(\cdot)) - \varphi(\tau, w(\cdot)) \\ &\quad = (t - \tau)\partial_\tau \varphi(\tau, w(\cdot)) + \langle x(t) - w(0), \nabla\varphi(\tau, w(\cdot))\rangle + o_{\{\tau, x(\cdot)\}}(t - \tau), \end{aligned} \tag{5}$$

where $x_t(\cdot) \in \mathrm{Lip}$ is defined by $x_t(\xi) = x(t + \xi)$, $\xi \in [-h, 0]$; $o_{\{\tau, x(\cdot)\}}(t - \tau)$ depends on the choice of τ and $x(\cdot)$, and $o_{\{\tau, x(\cdot)\}}(t - \tau)/(t - \tau) \to 0$ as $t \to \tau + 0$. The values $\partial_\tau \varphi(\tau, w(\cdot))$ and $\nabla \varphi(\tau, w(\cdot))$ are called *ci-derivatives* of φ at the point $(\tau, w(\cdot))$. □

Note that ci-derivatives and their close analogues were applied to the wide range of control problems for various functional differential systems (see, e.g., [1, 20, 2, 21, 5, 22]).

In accordance with Definition 1.1, a map $\mathbb{G} \ni (\tau, w(\cdot)) \mapsto \psi = (\psi_1, ..., \psi_n) \in \mathbb{R}^n$ is called *ci-differentiable* at the point $(\tau, w(\cdot))$ if the functionals $\psi_i\colon \mathbb{G} \mapsto \mathbb{R}$, $i = 1, \ldots, n$, are ci-differentiable at this point.

So, for the mapping $\psi(\tau, w(\cdot)) = g(\tau, w(-h))$, where g is taken from (1), we obtain

$$\partial_\tau g(\tau, w(-h)) = \partial_\tau \psi(\tau, w(\cdot)) = \frac{\partial g(\tau, w(-h))}{\partial \tau} + \nabla_x g(\tau, w(-h)) \cdot \frac{\mathrm{d}^+ w(-h)}{\mathrm{d}\tau},$$

$$\nabla g(\tau, w(-h)) = \nabla \psi(\tau, w(\cdot)) = 0,$$

if the partial derivative $\partial g(\tau, w(-h))/\partial \tau$ of g with respect to the first variable, the gradient matrix $\nabla_x g(\tau, w(-h))$ of g with respect to the second variable, and the right derivative $\mathrm{d}^+ w(-h)/\mathrm{d}\tau$ of w at the point $-h$ exist. Let us denote by $\mathbb{G}_*$ the set of points $(\tau, w(\cdot)) \in \mathbb{G}$, $\tau < \vartheta$, such that the ci-derivative $\partial_\tau g(\tau, w(-h))$ exists.

By system (1), define the Hamiltonian

$$H(\tau, x, y, s) = \min_{u \in \mathbb{U}} \langle f(\tau, x, y, u), s\rangle, \quad \tau \in [t_0, \vartheta], \quad x, y, s \in \mathbb{R}^n. \tag{6}$$

Consider the following Hamilton-Jacobi equation in ci-derivatives:

$$\partial_\tau \varphi(\tau, w(\cdot)) + \langle \partial_\tau g(\tau, w(-h)), \nabla \varphi(\tau, w(\cdot))\rangle + \\ + H(\tau, w(0), w(-h), \nabla \varphi(\tau, w(\cdot))) = 0, \quad (\tau, w(\cdot)) \in \mathbb{G}_*, \tag{7}$$

and the Cauchy problem for this equation with the right-end condition

$$\varphi(\vartheta, w(\cdot)) = \sigma(w(0)), \quad w(\cdot) \in \mathrm{Lip}. \tag{8}$$

Under certain conditions on the functions f and σ, one can show (see [7]) that, on the one hand, value functional (4) satisfies Hamilton-Jacobi equation (7) at the points of ci-differentiability, and, on the other hand, a sufficiently smooth (ci-differentiable and satisfying additional smoothness conditions) solution of problem (7), (8) coincides with the value functional. However, in many cases, Cauchy problem (7), (8) (as well as Cauchy problems for Hamilton-Jacobi equations in partial derivatives) does not have the sufficiently smooth solution. Therefore, generalized (minimax) solutions of such type problems were investigated [8, 16, 17]. In [16], existence and uniqueness conditions of the minimax solution were obtained. A coincidence of the minimax solution with the value functional was shown in [8]. In the present paper, we give an infinitesimal criterion of the minimax solution of problem (7), (8) by using a suitable directional derivatives. After proving Lipschitz continuity of the minimax solution (see Lemma 6.4), the proof of the criterion is based on results of [17].

2. Assumptions

Let $B(\alpha) = \{x \in \mathbb{R}^n \colon \|x\| \leq \alpha\}$, $\alpha \geq 0$. We assume that the functions $g(\tau, x) \in \mathbb{R}^n$, $H(\tau, x, y, s) \in \mathbb{R}$ and $\sigma(x) \in \mathbb{R}$, where $\tau \in [t_0, \vartheta]$, $x, y, s \in \mathbb{R}^n$, satisfy the following conditions:

(g) For every $\alpha > 0$, there exists $\lambda_g = \lambda_g(\alpha) > 0$ such that

$$\big\|g(\tau, x) - g(\tau', x')\big\| \leq \lambda_g\big(|\tau - \tau'| + \|x - x'\|\big), \quad \tau, \tau' \in [t_0, \vartheta], \quad x, x' \in B(\alpha).$$

(H_1) The function H is continuous.

(H_2) For every $\alpha > 0$, there exists $\lambda_H = \lambda_H(\alpha) > 0$ such that

$$\big|H(\tau, x, y, s) - H(\tau, x', y', s)\big| \leq \lambda_H \|s\| \big(\|x - x'\| + \|y - y'\|\big)$$

for any $\tau \in [t_0, \vartheta]$, $x, y, x', y' \in B(\alpha)$, and $s \in \mathbb{R}^n$.

(H_3) There exists a constant $c_H > 0$ such that, for all $\tau \in [t_0, \vartheta]$, $x, y, s, s' \in \mathbb{R}^n$,

$$\big|H(\tau, x, y, s) - H(\tau, x, y, s')\big| \leq c_H \|s - s'\| \big(1 + \|x\| + \|y\|\big).$$

(H_4) The following equality holds:

$$H(\tau, x, y, \alpha s) = \alpha H(\tau, x, y, s), \quad \tau \in [t_0, \vartheta], \quad x, y, s \in \mathbb{R}^n, \quad \alpha \geq 0.$$

(σ) For every $\alpha > 0$, there exists $\lambda_\sigma = \lambda_\sigma(\alpha) > 0$ such that

$$\big|\sigma(x) - \sigma(x')\big| \leq \lambda_\sigma \|x - x'\|, \quad x, x' \in B(\alpha).$$

Note that the conditions (H_1)–(H_4) are fulfilled for the Hamiltonian (H) defined by (6) if the function f from (1) is continuous and satisfies the conditions below:

(f_1) For every $\alpha > 0$, there exists $\lambda_f = \lambda_f(\alpha) > 0$ such that

$$\big\|f(\tau, x, y, u) - f(\tau, x', y', u)\big\| \leq \lambda_f \big(\|x - x'\| + \|y - y'\|\big)$$

for any $\tau \in [t_0, \vartheta]$, $x, y, x', y' \in B(\alpha)$, and $u \in \mathbb{U}$.

(f_2) There exists a constant $c_f > 0$ such that

$$\big\|f(\tau, x, y, u)\big\| \leq c_f \big(1 + \|x\| + \|y\|\big), \quad \tau \in [t_0, \vartheta], \quad x, y \in \mathbb{R}^n, \quad u \in \mathbb{U}.$$

3. Minimax solution

Let $\mathbb{P}$ and $\mathbb{Q}$ be nonempty sets, and let multivalued mappings $F_-(\tau, x, y, p) \subset \mathbb{R}^n$ and $F_+(\tau, x, y, q) \subset \mathbb{R}^n$, $\tau \in [t_0, \vartheta]$, $x, y \in \mathbb{R}^n$, $p \in \mathbb{P}$, $q \in \mathbb{Q}$, satisfy the following conditions:

(F_1) For every $\tau \in [t_0, \vartheta]$, $x, y \in \mathbb{R}^n$, $p \in \mathbb{P}$, and $q \in \mathbb{Q}$, the sets $F_-(\tau, x, y, p)$ and $F_+(\tau, x, y, q)$ are nonempty, convex and compact.

(F_2) For fixed $p \in \mathbb{P}$ and $q \in \mathbb{Q}$, the mappings $(\tau, x, y) \mapsto F_-(\tau, x, y, p)$ and $(\tau, x, y) \mapsto F_+(\tau, x, y, q)$ are continuous.

(F_3) There exists a constant $c_F > 0$ such that

$$\sup \big\{ \|l\| \,|\, l \in F_-(\tau, x, y, p) \cup F_+(\tau, x, y, q) \big\} \leq c_F \big(1 + \|x\| + \|y\|\big)$$

for any $\tau \in [t_0, \vartheta]$, $x, y \in \mathbb{R}^n$, $p \in \mathbb{P}$, and $q \in \mathbb{Q}$.

(F_4) The following equalities hold for all $\tau \in [t_0, \vartheta]$ and $x, y, s \in \mathbb{R}^n$:

$$H(\tau, x, y, s) = \inf_{p \in \mathbb{P}} \max_{l \in F_-(\tau, x, y, p)} \langle l, s \rangle = \sup_{q \in \mathbb{Q}} \min_{l \in F_+(\tau, x, y, q)} \langle l, s \rangle.$$

Denote by $\mathfrak{F}_-(H)$ and $\mathfrak{F}_+(H)$, respectively, the sets of pairs $\{\mathbb{P}, F_-\}$ and $\{\mathbb{Q}, F_+\}$ satisfying conditions (F_1)–(F_4). Under conditions (H_1)–(H_4), we have $\mathfrak{F}_-(H) \neq \emptyset$ and $\mathfrak{F}_+(H) \neq \emptyset$. In particular, conditions (F_1)–(F_4) are fulfilled (see, e.g., [24]) if $\mathbb{P} = \mathbb{Q} = \{s \in \mathbb{R}^n \colon \|s\| = 1\}$ and

$$\begin{aligned} F(x,y) &= \big\{l \in \mathbb{R}^n \colon \|l\| \leq \sqrt{2} c_H (1 + \|x\| + \|y\|)\big\}, \\ F_-(\tau,x,y,p) &= \big\{l \in F(x,y) \colon \langle l, p\rangle \leq H(\tau,x,y,p)\big\}, \\ F_+(\tau,x,y,q) &= \big\{l \in F(x,y) \colon \langle l, q\rangle \geq H(\tau,x,y,q)\big\}. \end{aligned} \tag{9}$$

Definition 3.1. A continuous functional $\varphi \colon \mathbb{G} \mapsto \mathbb{R}$ is called a *minimax solution* of problem (7), (8) if it satisfies right-end condition (8) and, for some $\{\mathbb{P}, F_-\} \in \mathfrak{F}_-(H)$ and $\{\mathbb{Q}, F_+\} \in \mathfrak{F}_+(H)$, the following conditions:

(φ_-) For every $(\tau, w(\cdot)) \in \mathbb{G}$, $t \in [\tau, \vartheta]$, and $p \in \mathbb{P}$, there exists a function $x(\cdot) \in \Lambda(\tau, w(\cdot))$ such that

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\big(x(\xi) - g(\xi, x(\xi - h))\big) \in F_-(\xi, x(\xi), x(\xi - h), p)$$

for a.e. $\xi \in [\tau, t]$, and $\varphi(t, x_t(\cdot)) \geq \varphi(\tau, w(\cdot))$.

(φ_+) For every $(\tau, w(\cdot)) \in \mathbb{G}$, $t \in [\tau, \vartheta]$, and $q \in \mathbb{Q}$, there exists a function $y(\cdot) \in \Lambda(\tau, w(\cdot))$ such that

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\big(y(\xi) - g(\xi, y(\xi - h))\big) \in F_+(\xi, y(\xi), y(\xi - h), q)$$

for a.e. $\xi \in [\tau, t]$, and $\varphi(t, y_t(\cdot)) \leq \varphi(\tau, w(\cdot))$.

It follows from [16] that, under conditions (g), (H_1)–(H_4), and (σ), a minimax solution of problem (7), (8) exists, is unique, and satisfies conditions (φ_-), (φ_+) for any $\{\mathbb{P}, F_-\} \in \mathfrak{F}_-(H)$, $\{\mathbb{Q}, F_+\} \in \mathfrak{F}_+(H)$. Also, the minimax solution satisfies Hamilton-Jacobi equation (7) at the points of ci-differentiability and if there exists a sufficiently smooth (ci-differentiable and satisfying additional smoothness conditions) solution of problem (7), (8) then it coincides with the minimax solution.

4. Main result

Let $(\tau, w(\cdot)) \in \mathbb{G}$, $\tau < \vartheta$, $z(\cdot) \in \mathrm{Lip}([\tau, \vartheta], \mathbb{R}^n)$, and $l \in \mathbb{R}^n$. Following [17], consider the lower and upper derivatives of a functional $\varphi \colon \mathbb{G} \mapsto \mathbb{R}$ at the point $(\tau, w(\cdot))$ in the direction $(z(\cdot), l)$:

$$\begin{aligned} \partial^-\{\varphi(\tau, w(\cdot)) \,|\, z(\cdot), l\} &= \liminf_{t \to \tau + 0} \big(\varphi(t, x_t(\cdot)) - \varphi(\tau, w(\cdot))\big)/(t - \tau), \\ \partial^+\{\varphi(\tau, w(\cdot)) \,|\, z(\cdot), l\} &= \limsup_{t \to \tau + 0} \big(\varphi(t, x_t(\cdot)) - \varphi(\tau, w(\cdot))\big)/(t - \tau). \end{aligned}$$

where the function $x(\cdot) \in \Lambda(\tau, w(\cdot))$ satisfies the equation $\mathrm{d}(x(t) - z(t))/\mathrm{d}t = l$ for almost every $t \in [\tau, \vartheta]$; $x_t(\xi) = x(t + \xi)$, $\xi \in [-h, 0]$.

Note that, these directional derivatives are essentially derivatives along extensions $x(\cdot) \in \Lambda(\tau, w(\cdot))$ (see, e.g., [14]). However, in order to obtain an infinitesimal criterion (next theorem) of the minimax solution (see Definition 3.1), it is convenient to define these extensions by pairs $(z(\cdot), l)$.

Theorem 4.1. *Let conditions* (g), (H_1)–(H_4), *and* (σ) *be fulfilled. Then the following statements hold:*

(1) *The minimax solution* $\varphi\colon \mathbb{G} \mapsto \mathbb{R}$ *of problem* (7), (8) *satisfies the inequalities*

$$\sup_{l \in F_-(\tau, w(0), w(-h), p)} \partial^+ \big\{ \varphi(\tau, w(\cdot)) \,\big|\, z^g(\cdot|\tau, w(\cdot)), l \big\} \geq 0,$$
$$\inf_{l \in F_+(\tau, w(0), w(-h), q)} \partial^- \big\{ \varphi(\tau, w(\cdot)) \,\big|\, z^g(\cdot|\tau, w(\cdot)), l \big\} \leq 0, \tag{10}$$

where $(\tau, w(\cdot)) \in \mathbb{G}$, $\tau < \vartheta$, $p \in \mathbb{P}$, $q \in \mathbb{Q}$, *for any* $\{\mathbb{P}, F_-\} \in \mathfrak{F}_-(H)$ *and* $\{\mathbb{Q}, F_+\} \in \mathfrak{F}_+(H)$.
Here $z^g(\cdot|\tau, w(\cdot))$ *is defined as follows:* $z^g(t|\tau, w(\cdot)) = g(t, w(t - \tau - h))$ *if* $\tau \leq t \leq \tau + h$, *and* $z^g(t|\tau, w(\cdot)) = g(t, w(0))$ *if* $t > \tau + h$.

(2) *If a continuous functional* $\varphi\colon \mathbb{G} \mapsto \mathbb{R}$ *satisfies right-end condition* (8) *and inequalities* (10) *for some* $\{\mathbb{P}, F_-\} \in \mathfrak{F}_-(H)$ *and* $\{\mathbb{Q}, F_+\} \in \mathfrak{F}_+(H)$ *then the functional* φ *is the minimax solution of problem* (7), (8).

Theorem 4.1 follows from results of [17] (see Theorem 1 and Proposition 4) and Lemma 6.4 below in Section 6. Before proving Lemma 6.4, let us consider an auxiliary Lyapunov-Krasovskii functional.

5. Lyapunov-Krasovskii functional

Let $\alpha, \varepsilon > 0$. Define the functional

$$V_\varepsilon^\alpha(\tau, s, w(\cdot)) = \kappa_\varepsilon^\alpha(s, w(\cdot)) e^{-2(\lambda_H + \lambda_g/h)(\tau - t_0)},$$
$$\kappa_\varepsilon^\alpha(s, w(\cdot)) = \sqrt{\varepsilon^2 + \|s\|^2} + \lambda_H \int_{-h}^{0} \left(1 - \frac{2\lambda_g \xi}{h}\right) \|w(\xi)\| \mathrm{d}\xi, \tag{11}$$
$$(\tau, s, w(\cdot)) \in [t_0, \vartheta] \times \mathbb{R}^n \times \mathrm{Lip},$$

where $\lambda_g = \lambda_g(\alpha) > 1$ and $\lambda_H = \lambda_H(\alpha) > 0$ are defined by conditions (g) and (H_2).

Lemma 5.1. *Let* $\alpha, \varepsilon > 0$ *and* $\tau \in [t_0, \vartheta]$. *Let the functions* $z(\cdot) \in \mathrm{Lip}([\tau - h, \vartheta], \mathbb{R}^n)$ *and* $s(\cdot) \in \mathrm{Lip}([\tau, \vartheta], \mathbb{R}^n)$ *satisfy the estimates*

$$\|s(t) - z(t)\| \leq \lambda_g \|z(t - h)\|, \quad t \in [\tau, \vartheta],$$
$$\langle \mathrm{d}s(t)/\mathrm{d}t, s(t) \rangle \leq \lambda_H \big(\|z(t)\| + \|z(t-h)\|\big) \|s(t)\| + \varepsilon^2 \text{ for a.e. } t \in [\tau, \vartheta]. \tag{12}$$

Then the following inequality holds:

$$V_\varepsilon^\alpha(t, s(t), z_t(\cdot)) \leq V_\varepsilon^\alpha(\tau, s(\tau), z_\tau(\cdot)) + (t - \tau)\varepsilon, \quad t \in [\tau, \vartheta]. \tag{13}$$

Proof. From (11), taking into account Lipschitz continuity of the functions $z(\cdot)$ and $s(\cdot)$, one can show Lipschitz continuity of the functions $\omega_1(t) = \kappa_\varepsilon^\alpha(s(t), z_t(\cdot))$ and $\omega_2(t) = V_\varepsilon^\alpha(t, s(t), z_t(\cdot))$, $t \in [\tau, \vartheta]$. Then, using (12), for almost every $t \in [\tau, \vartheta]$, we have

$$\begin{aligned}\frac{\mathrm{d}\omega_1(t)}{\mathrm{d}t} &= \frac{\langle \mathrm{d}s(t)/\mathrm{d}t, s(t)\rangle}{\sqrt{\varepsilon^2 + \|s(t)\|^2}} + \lambda_H\|z(t)\| - \lambda_H(1 + 2\lambda_g)\|z(t-h)\| + \\ &\qquad + \frac{2\lambda_H\lambda_g}{h}\int\limits_{t-h}^{t}\|z(\xi)\|\mathrm{d}\xi \\ &\le \varepsilon + 2\lambda_H\|z(t)\| - 2\lambda_H\lambda_g\|z(t-h)\| + \frac{2\lambda_H\lambda_g}{h}\int\limits_{t-h}^{t}\|z(\xi)\|\mathrm{d}\xi \\ &\le \varepsilon + 2\lambda_H\|s(t)\| + \frac{2\lambda_H\lambda_g}{h}\int\limits_{t-h}^{t}\|z(\xi)\|\mathrm{d}\xi.\end{aligned}$$

Hence, we derive

$$\frac{\mathrm{d}\omega_2(t)}{\mathrm{d}t} \le \Bigg(\varepsilon - 2(\lambda_g/h)\|s(t)\| - 2\lambda_H^2\int\limits_{t-h}^{t}\|z(\xi)\|\mathrm{d}\xi\Bigg)e^{-2(\lambda_H + \lambda_g/h)(t-t_0)} \le \varepsilon.$$

From this estimate, we obtain (13). □

Define the functional for $(\tau, w(\cdot)) \in \mathbb{G}$:

$$\psi(\tau, w(\cdot)) = \|w(0)\| + \|w(-h)\| + \|w(\vartheta - \tau - k(\tau)h)\| + \int\limits_{-h}^{0}\|w(\xi)\|\mathrm{d}\xi, \qquad (14)$$

where $k(\tau) = \min\{j \in \mathbb{N}\colon jh > \vartheta - \tau\}$, i.e. $\vartheta - \tau - k(\tau)h \in [-h, 0)$.

Lemma 5.2. *For every $\alpha > 0$, there exists $\lambda_* = \lambda_*(\alpha) > 0$ such that if $\varepsilon > 0$, $\tau \in [t_0, \vartheta]$, $z(\cdot) \in \mathrm{Lip}([\tau - h, \vartheta], \mathbb{R}^n)$, and $s(\cdot) \in \mathrm{Lip}([\tau, \vartheta], \mathbb{R}^n)$ satisfy estimates* (12) *then the following inequality holds:*

$$\|z(\vartheta)\| \le \lambda_*\big(\varepsilon + \psi(\tau, z_\tau(\cdot))\big). \qquad (15)$$

Proof. Let $\alpha > 0$. Define $\lambda_g = \lambda_g(\alpha) > 1$ and $\lambda_H = \lambda_H(\alpha) > 0$ by conditions (g) and (H_2). Put

$$\lambda_0 = e^{2(\lambda_H + \lambda_g/h)(\vartheta - t_0)}, \quad \lambda_* = k(t_0)\lambda_g^{k(t_0)}\lambda_0(1 + \lambda_H)(1 + 2\lambda_g)(1 + \vartheta - t_0). \quad (16)$$

Let $\varepsilon > 0$, $\tau \in [t_0, \vartheta]$, $z(\cdot) \in \mathrm{Lip}([\tau - h, \vartheta], \mathbb{R}^n)$, and $s(\cdot) \in \mathrm{Lip}([\tau, \vartheta], \mathbb{R}^n)$ satisfy (12). Let us prove by induction the inequality

$$\begin{aligned}\|z(t)\| \le (j+1)\lambda_g^j\lambda_0\big(V_\varepsilon^\alpha(\tau, s(\tau), z_\tau(\cdot)) + (\vartheta - \tau)\varepsilon\big) + \lambda_g^{j+1}\|z(t - (j+1)h)\|, \\ t \in [\tau + jh, \tau + (j+1)h] \cap [t_0, \vartheta], \quad j = 0, \ldots, k(\tau) - 1. \qquad (17)\end{aligned}$$

For $j=0$, from (11), (12) and Lemma 5.1, for every $t \in [\tau, \tau + h] \cap [t_0, \vartheta]$, we get

$$\|z(t)\| \le \|s(t)\| + \lambda_g\|z(t-h)\| \le \lambda_0\big(V_\varepsilon^\alpha(\tau, s(\tau), z_\tau(\cdot)) + (\vartheta - \tau)\varepsilon\big) + \lambda_g\|z(t-h)\|.$$

Assume that inequality (17) holds for $j = l$ and prove it for $j = l + 1$. Using the first estimate in (12), definition (11) of V_ε^α, inequality (17) for $j = l$, and Lemma 5.1, for $t \in [\tau + (l+1)h, \tau + (l+2)h] \cap [t_0, \vartheta]$, we obtain

$$\begin{aligned}\|z(t)\| &\leq \|s(t)\| + \lambda_g \|z(t-h)\| \leq \lambda_0 V_\varepsilon^\alpha(t, s(t), z_t(\cdot)) + \\ &\quad + (l+1)\lambda_g^{l+1}\lambda_0\big(V_\varepsilon^\alpha(\tau, s(\tau), z_\tau(\cdot)) + (\vartheta - \tau)\varepsilon\big) + \lambda_g^{l+2}\|z(t-(l+2)h)\| \\ &\leq (l+2)\lambda_g^{l+1}\lambda_0\big(V_\varepsilon^\alpha(\tau, s(\tau), z_\tau(\cdot)) + (\vartheta - \tau)\varepsilon\big) + \lambda_g^{l+2}\|z(t-(l+2)h)\|.\end{aligned}$$

Thus, inequality (17) is valid for any $j = 0, \ldots, k(\tau) - 1$.

According to (11) and (12), we have

$$V_\varepsilon^\alpha(\tau, s(\tau), z_\tau(\cdot)) \leq \varepsilon + \|z(\tau)\| + \lambda_g \|z(\tau - h)\| + \lambda_H(1 + 2\lambda_g) \int_{\tau-h}^{\tau} \|z(\xi)\| \mathrm{d}\xi.$$

Wherefrom, by (14), (16) and (17) for $j = k(\tau) - 1$, we conclude (15). □

6. Lipschitz continuity of minimax solution

Let F be defined as in (9). Define

$$X(\tau, w(\cdot)) = \Big\{x(\cdot) \in \Lambda(\tau, w(\cdot)) : \frac{\mathrm{d}}{\mathrm{d}t}\Big(x(t) - g(t, x(t-h))\Big) \in F(x(t), x(t-h)) \\ \text{for a.e. } t \in [\tau, \vartheta]\Big\}. \tag{18}$$

Lemma 6.1. *For every $\alpha, \lambda > 0$, there exist numbers $\alpha_X = \alpha_X(\alpha) > 0$ and $\lambda_X = \lambda_X(\alpha, \lambda) > 0$ such that if a pair $(\tau, w(\cdot)) \in \mathbb{G}$ satisfies the inequalities*

$$\|w(\xi)\| \leq \alpha, \quad \|w(\xi) - w(\xi')\| \leq \lambda|\xi - \xi'|, \quad \xi, \xi' \in [-h, 0], \tag{19}$$

then the estimates below hold:

$$\|x(t)\| \leq \alpha_X, \ \|x(t) - x(t')\| \leq \lambda_X|t - t'|, \ t, t' \in [\tau - h, \vartheta], \ x(\cdot) \in X(\tau, w(\cdot)). \tag{20}$$

Proof. Let $\alpha, \lambda > 0$ and $k \in \mathbb{N}$ be such that $(\vartheta - t_0)/k \leq h$. Define

$$\begin{aligned}&\alpha_0 = \alpha, \ \alpha_{i+1} = \big(\alpha_i + \lambda_g(\alpha_i)(\vartheta - t_0 + 2\alpha_i) + \sqrt{2}c_H(\vartheta - t_0)(1 + 2\alpha_i)\big)e^{\sqrt{2}c_H h}, \\ &\lambda_0 = \lambda, \ \lambda_{i+1} = \lambda_g(\alpha_i)(1 + \lambda_i) + \sqrt{2}c_H(1 + 2\alpha_{i+1}), \quad i = 0, \ldots, k-1,\end{aligned} \tag{21}$$

where c_H is taken from condition (H_3). Put $\alpha_X = \alpha_k$ and $\lambda_X = \lambda_k$. Let $x(\cdot) \in X(\tau, w(\cdot))$. Denote $\tau_i = \min\{\tau + ih, \vartheta\}$, $i = 0, \ldots, k$. Let us prove by induction the inequalities

$$\|x(t)\| \leq \alpha_i, \ \|x(t) - x(t')\| \leq \lambda_i|t - t'|, \ t, t' \in [\tau - h, \tau_i], \ i = 0, \ldots, k. \tag{22}$$

For $i = 0$, these inequalities follow directly from (19) and (21).

Assume that these inequalities hold for $j = l$. Then, using (g) and (18), we get

$$\begin{aligned}\|x(t)\| &\le \|w(0)\| + \|g(t, x(t-h)) - g(\tau, w(-h))\| + \\ &\quad + \sqrt{2} c_H \int\limits_{\tau}^{t} \big(1 + \|x(\xi)\| + \|x(\xi - h)\|\big) \mathrm{d}\xi \\ &\le \alpha_l + \lambda_g(\alpha_l)(\vartheta - t_0 + 2\alpha_l) + \sqrt{2} c_H (\vartheta - t_0)(1 + 2\alpha_l) \\ &\quad + \sqrt{2} c_H \int\limits_{\tau_l}^{t} \|x(\xi)\| \mathrm{d}\xi, \quad t \in [\tau_l, \tau_{l+1}].\end{aligned}$$

Wherefrom, applying Bellman-Gronwall's lemma (see, e.g., [3, p. 31]) and taking into account definition (21) of α_{l+1}, we obtain the first inequality in (22) for $i = l + 1$. Further, by using the obtained inequality together with condition (g), definition (18), and the second inequality in (22) for $i = l$, we derive

$$\begin{aligned}\|x(t) - x(t')\| &\le \|g(t, x(t-h)) - g(t', x(t'-h))\| \\ &\quad + \sqrt{2} c_H \int\limits_{t}^{t'} \big(1 + \|x(\xi)\| + \|x(\xi - h)\|\big) \mathrm{d}\xi \\ &\le \big(\lambda_g(\alpha_l)(1 + \lambda_l) + \sqrt{2} c_H (1 + 2\alpha_{l+1})\big)|t - t'| = \lambda_{l+1}|t - t'|,\end{aligned}$$

where $\tau_l \le t \le t' \le \tau_{l+1}$. Thus, inequalities (22) hold for any $i = 0, \ldots, k$. In accordance with the choice of α_X and λ_X, this proves the lemma. $\square$

Corollary 6.2. *For every $(\tau, w(\cdot)) \in \mathbb{G}$, there exist $\alpha_X = \alpha_X(\tau, w(\cdot)) > 0$ and $\lambda_X = \lambda_X(\tau, w(\cdot)) > 0$ such that estimates* (20) *hold.*

Corollary 6.3. *For every $\alpha > 0$, there exists $\alpha_X = \alpha_X(\alpha) > 0$ such that*

$$\|x(t)\| \le \alpha_X, \ t \in [\tau - h, \vartheta], \ x(\cdot) \in X(\tau, w(\cdot)), \ \tau \in [t_0, \vartheta], \ w(\cdot) \in P(\alpha),$$

where
$$P(\alpha) = \big\{w(\cdot) \in \mathrm{Lip}\colon \|w(\cdot)\|_\infty \le \alpha\big\}.$$

Lemma 6.4. *Let a functional $\varphi\colon \mathbb{G} \mapsto \mathbb{R}$ be a minimax solution of problem* (7), (8)*. Then, for every $\alpha > 0$, there exists $\lambda_\varphi = \lambda_\varphi(\alpha) > 0$ such that*

$$\big|\varphi(\tau, w(\cdot)) - \varphi(\tau, r(\cdot))\big| \le \lambda_\varphi \psi(\tau, w(\cdot) - r(\cdot)), \ \tau \in [t_0, \vartheta], \ w(\cdot), r(\cdot) \in P(\alpha), \quad (23)$$

where the functional ψ is defined by (14)*.*

Proof. Let $\alpha > 0$. In accordance with Corollary 6.3 and conditions (g), (H_2), (σ), define $\alpha_X = \alpha_X(\alpha) > 0$ and $\lambda_g = \lambda_g(\alpha_X) > 0$, $\lambda_H = \lambda(\alpha_X) > 0$, $\lambda_\sigma = \lambda_\sigma(\alpha_X) > 0$. Then, for every $\tau \in [t_0, \vartheta]$, $w(\cdot), r(\cdot) \in P(\alpha)$, and any $x(\cdot) \in X(\tau, w(\cdot))$, $y(\cdot) \in X(\tau, r(\cdot))$, $t \in [\tau, \vartheta]$, we have

$$\big\|g(t, x(t-h)) - g(t, y(t-h))\big\| \le \lambda_g \|z(t-h)\|,$$

$$\begin{aligned}&\big|H(t,x(t),x(t-h),s(t))-H(t,y(t),y(t-h),s(t))\big|\le\\&\qquad\le\lambda_H\big(\|z(t)\|+\|z(t-h)\|\big)\|s(t)\|,\qquad(24)\\&\big|\sigma(x(\vartheta))-\sigma(y(\vartheta))\big|\le\lambda_\sigma\|z(\vartheta)\|,\end{aligned}$$

where
$$z(t)=x(t)-y(t),\quad t\in[\tau-h,\vartheta],$$
$$s(t)=z(t)-g(t,x(t-h))+g(t,y(t-h)),\quad t\in[\tau,\vartheta]. \qquad (25)$$

In accordance with Lemma 5.2, define $\lambda_*=\lambda_*(\alpha_X)>0$. Put

$$\lambda_\varphi=\lambda_*\lambda_\sigma. \qquad (26)$$

Let $\tau\in[t_0,\vartheta]$, $w(\cdot),r(\cdot)\in P(\alpha)$, and $\varepsilon>0$. Define the set

$$\begin{aligned}W_\varepsilon=\Big\{&(x(\cdot),y(\cdot))\in X(\tau,w(\cdot))\times X(\tau,r(\cdot)):\ \text{for a.e. } t\in[\tau,\vartheta],\\&\langle \mathrm{d}s(t)/\mathrm{d}t,s(t)\rangle\le\lambda_H\big(\|(z(t)\|+\|z(t-h)\|\big)\|s(t)\|+\varepsilon^2\Big\}\end{aligned}\qquad (27)$$

where the functions $z(\cdot)$ and $s(\cdot)$ are defined by (25). Define

$$M_\varepsilon(t)=\big\{(x(\cdot),y(\cdot))\in W_\varepsilon\colon \varphi(t,x_t(\cdot))\ge\varphi(\tau,w(\cdot)),\ \varphi(t,y_t(\cdot))\le\varphi(\tau,r(\cdot))\big\},$$
$$T_\varepsilon=\big\{t\in[\tau,\vartheta]\colon M_\varepsilon(t)\ne\emptyset\big\},\quad t_\varepsilon=\sup T_\varepsilon. \qquad (28)$$

We show that $W_\varepsilon\ne\emptyset$ and $T_\varepsilon\ne\emptyset$. If we take the functions $x(\cdot)\in\Lambda(\tau,w(\cdot))$ and $y(\cdot)\in\Lambda(\tau,r(\cdot))$ such that we have for $t\in[\tau,\vartheta]$

$$x(t)=g(t,x(t-h))+w(0)-g(\tau,w(-h)),\ \ y(t)=g(t,y(t-h))+r(0)-g(\tau,r(-h)),$$

then we have $x(\cdot)\in X(\tau,w(\cdot))$, $y(\cdot)\in X(\tau,r(\cdot))$, and $\mathrm{d}s(t)/\mathrm{d}t=0$ for almost every $t\in[\tau,\vartheta]$. This implies the inclusions $(x(\cdot),y(\cdot))\in W_\varepsilon$ and $\tau\in T_\varepsilon$.

Let us prove that the supremum in (28) is achieved. Let a sequence $t_k\in[\tau,t_\varepsilon]$, $k=1,2,\ldots$ be such that $M_\varepsilon(t_k)\ne\emptyset$, $k=1,2,\ldots$ and $t_k\to t_\varepsilon$ as $k\to\infty$. It means that there exist the functions $(x^{(k)}(\cdot),y^{(k)}(\cdot))\in W_\varepsilon$ such that

$$\varphi(t_k,x^{(k)}_{t_k}(\cdot))\ge\varphi(\tau,w(\cdot)),\quad \varphi(t_k,y^{(k)}_{t_k}(\cdot))\le\varphi(\tau,r(\cdot)). \qquad (29)$$

Without loss of generality, taking into account Corollary 6.2 and Arzela-Ascoli theorem (see, e.g., [18]), we can suppose that the sequence $(x^{(k)}(\cdot),y^{(k)}(\cdot))$, $k=1,2,\ldots$ converges uniformly. Let $(x^\circ(\cdot),y^\circ(\cdot))$ be the limit of this sequence. Similar to Lemma 1 of [6, p. 76], one can prove the inclusion $(x^\circ(\cdot),y^\circ(\cdot))\in W_\varepsilon$.

Passing to the limit in (29) as $k\to\infty$, due to continuity of the functional φ, we obtain $(x^\circ(\cdot),y^\circ(\cdot))\in M_\varepsilon(t_\varepsilon)$, i.e. the supremum in (28) is achieved.

Now, let us prove the equality $t_\varepsilon=\vartheta$ for any $\varepsilon>0$. For the sake of a contradiction, suppose that there exists $\varepsilon>0$ such that $t_\varepsilon<\vartheta$. Let $(x(\cdot),y(\cdot))\in X(\tau,w(\cdot))\times X(\tau,r(\cdot))$ be such that

$$\langle \mathrm{d}s(t)/\mathrm{d}t,s(t)\rangle\le\lambda_H\big(\|(z(t)\|+\|z(t-h)\|\big)\|s(t)\|+\varepsilon^2 \text{ for a.e. } t\in[\tau,\vartheta], \qquad (30)$$

where the functions $z(\cdot)$ and $s(\cdot)$ are defined by (25), and

$$\varphi(t_\varepsilon, x_{t_\varepsilon}(\cdot)) \geq \varphi(\tau, w(\cdot)), \quad \varphi(t_\varepsilon, y_{t_\varepsilon}(\cdot)) \leq \varphi(\tau, r(\cdot)). \tag{31}$$

Let $\{\mathbb{P}, F_-\} \in \mathfrak{F}_-(H)$ and $\{\mathbb{Q}, F_+\} \in \mathfrak{F}_+(H)$ be defined by (9). According to (F_4), there exist $p^\circ \in \mathbb{P}$ and $q^\circ \in \mathbb{Q}$ such that

$$\begin{aligned} \max_{l \in F_-(t_\varepsilon, x(t_\varepsilon), x(t_\varepsilon - h), p^\circ)} \langle l, s(t_\varepsilon)\rangle &\leq H(t_\varepsilon, x(t_\varepsilon), x(t_\varepsilon - h), s(t_\varepsilon)) + \varepsilon^2/4, \\ \min_{l \in F_+(t_\varepsilon, y(t_\varepsilon), y(t_\varepsilon - h), q^\circ)} \langle l, s(t_\varepsilon)\rangle &\geq H(t_\varepsilon, y(t_\varepsilon), y(t_\varepsilon - h), s(t_\varepsilon)) - \varepsilon^2/4. \end{aligned} \tag{32}$$

Define the sets

$$\begin{aligned} \widetilde{X}_- &= \Big\{ \tilde{x}(\cdot) \in X(t_\varepsilon, x_{t_\varepsilon}(\cdot)) : \text{ for a.e. } t \in [\tau, \vartheta] \\ &\qquad \frac{\mathrm{d}}{\mathrm{d}t}\Big(\tilde{x}(t) - g(t, \tilde{x}(t-h))\Big) \in F_-(t, \tilde{x}(t), \tilde{x}(t-h), p^\circ)\Big\}, \\ \widetilde{X}_+ &= \Big\{ \tilde{y}(\cdot) \in X(t_\varepsilon, y_{t_\varepsilon}(\cdot)) : \text{ for a.e. } t \in [\tau, \vartheta] \\ &\qquad \frac{\mathrm{d}}{\mathrm{d}t}\Big(\tilde{y}(t) - g(t, \tilde{y}(t-h))\Big) \in F_+(t, \tilde{y}(t), \tilde{y}(t-h), q^\circ)\Big\}. \end{aligned}$$

Similar to Theorem 4.2 in [4] one can show that $\widetilde{X}_- \neq \emptyset$ and $\widetilde{X}_+ \neq \emptyset$. Taking into account conditions (H_1), (F_2), Corollary 6.2, and inequalities (32), one can prove that there exists $t^* \in (t_\varepsilon, \vartheta]$ such that, for every functions $\tilde{x}(\cdot) \in \widetilde{X}_-$ and $\tilde{y}(\cdot) \in \widetilde{X}_+$, the following inequalities holds for a.e. $t \in [t_\varepsilon, t^*]$:

$$\begin{aligned} &\langle \frac{\mathrm{d}}{\mathrm{d}t}\Big(\tilde{x}(t) - g(t, \tilde{x}(t-h))\Big), \tilde{s}(t)\rangle \leq H(t, \tilde{x}(t), \tilde{x}(t-h), \tilde{s}(t)) + \varepsilon^2/2, \\ &\langle \frac{\mathrm{d}}{\mathrm{d}t}\Big(\tilde{y}(t) - g(t, \tilde{y}(t-h))\Big), \tilde{s}(t)\rangle \geq H(t, \tilde{y}(t), \tilde{y}(t-h), \tilde{s}(t)) - \varepsilon^2/2, \end{aligned} \tag{33}$$

where $\tilde{s}(t) = \tilde{x}(t) - g(t, \tilde{x}(t-h)) - \tilde{y}(t) + g(t, \tilde{y}(t))$. Moreover, since φ is a minimax solution of problem (7), (8), we can find $\tilde{x}(\cdot) \in \widetilde{X}_-$ and $\tilde{y}(\cdot) \in \widetilde{X}_+$ such that

$$\varphi(t^*, \tilde{x}_{t^*}(\cdot)) \geq \varphi(t_\varepsilon, x_{t_\varepsilon}(\cdot)), \quad \varphi(t^*, \tilde{y}_{t^*}(\cdot)) \leq \varphi(t_\varepsilon, y_{t_\varepsilon}(\cdot)). \tag{34}$$

From (24), (33) we derive

$$\langle \mathrm{d}\tilde{s}(t)/\mathrm{d}t, \tilde{s}(t)\rangle \leq \lambda_H\big(\|\tilde{z}(t)\| + \|\tilde{z}(t-h)\|\big)\|\tilde{s}(t)\| + \varepsilon^2 \text{ for a.e. } t \in [t_\varepsilon, t^*]. \tag{35}$$

Define the functions $\hat{x}(\cdot) \in \Lambda(\tau, w(\cdot))$ and $\hat{y}(\cdot) \in \Lambda(\tau, r(\cdot))$ by the relations

$$\hat{x}(t) = x(t),\ \ \hat{y}(t) = y(t),\ \ t \in [\tau, t_\varepsilon); \quad \hat{x}(t) = \tilde{x}(t),\ \ \hat{y}(t) = \tilde{y}(t),\ \ t \in [t_\varepsilon, t^*),$$

and
$$\begin{aligned} \hat{x}(t) &= g(t, \hat{x}(t-h)) + \hat{x}(t^*) - g(t^*, \hat{x}(t^*-h)), \\ \hat{y}(t) &= g(t, \hat{y}(t-h)) + \hat{y}(t^*) - g(t^*, \hat{y}(t^*-h)), \end{aligned} \quad t \in [t^*, \vartheta].$$

Then, according to (30) and (35), we have $(\hat{x}(\cdot), \hat{y}(\cdot)) \in W_\varepsilon$. Moreover, due to (31) and (34), we obtain $(\hat{x}(\cdot), \hat{y}(\cdot)) \in M(t^*)$, which contradicts the definition of t_ε in (28). Thus, the equality $t_\varepsilon = \vartheta$ holds for any $\varepsilon > 0$.

Since $t_\varepsilon = \vartheta$ and φ satisfies (8), there exists $(x(\cdot), y(\cdot)) \in W_\varepsilon$ such that

$$\sigma(x(\vartheta)) = \varphi(\vartheta, x_\vartheta(\cdot)) \geq \varphi(\tau, w(\cdot)), \quad \sigma(y(\vartheta)) = \varphi(\vartheta, y_\vartheta(\cdot)) \leq \varphi(\tau, r(\cdot)).$$

From the inclusion $(x(\cdot), y(\cdot)) \in W_\varepsilon$, the first estimate in (24), and (25), we obtain that inequalities (12) hold if $\alpha = \alpha_X$, $\lambda_g = \lambda_g(\alpha_X)$, and $\lambda_H = \lambda_H(\alpha_X)$. Then, due to Lemma 5.2, using the third estimate in (24), we derive

$$\varphi(\tau, w(\cdot)) - \varphi(\tau, r(\cdot)) \leq \lambda_\sigma \|x(\vartheta) - y(\vartheta)\| \leq \lambda_\sigma \lambda_*(\varepsilon + \psi(\tau, w(\cdot) - r(\cdot))).$$

Since this relation is valid for any $\varepsilon > 0$, it is valid for $\varepsilon = 0$. Thus, taking into account (26), we obtain (23). □

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