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On Almost Periodic Viscosity Solutions to Hamilton-Jacobi Equations

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We establish that a viscosity solution to a multidimensional Hamilton-Jacobi equation with Bohr almost periodic initial data remains to be spatially almost periodic and the additive subgroup generated by its spectrum does not increase in time. In the case of one space variable and a non-degenerate hamiltonian we prove the decay property of almost periodic viscosity solutions when time $t \to +\infty$. For convex hamiltonian we also provide another proof of this property using the Hopf-Lax-Oleinik formula. For periodic solutions the more general result is proved on unconditional asymptotic convergence of a viscosity solution to a traveling wave.

Keywords: Hamilton-Jacobi equations, viscosity solutions, almost periodic functions, long time behavior.

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1. Introduction

In the half-space $\Pi = \mathbb{R}_+ \times \mathbb{R}^n$, $\mathbb{R}_+ = (0, +\infty)$, we consider the Cauchy problem for a first order Hamilton-Jacobi equation

$$u_t + H(\nabla_x u) = 0 \tag{1}$$

with merely continuous hamiltonian function $H(p) \in C(\mathbb{R}^n)$, and with initial condition

$$u(0, x) = u_0(x) \in BUC(\mathbb{R}^n), \tag{2}$$

where $BUC(\mathbb{R}^n)$ denotes the Banach space of bounded uniformly continuous functions on $\mathbb{R}^n$ equipped with the uniform norm $\|u\|_\infty = \sup |u(x)|$.

We recall the notions of *superdifferential* D^+u and *subdifferential* D^-u of a continuous function $u(t, x) \in C(\Pi)$:

$$D^+u(t_0, x_0) = \left\{ \nabla\varphi(t_0, x_0) \;\middle|\; \begin{array}{l} \varphi(t,x) \in C^1(\Pi),\ (t_0, x_0) \text{ is a point} \\ \text{of local maximum of } u - \varphi \end{array} \right\},$$

$$D^-u(t_0, x_0) = \left\{ \nabla\varphi(t_0, x_0) \;\middle|\; \begin{array}{l} \varphi(t,x) \in C^1(\Pi),\ (t_0, x_0) \text{ is a point} \\ \text{of local minimum of } u - \varphi \end{array} \right\}.$$

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We denote by $BUC_{loc}(\bar{\Pi})$ the space of continuous functions on the product space $\bar{\Pi} = \mathrm{Cl}\,\Pi = [0, +\infty) \times \mathbb{R}^n$, which are bounded and uniformly continuous in any layer $[0, T) \times \mathbb{R}^n$, $T > 0$.

Definition 1.1. (see [7, 6]) A function $u(t, x) \in BUC_{loc}(\bar{\Pi})$ is called a *viscosity subsolution* (v.subs. for short) of problem (1), (2) if $u(0, x) \le u_0(x)$ and $s + H(v) \le 0$ for all $(s, v) \in D^+u(t, x)$, $(t, x) \in \Pi$.

A function $u(t, x) \in BUC_{loc}(\bar{\Pi})$ is called a *viscosity supersolution* (v.supers.) of problem (1), (2) if $u(0, x) \ge u_0(x)$ and $s + H(v) \ge 0$ for all $(s, v) \in D^-u(t, x)$, $(t, x) \in \Pi$.

Finally, $u(t, x) \in BUC_{loc}(\bar{\Pi})$ is called a *viscosity solution* (v.s.) of (1), (2) if it is a v.subs. and a v.supers. of this problem simultaneously. □

The theory of viscosity solutions was developed in [6, 7] for general multidimensional Hamilton-Jacobi equations; this theory extended the earlier results of S. N. Kruzhkov [14, 15]. An alternative theory of minimax solutions was introduced by A. I. Subbotin in [30, 31], where in particular the equivalence of minimax and viscosity solutions was established.

It is known that for each $u_0(x) \in BUC(\mathbb{R}^n)$ there exists a unique v.s. of problem (1), (2). The uniqueness readily follows from the more general comparison principle, formulated in the next section.

In this paper we study v.s. of problem (1), (2) with almost periodic initial function and its asymptotic behavior for large time. The large-time behavior of viscosity solution has been studied by many authors, see for instance papers [1, 2, 4, 5, 9, 10, 11, 12, 13, 21, 29] and reference therein, where mainly the case of convex hamiltonian is studied. In the present paper we use different methods based on connection of viscosity solutions and entropy solutions to conservation laws, known in the case of single space variable. We also suggest a procedure of reduction of the study of spatially almost periodic v.s. to the periodic case.

The paper is organized as follows. In the next Section 2 we establish some simple properties of v.s., which will be useful in the sequel. In Section 3 we prove that a v.s. $u = u(t, x)$ of (1), (2) is space almost periodic and the spectrum of $u(t, \cdot)$ is contained in the additive subgroup generated by the spectrum of initial data, see Theorem 3.1 below.

In the last Section 4 we investigate the case of single space variable $n = 1$. In this case we essentially use the tight connection of v.s. and entropy solutions to conservation laws. Our main results are about long-time asymptotic behavior of v.s. Firstly, we consider periodic initial function. We prove that if $H(0) = 0$ than v.s. asymptotically converges to a traveling wave. The exact formulation is given in Theorem 4.4. In particular, it follows from result of Theorem 4.4 that, under the assumption that the hamiltonian $H(u)$ is not linear in any vicinity of 0, a v.s. uniformly converges to a constant (*decay property*). We then extend this decay property to the case of almost periodic initial data, in Theorem 4.6. In the conclusion we show that in the case of convex hamiltonian the decay property can be derived with the help of Hopf-Lax-Oleinik formula. Moreover, in this case we can indicate the exact value of limit constant, it equals the infimum of initial function, see Theorem 4.7.

2. Some properties of v.s.

We recall the following *comparison principle.*

Theorem 2.1. *Let* $u_1(t,x), u_2(t,x) \in BUC_{loc}(\bar{\Pi})$ *be a v.subs. and a v.supers. of* (1), (2) *with initial data* $u_{10}(x), u_{20}(x)$*, respectively. Assume that* $u_{10}(x) \le u_{20}(x)$ $\forall x \in \mathbb{R}^n$*. Then* $u_1(t,x) \le u_2(t,x)$ $\forall (t,x) \in \Pi$*.*

For the proof of Theorem 2.1, we refer to [6].

Corollary 2.2. *Let* $u_1(t,x), u_2(t,x) \in BUC_{loc}(\bar{\Pi})$ *be v.s. of* (1), (2) *with initial data* $u_{10}(x), u_{20}(x)$*, respectively. Then for all* $t > 0$

$$\inf(u_{10}(x) - u_{20}(x)) \le u_1(t,x) - u_2(t,x) \le \sup(u_{10}(x) - u_{20}(x)).$$

In particular, $\|u_1 - u_2\|_\infty \le \|u_{10} - u_{20}\|_\infty$.

Proof. We define $a = \inf(u_{10}(x) - u_{20}(x))$ and $b = \sup(u_{10}(x) - u_{20}(x))$, and observe that the functions $a + u_2(t,x)$, $b + u_2(t,x)$ a v.s. of (1), (2) with initial data $a + u_{20}(x)$, $b + u_{20}(x)$, respectively. Since $a + u_{20}(x) \le u_{10}(x) \le b + u_{20}(x)$, then by Theorem 2.1 $a + u_2(t,x) \le u_1(t,x) \le b + u_2(t,x)$ $\forall (t,x) \in \Pi$, which completes the proof. □

Lemma 2.3. *Let* $H(p,q) \in C(\mathbb{R}^n \times \mathbb{R}^m)$*. We consider the equation*

$$U_t + H(\nabla_x U, \nabla_y U) = 0 \tag{3}$$

in the half-space $\{ (t,x,y) \mid t > 0, x \in \mathbb{R}^n, y \in \mathbb{R}^m \}$*. Then* $U(t,x,y) = u(t,y)$ *is a non-depending on* x *v.s. of* (3) *if and only if* $u(t,y)$ *is a v.s. of the reduced equation*

$$u_t + H(0, \nabla_y u) = 0, \quad (t,y) \in \mathbb{R}_+ \times \mathbb{R}^m.$$

Proof. The assertion of the lemma readily follows from the evident equalities

$$D^\pm U(t,x,y) = \{(s,0,v) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \mid (s,v) \in D^\pm u(t,y) \}. \qquad \square$$

Lemma 2.4. *Let* $y = Ax$, $y_i = \sum_{j=1}^n a_{ij} x_j$, $i = 1, \dots, n$, *be a non-degenerate linear operator on* $\mathbb{R}^n$, $v_0(y) \in BUC(\mathbb{R}^n)$, $v(t,y) \in BUC_{loc}(\Pi)$*. Then the function* $u(t,x) = v(t,Ax)$ *is a v.s. of* (1) *with initial data* $u_0(x) = v_0(Ax)$ *if and only if* $v(t,y)$ *is a v.s. of the problem*

$$v_t + H(A^* \nabla_y v) = 0, \quad v(0,y) = v_0(y),$$

where A^* *is a conjugate operator (so that* $(A^* \nabla_y v)_j = \sum_{i=1}^n a_{ij} \partial_{y_i} v$, $j = 1, \dots, n$).

Proof. The statement of Lemma 2.4 follows from the fact that (t_0, x_0) is a point of local maximum (minimum) of $u(t,x) - \psi(t, Ax)$, with $\psi(t,y) \in C^1(\Pi)$, if and only if (t_0, Ax_0) is a point of local maximum (minimum) of $v(t,y) - \psi(t,y)$ and from the classical identity $A^*\nabla_y\psi(t,y) = \nabla_x\psi(t,Ax)$, $y = Ax$. □

In the half space $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m$ we consider the Cauchy problem for equation

$$u_t + H(\nabla_x u) = 0, \quad u = u(t,x,y), \quad (t,x,y) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m, \tag{4}$$

with the initial condition

$$u(0,x,y) = u_0(x,y) \in BUC(\mathbb{R}^n \times \mathbb{R}^m). \tag{5}$$

Lemma 2.5. *A function $u(t,x,y) \in BUC_{loc}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m)$ is a v.s. of* (4), (5) *if and only if for all fixed $y \in \mathbb{R}^m$ the functions $u^y(t,x) = u(t,x,y)$ is a v.s. of* (1), (2) *with initial data $u_0^y(x) = u_0(x,y)$.*

Proof. Let $u(t,x,y)$ be a v.s. of (4), (5), and $y_0 \in \mathbb{R}^m$. We assume that $\varphi(t,x) \in C^1(\Pi)$ and $(t_0,x_0) \in \Pi$ is a point of local maximum of $u^{y_0} - \varphi$. Moreover, replacing φ by $\varphi(t,x) + (t-t_0)^2 + |x-x_0|^2 + u(t_0,x_0,y_0) - \varphi(t_0,x_0)$ (here and in the sequel we denote by $|z|$ the Euclidean norm of a finite-dimensional vector z), we can suppose, without loss of generality, that $(t_0,x_0) \in \Pi$ is a point of strict local maximum of $u^{y_0} - \varphi$, and that in this point $u^{y_0}(t_0,x_0) - \varphi(t_0,x_0) = 0$. Therefore, there exists $c > 0$ such that

$$\varphi(t,x) - u(t,x,y_0) > c \quad \forall (t,x) \in \Pi,\ (t-t_0)^2 + |x-x_0|^2 = r^2,$$

for some $r \in (0,t_0)$. Because of the continuity there exists $h > 0$ such that $\varphi(t,x) - u(t,x,y) > c/2$ for all $(t,x,y) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m$, $(t-t_0)^2 + |x-x_0|^2 = r^2$, $|y-y_0| \le h$. We can choose such $C_0 > 0$ that

$$C_0h^2 - c > \max\{\ \varphi(t,x) - u(t,x,y) \mid (t-t_0)^2 + |x-x_0|^2 \le r^2,\ |y-y_0| = h\ \}.$$

Then for each natural $k > C_0$ the function

$$p_k(t,x,y) = \varphi(t,x) + k|y-y_0|^2 \in C^1(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m)$$

satisfies the property

$$p_k(t,x,y) - u(t,x,y) > c/2 > 0 = p_k(t_0,x_0,y_0) - u(t_0,x_0,y_0) \tag{6}$$

for all $(t,x,y) \in \partial V_{r,h}$, where we denote by V_{rh} the domain

$$V_{rh} = \{\ (t,x,y) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \mid (t-t_0)^2 + |x-x_0|^2 < r^2,\ |y-y_0| < h\ \}.$$

In view of (6) the point (t_k, x_k, y_k) such that

$$p_k(t_k,x_k,y_k) - u(t_k,x_k,y_k) = \min_{(t,x,y) \in \operatorname{Cl} V_{rh}} (p_k(t,x,y) - u(t,x,y))$$

lies in V_{rh} and is, in consequence, a point of local maximum of the difference $u(t,x,y) - p_k(t,x,y)$.

Since $\nabla p_k(t,x,y) = (\partial_t\varphi(t,x), \nabla_x\varphi(t,x), 2k(y-y_0))$, then by the definition of v.s. of (4)

$$\partial_t\varphi(t_k,x_k) + H(\nabla_x\varphi(t_k,x_k)) \le 0. \tag{7}$$

Since $\min_{(t,x,y)\in \operatorname{Cl} V_{rh}} (p_k(t,x,y) - u(t,x,y)) \le p_k(t_0,x_0,y_0) - u(t_0,x_0,y_0) = 0$, then $k|y_k - y_0|^2 \le m = \max_{(t,x,y)\in \operatorname{Cl} V_{rh}} (u(t,x,y) - \varphi(t,x))$. In particular $y_k \to y_0$ as $k \to \infty$. Taking into account that (t_0,x_0) is a point of strict local maximum of $u(t,x,y_0) - \varphi(t,x)$, we derive that $(t_k,x_k) \to (t_0,x_0)$ as $k \to \infty$. Therefore, it follows from (7) in the limit as $k \to \infty$ that

$$\partial_t\varphi(t_0,x_0) + H(\nabla_x\varphi(t_0,x_0)) \le 0.$$

This means that $u(t,x,y_0)$ is a v.subs. of (1). By the similar reasons we obtain that

$$\partial_t\varphi(t_0,x_0) + H(\nabla_x\varphi(t_0,x_0)) \ge 0$$

whenever (t_0,x_0) is a point of strict local minimum of $u(t,x,y_0) - \varphi(t,x)$, where $\varphi(t,x) \in C^1(\Pi)$, that is, $u(t,x,y_0)$ is a v.supers. of (1). Thus, $u(t,x,y_0)$ is a v.s. of (1) for each $y_0 \in \mathbb{R}^m$.

Conversely, assume that $u^y(t,x)$ is a v.s. of (1) for every $y \in \mathbb{R}^m$. Suppose $\varphi(t,x,y) \in C^1(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m)$ and that (t_0,x_0,y_0) is a point of local maximum (minimum) of $u(t,x,y) - \varphi(t,x,y)$. Then the point $(t_0,x_0) \in \Pi$ is a point of local maximum (minimum) of $u^{y_0}(t,x) - \varphi(t,x,y_0)$.

Since u^{y_0} is a v.s. of (1) then $\varphi_t(t_0,x_0,y_0) + H(\nabla_x\varphi(t_0,x_0,y_0)) \le 0$ (respectively, $\varphi_t(t_0,x_0,y_0) + H(\nabla_x\varphi(t_0,x_0,y_0)) \ge 0$). Hence, $u(t,x,y)$ is a v.s. of (4). To complete the proof, it only remains to notice that initial condition (5) is satisfied if and only if $u^y(t,x)$ satisfies (2) with initial data u_0^y for all $y \in \mathbb{R}^m$. □

3. Almost periodic viscosity solutions

Recall that the space $AP(\mathbb{R}^n)$ of Bohr (or uniform) almost periodic functions is a closure of trigonometric polynomials, i.e. finite sums $\sum a_\lambda e^{2\pi i\lambda\cdot x}$, in the space $BUC(\mathbb{R}^n)$ (by $\cdot$ we denote the inner product in $\mathbb{R}^n$). It is clear that $AP(\mathbb{R}^n)$ contains continuous periodic functions (with arbitrary lattice of periods). Originally, almost periodic functions are defined with the help of notion of almost-periods, see [20] for details. Let C_R be the cube

$$\{\ x = (x_1,\dots,x_n) \in \mathbb{R}^n \mid |x|_\infty = \max_{i=1,\dots,n} |x_i| \le R/2\ \}, \quad R > 0.$$

It is known (see for instance [20]) that for each function $u \in AP(\mathbb{R}^n)$ there exists the mean value

$$\bar{u} = ⨍_{\mathbb{R}^n} u(x)dx \doteq \lim_{R\to+\infty} R^{-n} \int_{C_R} u(x)dx$$

and, more generally, the Bohr-Fourier coefficients

$$a_\lambda = ⨍_{\mathbb{R}^n} u(x)e^{-2\pi i\lambda\cdot x}dx, \quad \lambda \in \mathbb{R}^n.$$

The set
$$Sp(u) = \{ \lambda \in \mathbb{R}^n \mid a_\lambda \neq 0 \}$$
is called the *spectrum* of an almost periodic function $u(x)$. It is known [20], that the spectrum $Sp(u)$ is at most countable.

Now we assume that the initial function $u_0(x) \in AP(\mathbb{R}^n)$. Let M_0 be the smallest additive subgroup of $\mathbb{R}^n$ containing $Sp(u_0)$. Notice that in the case when u_0 is a continuous periodic function M_0 coincides with the dual lattice to the group of periods.

Our first result is the following.

Theorem 3.1. *Let $u(t,x)$ be a unique v.s. of* (1), (2).
Then $u(t,\cdot) \in C([0,+\infty), AP(\mathbb{R}^n))$ and for all $t > 0$ we have $Sp(u(t,\cdot)) \subset M_0$.

Proof. We first assume that the initial function is a trigonometric polynomial $u_0(x) = \sum_{\lambda \in S} a_\lambda e^{2\pi i \lambda \cdot x}$. Here $S = Sp(u_0) \subset \mathbb{R}^n$ is a finite set. Then the subgroup M_0 is a finite generated torsion-free abelian group and therefore it is a free abelian group of finite rank (see [19]). Hence, there is a basis $\lambda_j \in M_0$, $j = 1, \dots, m$, such that every element $\lambda \in M_0$ can be uniquely represented as $\lambda = \lambda(\bar{k}) = \sum_{j=1}^m k_j \lambda_j$, $\bar{k} = (k_1, \dots, k_m) \in \mathbb{Z}^m$. In particular, we can represent the initial function as
$$u_0(x) = \sum_{\bar{k} \in J} a_{\bar{k}} e^{2\pi i \sum_{j=1}^m k_j \lambda_j \cdot x}, \quad a_{\bar{k}} \doteq a_{\lambda(\bar{k})},$$
where $J = \{ \bar{k} \in \mathbb{Z}^m \mid \lambda(\bar{k}) \in S \}$ is a finite set. By this representation we have $u_0(x) = v_0(y(x))$, where
$$v_0(y) = \sum_{\bar{k} \in J} a_{\bar{k}} e^{2\pi i \bar{k} \cdot y}$$
is a periodic function on $\mathbb{R}^m$ with the standard lattice of periods $\mathbb{Z}^m$ while $y = \Lambda x$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ defined by the equalities $y_j = \lambda_j \cdot x = \sum_{i=1}^n \lambda_{ji} x_i$, λ_{ji}, $i = 1, \dots, n$, being coordinates of the vectors λ_j, $j = 1, \dots, m$. We consider the Hamilton-Jacobi equation
$$v_t + H(\Lambda^* \nabla_y v) = 0, \quad v = v(t,y),\ t > 0,\ y \in \mathbb{R}^m, \tag{8}$$
where $(\Lambda^* \nabla_y v)_i = \sum_{j=1}^m \lambda_{ji} \partial_{y_j} v$, $i = 1, \dots, n$. Let $v(t,y)$ be a v.s. of the Cauchy problem for equation (8) with initial function $v_0(y)$. By the periodicity of v_0 the function $v(t, y + e)$ is a v.s. of the same problem for each vector $e \in \mathbb{Z}^m$. In view of uniqueness of v.s. we conclude that $v(t, y + e) \equiv v(t,y)$ $\forall e \in \mathbb{Z}^m$, i.e., $v(t,y)$ is a periodic function with respect to y (with the lattice of periods $\mathbb{Z}^m$). In view of periodicity $v(t,\cdot) \in C([0,+\infty), C(\mathbb{T}^m))$, where $\mathbb{T}^m = \mathbb{R}^m / \mathbb{Z}^m$ is a torus (it can be identified with the periodicity cell $[0,1)^m$).

We demonstrate that $u(t,x) = v(t, \Lambda x)$. We introduce the invertible linear operator $\tilde{\Lambda}$ on the extended space $\mathbb{R}^{n+m}$, defined by the equality $\tilde{\Lambda}(x,z) = (x, z + \Lambda x)$. Since $\tilde{\Lambda}^*(x,y) = (x + \Lambda^* y, y)$, equation (8) can also be rewritten in the form $v_t + H(\tilde{\Lambda}^*(0, \nabla_y v)) = 0$, where $H(p,q) = H(p)$, $p \in \mathbb{R}^n$, $q \in \mathbb{R}^m$. By Lemma 2.3 the function $v = v(t,y)$ is a v.s. of equation $v_t + H(\tilde{\Lambda}^*(\nabla_x v, \nabla_y v)) = 0$ in the extended domain $(t,x,y) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m$. Then, by Lemma 2.4 the function $u(t,x,z) = v(t, z+\Lambda x)$ is a v.s. of (1) considered in the extended domain $(t,x,z) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m$. Applying Lemma 2.5 we conclude that $u^z(t,x) = u(t,x,z)$ is a v.s. of (1) for all $z \in \mathbb{R}^m$. Taking $z=0$ we find that $u(t,x) = v(t,\Lambda x)$ is a v.s. of (1). It is clear that $u(0,x) = v_0(\Lambda x) = u_0(x)$, that is, $u(t,x)$ is a v.s. of original problem (1), (2).

For a fixed $t > 0$ a continuous periodic function $v(t,y)$ can be uniformly approximated by finite sums $v_m(y) = \sum_{|\bar{k}|\le m} a_{m,\bar{k}} e^{2\pi i \bar{k}\cdot y}$ (for instance, one can take Fejér sums), so that $v_m \rightrightarrows v(t,y)$ as $m \to \infty$. This implies that $v_m(\Lambda x) \rightrightarrows u(t,x)$ as $m \to \infty$ and since

$$v_m(\Lambda x) = \sum_{|\bar{k}|\le m} a_{m,\bar{k}} e^{2\pi i \lambda(\bar{k})\cdot x}$$

are trigonometric polynomials while $\lambda(\bar{k}) = \sum_{j=1}^m k_j \lambda_j \in M_0$, we find that the limit function $u(t,\cdot) \in AP(\mathbb{R}^n)$ and $Sp(u(t,\cdot)) \subset M_0$.

In the general case $u_0 \in AP(\mathbb{R}^n)$ there exists a sequence of trigonometric polynomials $u_{0m}(x)$, $m \in \mathbb{N}$, such that $Sp(u_{0m}) \subset M_0$ and $u_{0m} \rightrightarrows u_0$ as $m \to \infty$. We can choose u_{0m} as the sequence of Bochner-Fejér trigonometric polynomials, see [20]. As it has been already established, a unique v.s. $u_m(t,x)$ of (1), (2) with initial data u_{0m} belongs to the space $C([0,+\infty), AP(\mathbb{R}^n))$ and $Sp(u_m(t,\cdot)) \subset M_0$ for all $t \ge 0$. Since $\|u_m - u\|_\infty \le \|u_{0m} - u_0\|_\infty \to 0$ as $m \to \infty$, then $u_m \rightrightarrows u$ as $m \to \infty$ and, therefore, the limit v.s. $u(t,x) \in C([0,+\infty), AP(\mathbb{R}^n))$ and $Sp(u(t,\cdot)) \subset M_0$ for all $t \ge 0$. The proof is complete. □

Remark 3.2. In view of Corollary 2.2 and the translation invariance of v.s.

$$\|u(t,x+l) - u(t,x)\|_\infty \le \|u_0(x+l) - u_0(x)\|_\infty \quad \text{for all } t > 0.$$

This estimate implies that any ε-almost-period l of $u_0(x) \in AP(\mathbb{R}^n)$ is a common ε-almost period of $u(t,\cdot)$ for all $t \ge 0$. From this and the known relation between the ε-almost-periods and the spectrum it readily follows the assertion of Theorem 3.1. This proof seems to be more simple but we preferred the proof which does not use the notion of almost-periods and based on reduction to the periodic case because this reduction will be utilized also in the proof of Theorem 4.6 below.

4. The case of single space variable

4.1. Long time behavior of periodic v.s.

Now we consider the case of a single space variable when our equation (1) has the form

$$u_t + H(u_x) = 0, \tag{9}$$

$u = u(t,x)$, $(t,x) \in \Pi = \mathbb{R}_+ \times \mathbb{R}$. We are going to investigate the long time behavior of almost periodic viscosity solutions of the Cauchy problem for equation (9) with initial condition

$$u(0,x) = u_0(x). \tag{10}$$

In the one-dimensional case there is a direct connection between v.s. of (9), (10) and entropy solutions (in Kruzhkov sense [16]) of the corresponding Cauchy problem for the conservation law

$$v_t + H(v)_x = 0 \tag{11}$$

with initial data

$$v(0,x) = v_0(x). \tag{12}$$

More precisely, assume that $u_0(x)$ is Lipshitz continuous, i.e. its generalized derivative $u_0'(x) \in L^\infty(\mathbb{R})$. Then the unique v.s. $u(t,x)$ of (9), (10) also satisfies the Lipschitz condition with respect to the space variable and the derivative $v = u_x(t,x) \in L^\infty(\Pi)$ is a unique entropy solution of (11), (12) with $v_0(x) = u_0'(x)$, cf. [7]. Observe that (11) can be derived from (9) by formal differentiation: $v_t + H(v)_x = u_{xt} + H(u_x)_x = \frac{\partial}{\partial x}(u_t + H(u_x)) = 0$, $v = u_x$.

Recall the notion of *entropy solution* (e.s. for short) of the Cauchy problem for a multidimensional conservation law

$$v_t + \mathrm{div}_x f(v) = 0, \tag{13}$$

$v = v(t,x)$, $(t,x) \in \Pi = \mathbb{R}_+ \times \mathbb{R}^n$, with merely continuous flux vector $f(v) = (f_1(v), \ldots, f_n(v)) \in C(\mathbb{R}, \mathbb{R}^n)$ and with initial condition

$$v(0,x) = v_0(x) \in L^\infty(\mathbb{R}^n). \tag{14}$$

Definition 4.1. A bounded measurable function $v = v(t,x) \in L^\infty(\Pi)$ is called an *entropy solution* (e.s.) of (13), (14) if for all $k \in \mathbb{R}$

$$\frac{\partial}{\partial t}|v-k| + \mathrm{div}_x[\mathrm{sign}(v-k)(f(v)-f(k))] \le 0 \tag{15}$$

in the sense of distributions on Π (in $\mathcal{D}'(\Pi)$), and

$$\operatorname*{ess\,lim}_{t\to 0+} v(t,\cdot) = v_0 \quad \text{in } L^1_{loc}(\mathbb{R}^n). \qquad \square$$

Here

$$\mathrm{sign}\, u = \begin{cases} 1, & u > 0 \\ -1, & u \le 0 \end{cases},$$

and relation (15) means that for each test function $h = h(t,x) \in C_0^1(\Pi)$, $h \ge 0$,

$$\int_\Pi [|v-k|h_t + \mathrm{sign}(v-k)(f(v)-f(k)) \cdot \nabla_x h] dt dx \ge 0.$$

Taking in (15) $k = \pm R$, where $R \ge \|v\|_\infty$, we obtain that $v_t + \mathrm{div}_x f(v) = 0$ in $\mathcal{D}'(\Pi)$, that is, an e.s. $v = v(t,x)$ is a weak solutions of this equation as well. It was also established in [23, Corollary 7.1] that, after possible correction on a set of null measure, an e.s. $u(t,x)$ is continuous on $[0,+\infty)$ as a map $t \mapsto u(t,\cdot)$ into $L^1_{loc}(\mathbb{R}^n)$. In the sequel we will always assume that this property is satisfied.

Suppose that the initial function v_0 is periodic with a lattice of periods $\mathcal{L}$, i.e., $v_0(x+e) = v_0(x)$ a.e. on $\mathbb{R}^n$ for every $e \in \mathcal{L}$. Let $\mathbb{T}^n = \mathbb{R}^n/\mathcal{L}$ be the corresponding n-dimensional torus, and $\mathcal{L}'$ be the dual lattice

$$\mathcal{L}' = \{ \xi \in \mathbb{R}^n \mid \xi \cdot x \in \mathbb{Z} \ \forall x \in \mathcal{L} \}.$$

In the case under consideration when the flux vector is merely continuous the property of finite speed of propagation for initial perturbation may be violated, which, in the multidimensional situation $n > 1$, may even lead to the nonuniqueness of e.s. to Cauchy problem (13), (14), see examples in [17, 18]. But for a periodic initial function $v_0(x)$, an e.s. $v(t,x)$ of (13), (14) is unique (in the class of all e.s., not necessarily periodic) and space-periodic, the proof can be found in [22]. It is also shown in [22] that the mean value of e.s. over the period does not depend on time:

$$\int_{\mathbb{T}^n} v(t,x)dx = I \doteq \int_{\mathbb{T}^n} v_0(x)dx,$$

where dx is the normalized Lebesgue measure on $\mathbb{T}^n$.

The following theorem, proven in [25], generalizes the previous results of [8, 24].

Theorem 4.2. *Suppose that*

$$\begin{cases} \forall \xi \in \mathcal{L}',\ \xi \neq 0,\ \textit{the function } u \to \xi \cdot f(v) \\ \textit{is not affine on any vicinity of } I. \end{cases} \tag{16}$$

Then

$$\lim_{t\to+\infty} v(t,\cdot) = I \quad \textit{in } L^1(\mathbb{T}^n). \tag{17}$$

Actually, condition (16) is necessary and sufficient for the decay property (17). Now we return to the case of one space variable and assume that initial function $v_0 \in L^\infty(\mathbb{R})$ is periodic: $v_0(x+1) = v_0(x)$ a.e. in $\mathbb{R}$. This can be written as $v_0 \in L^\infty(\mathbb{T})$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is a circle (it can be identified with the periodicity cell $[0,1)$). Denote by I the mean value of v_0:

$$I = \int_{\mathbb{T}} v_0(x)dx = \int_0^1 v_0(x)dx.$$

The unique e.s. $v = v(t,x)$ of (11), (12) is also x-periodic, $v \in C([0,+\infty), L^1(\mathbb{T}))$, and, as was shown in paper [26], the following long time asymptotic property holds.

Theorem 4.3. *There exist a periodic function $w(y) \in L^\infty(\mathbb{T})$ (profile) and a constant $c \in \mathbb{R}$ (speed) such that*

$$v(t,x) - w(x-ct) \underset{t\to+\infty}{\to} 0 \quad \textit{in } L^1(\mathbb{T}). \tag{18}$$

Moreover, $\int_{\mathbb{T}} w(y)dy = I$, and the function $H(v) - cv \equiv \mathrm{const}$ on the interval $(\operatorname{ess\,inf} w(y), \operatorname{ess\,sup} w(y))$.

Notice that in the case when the function $H(v)$ is not affine in any vicinity of I it follows from Theorem 4.3 that $w(y) \equiv I$ and (18) reduces to the decay property

$$v(t,x) \underset{t\to+\infty}{\to} I \text{ in } L^1(\mathbb{T})$$

from the previous Theorem 4.2 (in the case $n = 1$).

Making the change $\tilde{u} = u + H(0)t$, which transforms the equation (9) to the equation $\tilde{u}_t + H(\tilde{u}_x) - H(0) = 0$, we may suppose that $H(0) = 0$. Then constants are v.s. of (9), and by Corollary 2.2 with $u_1 = u$, $u_2 = 0$ we find that a v.s. $u = u(t,x)$ is bounded, namely $\|u\|_\infty \le \|u_0\|_\infty$.

Let (a,b), $-\infty \le a < 0 < b \le +\infty$, be the maximal neighborhood of zero such that $H(u)$ is linear on (a,b): $H(u) = cu$ on (a,b) for some $c \in \mathbb{R}$. If such an interval does not exist, i.e. $H(u)$ is not linear in any vicinity of zero, we set $a = b = 0$. Assume that the initial function $u_0(x)$ is periodic, $u_0(x) \in C(\mathbb{T})$, and $u(t,x)$ is a unique v.s. of (9), (10). Since $u(t,x+1)$ is a v.s. of the same problem, then $u(t,x+1) = u(t,x)$ by the uniqueness of v.s. The next our result is similar to Theorem 4.3, this is about the long time convergence of $u(t,x)$ to a traveling wave.

Theorem 4.4. *There exist a continuous periodic function $p(y) \in C(\mathbb{T})$ and a real constant c such that*

$$u(t,x) - p(x-ct) \rightrightarrows 0 \quad \text{as } t \to +\infty. \tag{19}$$

Moreover, the profile $p(y)$ satisfies the one-sided Lipschitz estimates

$$a(y_2 - y_1) \le p(y_2) - p(y_1) \le b(y_2 - y_1) \quad \forall y_1, y_2 \in \mathbb{R}, y_2 > y_1, \tag{20}$$

while the speed c is determined by the condition that $H(u) = cu$ on (a,b). In the case $a = b = 0$ it follows from (20) that $p(y) \equiv p_0 = \text{const}$ and (19) reduces to the following decay property

$$u(t,x) \rightrightarrows p_0 \quad \text{as } t \to +\infty$$

(in this case the value of c does not matter).

Proof. First, we consider the case when u_0 is Lipschitz. Then $v = u_x(t,x)$ is an e.s. of (11), (12) with initial data $v_0(x) = u_0'(x) \in L^\infty(\mathbb{T})$. Obviously, $I = \int_{\mathbb{T}} v_0(x)dx = 0$ and by Theorem 4.3 $v(t,x)$ converges as $t \to +\infty$ to a traveling wave $w(x-ct)$ in the sense of relation (18). In view of Theorem 4.3, we also find that $\int_{\mathbb{T}} w(y)dy = 0$ and that $a \le w(y) \le b$. Therefore, the function $\tilde{p}(y) = \int_0^y w(s)ds$ is periodic continuous function satisfying (20) (with $\tilde{p}$ instead of p). Notice that $\tilde{p} \equiv 0$ in the case $a = b = 0$. Let $c \in \mathbb{R}$ is determined by the condition $H(u) = cu$ on (a,b) and is chosen arbitrarily if $a = b = 0$. Now it follows from (18) that

$$\frac{\partial}{\partial x}(u(t,x) - \tilde{p}(x-ct)) = v(t,x) - w(x-ct) \underset{t\to+\infty}{\to} 0 \text{ in } L^1(\mathbb{T}). \tag{21}$$

Since $H(u) \equiv cu$ on the range of $\tilde{p}_x$, the traveling wave $\tilde{p}(x-ct)$ is a v.s. of equation (9), $\tilde{p}_t + H(\tilde{p}_x) = \tilde{p}_t + c\tilde{p}_x = 0$. Then, as is readily follows from Corollary 2.2, where we replace the initial time $t = 0$ by $t = \tau$,

$$\min_{x\in\mathbb{T}}(u(\tau,x) - p(x-c\tau)) \le u(t,x) - p(x-ct) \le \max_{x\in\mathbb{T}}(u(\tau,x) - p(x-c\tau))$$

for all $x \in \mathbb{T}$, $t, \tau \ge 0$, $t > \tau$. Therefore, the functions

$$m(t) = \min_{x\in\mathbb{T}}(u(t,x) - p(x-ct)), \quad M(t) = \max_{x\in\mathbb{T}}(u(t,x) - p(x-ct))$$

are, respectively, increasing and decreasing on $[0, +\infty)$, and $m(t) \le M(t)$. This implies existence of limits

$$m_\infty = \lim_{t\to+\infty} m(t), \quad M_\infty = \lim_{t\to+\infty} M(t).$$

In view of (21) we also find that

$$M(t) - m(t) \le \int_{\mathbb{T}} \left| \frac{\partial}{\partial x}(u(t,x) - \tilde{p}(x-ct)) \right| dx \to 0$$

as $t \to +\infty$. Hence $m_\infty = M_\infty = m_*$. It now follows from the estimate $m(t) \le u(t,x) - \tilde{p}(x-ct) \le M(t)$ and the limit relation

$$\lim_{t\to+\infty} m(t) = \lim_{t\to+\infty} M(t) = m_*,$$

that $u(t,x) - \tilde{p}(x-ct) \rightrightarrows m_*$ as $t \to +\infty$. Taking $p(y) = m_* + \tilde{p}(y)$, we conclude that (19) holds. Clearly, condition (20) is also satisfied.

In the general case $u_0 \in C(\mathbb{T})$ we consider a sequence $u_{0n} \in C(\mathbb{T})$, $n \in \mathbb{N}$ of Lipschitz functions, such that $u_{0n} \to u_0$ as $n \to \infty$ in $C(\mathbb{T})$. Let $u_n = u_n(t,x)$ be a v.s. of (9), (10) with initial data u_{0n}. Then, by Corollary 2.2,

$$\|u_n(t,\cdot) - u(t,\cdot)\|_\infty \le \|u_{0n} - u_0\|_\infty \underset{n\to\infty}{\to} 0. \tag{22}$$

As we have already established, there exist functions $p_n(y) \in C(\mathbb{T})$ such that

$$u_n(t,x) - p_n(x-ct) \rightrightarrows 0 \quad \text{ as } t \to +\infty, \tag{23}$$

$$a(y_2 - y_1) \le p_n(y_2) - p_n(y_1) \le b(y_2 - y_1) \quad \forall y_1, y_2 \in \mathbb{R}, y_2 > y_1. \tag{24}$$

Notice that the speed c does not depend on n. It follows from (23) and (22) that for all $m, n \in \mathbb{N}$

$$\|p_n - p_m\|_\infty \le \|u_n - u_m\|_\infty \to 0 \text{ as } n, m \to \infty,$$

that is $p_n(y)$ is a Cauchy sequence in $C(\mathbb{T})$. Since this space is complete, there exists a limit function $p(y) \in C(\mathbb{T})$, so that $p_n(y) \rightrightarrows p(y)$ as $n \to \infty$. Now (19), (20) follows from (23), (24) in the limit as $n \to \infty$. The proof is complete. □

Remark 4.5. As is easy to see, in the case when $H(0)$ may be arbitrary, relation (19) should be replaced by the following one

$$u(t,x) + H(0)t - p(x-ct) \rightrightarrows 0 \quad \text{ as } t \to +\infty.$$

4.2. Decay of almost periodic v.s.

Now we assume that initial function in (10) is almost periodic: $u_0 \in AP(\mathbb{R})$. Suppose also that $H(0) = 0$. By Theorem 3.1 a unique v.s. $u(t,x)$ of (9), (10) is also almost periodic over the space variables: $u(t,x) \in C([0,+\infty), AP(\mathbb{R}))$, and $Sp(u(t,\cdot)) \subset M_0$, where M_0 is an additive subgroup of $\mathbb{R}$ spanned by $Sp(u_0)$. It turns out that this v.s. satisfies the same decay property as in periodic case, cf. Theorem 4.4.

Theorem 4.6. *Assume that $H(u)$ is not linear in any vicinity of zero. Then*

$$u(t,\cdot) \rightrightarrows c = \text{const} \quad \textit{as } t \to +\infty.$$

Proof. It seems natural to reduce the Hamilton-Jacobi equation to conservation law (11), like in the proof of Theorem 4.4 above, and to use the decay properties of almost periodic e.s. Unfortunately, the known decay property of almost periodic e.s. of conservation laws (see, for example, [27]) asserts the decay in the Besicovitch norm, which is too weak for our aims. We will follow another approach similar to one used in the proof of Theorem 3.1, which is based on reduction to the periodic case.

Let us first consider the case when $u_0(x) = \sum_{\lambda\in S} a_\lambda e^{2\pi i\lambda x}$ is a trigonometric polynomial. In this case the set $S = Sp(u_0)$ is finite and the group $M_0 \subset \mathbb{R}$ generated by the spectrum S is a free abelian group of finite rank. Let $\lambda_j \in \mathbb{R}$, $j = 1,\dots,m$ be a basis of M_0, $\Lambda = (\lambda_1,\dots,\lambda_m) \in \mathbb{R}^m$.

Then, see the proof of Theorem 3.1, $u_0(x) = v_0(y(x))$, $u(t,x) = v(t,y(x))$, where $v_0(y) = \sum_{\bar{k}\in J} a_{\bar{k}} e^{2\pi i\bar{k}\cdot y} \in C(\mathbb{T}^m)$ is a periodic trigonometric polynomial on $\mathbb{R}^m$ with the standard lattice of periods $\mathbb{Z}^m$,

$$J = \{\bar{k} = (k_1,\dots,k_m) \in \mathbb{Z}^m \mid \lambda(\bar{k}) = \Lambda\cdot\bar{k} \in S\ \},$$

$y(x) = x\Lambda$ is a linear map from $\mathbb{R}$ into $\mathbb{R}^m$, and $v(t,y) \in C([0,+\infty), C(\mathbb{T}^m))$ is a unique v.s. of the problem

$$v_t + H(\Lambda\cdot\nabla_y v) = 0, \quad v(0,y) = v_0(y). \tag{25}$$

Since $v_0(y)$ is a trigonometric polynomial, it satisfies the Lipschitz condition $|v_0(y+z) - v_0(y)| \le L|z|\ \forall y,z \in \mathbb{R}^m$, where $L > 0$ is a Lipschitz constant. Obviously, $v(t,y+z)$ is a v.s. of (25) with initial function $v_0(y)$ replaced by $v_0(y+z)$. By Corollary 2.2

$$|v(t,y+z) - v(t,y)| \le \|v_0(\cdot+z) - v_0\|_\infty \le L|z|\ \ \forall t \ge 0,\ y,z \in \mathbb{R}^m,$$

that is, $v(t,y)$ is Lipschitz continuous with respect to the space variables y. In particular, $\nabla_y v(t,y) \in L^\infty(\Pi,\mathbb{R}^m)$, and $\|\nabla_y v(t,y)\|_\infty \le L$. By our assumptions $H(0) = 0$, which guarantees the estimate $\|v\|_\infty \le \|v_0\|_\infty$. We see that the family $v(t,\cdot)$, $t > 0$ is bounded and equicontinuous in $C(\mathbb{T}^m)$. By the Arzelà-Ascoli theorem there exist a sequence $t_r > 0$, $r \in \mathbb{N}$ such that $t_r \to +\infty$ as $r \to \infty$ and a function $v_\infty = v_\infty(y) \in C(\mathbb{T}^m)$ with the property $v(t_r,\cdot) \rightrightarrows v_\infty$ as $r \to \infty$.

Let us apply the directional derivative $D_\Lambda = \Lambda \cdot \nabla$ to equation (25). We obtain

$$(D_\Lambda v)_t + D_\Lambda H(D_\Lambda v) = 0.$$

After the change $w = D_\Lambda v$ we arrive at the conservation law

$$w_t + \operatorname{div}_y(H(w)\Lambda) = w_t + \sum_{j=1}^{m} (\lambda_j H(w))_{y_j} = 0 \tag{26}$$

with the flux vector $f(w) = H(w)\Lambda$, equipped with the corresponding initial condition

$$w(0, y) = w_0(y) \doteq D_\Lambda v_0(y) \in L^\infty(\mathbb{T}^m). \tag{27}$$

Using the vanishing viscosity approximations like in [7], one can claim that $w = D_\Lambda v(t, y)$ is an e.s. of (26), (27). For the sake of completeness we provide more details. As was shown in [7], the v.s. $v(t, y)$ is an uniform limit of the sequence $v_l(t, y) \in C^2(\Pi)$, $l \in \mathbb{N}$, of classical solutions to the parabolic problems

$$v_t + H_l(\Lambda \cdot \nabla_y v) = \varepsilon_l \Delta v, \quad v(0, y) = v_0(y), \tag{28}$$

where $0 < \varepsilon_l \to 0$ as $l \to \infty$, and $H_l(s)$ is a sufficiently regular approximation of the hamiltonian $H(s)$, Applying the derivative D_Λ to equation (28), we find that the function $w_l(t, y) = D_\Lambda v_l$ is a classical solution to the problem

$$w_t + \sum_{j=1}^{m} (\lambda_j H_l(w))_{y_j} = \varepsilon_l \Delta w, \quad w(0, y) = w_0(y).$$

In the correspondence with vanishing viscosity method (see, for example, [16]), the sequence $w_l(t, y) = D_\Lambda v_l \to w(t, y)$ as $l \to \infty$ in $L^1_{loc}(\Pi)$, where $w(t, y)$ is an e.s. of (26), (27). As was noticed above, $v_l \rightrightarrows v$ as $l \to \infty$ and we conclude that $D_\lambda v = w$ in $\mathcal{D}'(\Pi)$. Hence, $w = D_\lambda v$ is an e.s. of (26), (27).

Observe that $w_0(y) = D_\Lambda v_0(y)$ is a periodic function with zero mean value. Further, for each $\xi \in \mathbb{Z}^m$, $\xi \neq 0$, we have

$$\xi \cdot H(w)\Lambda = (\xi \cdot \Lambda) H(w)$$

and since λ_j, $j = 1, \ldots, m$, is a basis of M_0, it follows

$$\xi \cdot \Lambda = \sum_{j=1}^{m} \xi_j \lambda_j \neq 0.$$

Therefore, $\xi \cdot (H(w)\Lambda) = (\xi \cdot \Lambda) H(w)$ is not linear in any vicinity of zero for all $\xi \in \mathbb{Z}^m$, $\xi \neq 0$. By Theorem 4.2 (with $\mathcal{L}' = \mathcal{L} = \mathbb{Z}^m$) $w(t, y) \to 0$ as $t \to \infty$ in $L^1(\mathbb{T}^m)$. Taking $t = t_r$, we find that

$$w(t_r, y) = D_\Lambda v(t_r, y) \underset{r\to\infty}{\to} 0 \ \text{ in } L^1(\mathbb{T}^m), \ \text{ and } \ v(t_r, y) \underset{r\to\infty}{\to} v_\infty(y) \ \text{ in } C(\mathbb{T}^m).$$

From these equalities it follows that $D_\Lambda v_\infty(y) = 0$ in $\mathcal{D}'(\mathbb{R}^m)$. Taking also into account that $v_\infty(y)$ is a continuous function we find that $v_\infty(x\Lambda) = c = \text{const}$ $\forall x \in \mathbb{R}$. Since the numbers λ_j, $j = 1, \dots, m$, are linear independent over the ring $\mathbb{Z}$, then by the Kronecker's theorem (see [20]) the curve $y = x\Lambda$ is dense in torus $\mathbb{T}^m$. This, together with continuity of v_∞, implies that $v_\infty \equiv c$. By Corollary 2.2 we find

$$\|v(t,\cdot) - c\|_\infty \le \|v(t_r,\cdot) - c\|_\infty \quad \forall t > t_r.$$

From this it follows that

$$\lim_{t\to+\infty} \|v(t,\cdot) - c\|_\infty = \lim_{r\to\infty} \|v(t_r,\cdot) - c\|_\infty = 0.$$

To complete the proof, it only remains to notice that $u(t,x) = v(t, x\Lambda)$.

In the general case of arbitrary almost periodic initial function $u_0(x)$ we construct a sequence u_{0n}, $n \in \mathbb{N}$, of trigonometric polynomials such that $u_{0n} \to u_0$ as $n \to \infty$ in $AP(\mathbb{R})$. Let $u_n = u_n(t,x)$ be a v.s. of (9), (10) with initial data u_{0n}. Then, as follows from Corollary 2.2,

$$\|u_n(t,\cdot) - u(t,\cdot)\|_\infty \le \|u_{0n} - u_0\|_\infty \underset{n\to\infty}{\to} 0. \tag{29}$$

As we have already established, there exist constants c_n such that

$$u_n(t,x) \rightrightarrows c_n \quad \text{ as } t \to +\infty. \tag{30}$$

It follows from (29) and (30) that for all $m, n \in \mathbb{N}$

$$|c_n - c_m| \le \|u_n - u_m\|_\infty \to 0 \ \text{ as } n, m \to \infty,$$

that is, c_n is a Cauchy sequence in $\mathbb{R}$. Therefore, $c_n \to c$ as $n \to \infty$, where c is some constant. It now follows from (29), (30) in the limit as $n \to \infty$ that $u(t,\cdot) \rightrightarrows c$ as $t \to +\infty$. The proof is complete. □

4.3. The case of convex hamiltonian

In the case of convex H the decay property may be derived with the help of Hopf-Lax-Oleinik formula. Moreover, in this case we can indicate the exact value of limit constant. So, assume that $H(v)$ is a convex function, which is not linear in any vicinity of 0. As above, we also suppose that $H(0) = 0$.

Theorem 4.7. *Let $u(t,x)$ be a v.s. of* (9), (10). *Then*

$$u(t,\cdot) \rightrightarrows c = \inf u_0(x) \quad \text{ as } t \to +\infty. \tag{31}$$

Proof. Assume first that the initial function $u_0(x)$ is Lipschitz:

$$|u_0(x) - u_0(y)| \le K|x - y| \quad \forall x, y \in \mathbb{R},$$

$K > 0$ is a Lipschitz constant. By Corollary 2.2 we have

$$|u(t, x+\delta) - u(t,x)| \le \sup |u_0(x+\delta) - u_0(x)| \le K\delta \quad \forall x \in \mathbb{R}, \delta > 0.$$

Thus, the functions $u(t,\cdot)$ satisfy the Lipschitz condition with the constant K. Therefore, the generalized derivative $u_x \in L^\infty(\Pi)$, $\|u_x\|_\infty \le K$. This readily implies that $|v| \le K$ whenever $(s,v) \in D^\pm u(t,x)$, $(t,x) \in \Pi$. We see that the behavior of $H(v)$ for $|v| > K$ does not matter and we can suppose that the convex hamiltonian $H(v)$ is superlinear, i.e., $H(v)/|v| \to +\infty$ as $v \to \infty$. Then the Legendre transform

$$H^*(p) = \max_{v\in\mathbb{R}}(pv - H(v))$$

is everywhere defined convex and superlinear function of real variable p. According to the Hopf-Lax-Oleinik formula

$$u(t,x) = \min_{y\in\mathbb{R}} \left(u_0(y) + tH^*((x-y)/t)\right), \tag{32}$$

see for example [3, 15].

Let $c = \inf u_0(x)$ and $\varepsilon > 0$. Recall that $h \in \mathbb{R}$ is an ε-almost-period of the function $u_0(x) \in AP(\mathbb{R})$ if $|u_0(x+h) - u_0(x)| \le \varepsilon$ for all $x \in \mathbb{R}$. By the definition of almost periodic functions (see [20]) there exists such $l = l(\varepsilon) > 0$ that any segment $[a, a+l]$, $a \in \mathbb{R}$, of length l contains at least one ε-almost-period of $u_0(x)$. Let us demonstrate that for arbitrary $a \in \mathbb{R}$

$$\min_{y\in[a,a+l]} u_0(y) \le c + \varepsilon. \tag{33}$$

For that, we find $x_\delta \in \mathbb{R}$ such that $u_0(x_\delta) < c + \delta$, where $\delta > 0$.

Let $h \in [a - x_\delta, a - x_\delta + l]$ be an ε-almost-period of u_0. Then

$$u_0(x_\delta + h) \le u_0(x_\delta) + \varepsilon < c + \delta + \varepsilon. \tag{34}$$

We notice that $x_\delta + h \in [a, a+l]$. Therefore,

$$\min_{y\in[a,a+l]} u_0(y) \le u_0(x_\delta + h), \quad \text{and} \quad \min_{y\in[a,a+l]} u_0(y) < c + \delta + \varepsilon$$

by inequality (34). Obviously, the last inequality implies (33) in view of the arbitrariness of $\delta > 0$.

Now, since $H(0) = 0$, then $H^*(p) \ge 0$ and $H^*(p) = 0$ if and only if $p \in D^- H(0)$. Observe that the set $D^- H(v)$ coincides with the nonempty segment $[H'(v-), H'(v+)]$, where $H'(v-)$, $H'(v+)$, are, respectively, the left and right derivative of H at the point $v \in \mathbb{R}$. We fix $p_0 \in D^- H(0)$. Then $0 = H^*(p_0) = \min H^*(p)$. It is easily verified that

$$H(v) = \max_{p\in\mathbb{R}}(vp - H^*(p)) = p_0 v \quad \forall v \in [v_-, v_+] = D^- H^*(p_0), \tag{35}$$

where $v_\pm = (H^*)'(p_0\pm)$ are one-sided derivatives of $H^*(p)$ at p_0. Since p_0 is a minimum point of $H^*(p)$, then $0 \in D^- H^*(p_0)$ and therefore $v_- \le 0 \le v_+$. If $v_- v_+ < 0$ then (v_-, v_+) is a neighborhood of 0 and $H(v)$ is linear in this neighborhood by (35). But this contradicts to our assumptions. Hence, at least one of the values $v_\pm$ is zero. In view of (33) we can find such $y_+ = y_+(t,x) \in [x - p_0 t - l, x - p_0 t]$ and $y_- = y_-(t,x) \in [x - p_0 t, x - p_0 t + l]$ that $u_0(y_\pm) \le c + \varepsilon$.

We denote $z_\pm = z_\pm(t,x) = x - p_0 t - y_\pm(t,x)$. Then $y_\pm = x - p_0 t - z_\pm$, while $z_+ \in [0,l]$, $z_- \in [-l,0]$. By (32)

$$u(t,x) \le u_0(y_\pm) + tH^*((x-y_\pm)/t) = u_0(y_\pm) + tH^*(p_0 + t^{-1}z_\pm)$$
$$\le c + tH^*(p_0 + t^{-1}z_\pm) + \varepsilon \le c + tH^*(p_0 \pm t^{-1}l) + \varepsilon. \quad (36)$$

By Corollary 2.2 (with $u_1 = u$, $u_2 = 0$) we also find that $u(t,x) \ge \inf u_0(x) = c$. From this estimate and (36) it follows that

$$c \le u(t,x) \le c + \varepsilon + t\min\left(H^*(p_0 + t^{-1}l), H^*(p_0 - t^{-1}l)\right). \quad (37)$$

Since $\lim\limits_{t\to+\infty} t\min\left(H^*(p_0 + t^{-1}l), H^*(p_0 - t^{-1}l)\right) = l\min(v_+, -v_-) = 0,$

while $\varepsilon > 0$ is arbitrary, it follows from (37) that $u(t,\cdot) \rightrightarrows c$ as $t \to +\infty$.

This completes the proof in the case of Lipschitz initial data.

In the general case we construct the sequence $u_{0n} \in AP(\mathbb{R})$, $n \in \mathbb{N}$ of Lipschitz functions such that $u_{0n} \to u_0$ as $n \to \infty$ in $AP(\mathbb{R})$. Let $u_n = u_n(t,x)$ be a v.s. of (9), (10) with initial data u_{0n}. Then (see the proof of Theorem 4.6), as $n \to \infty$,

$$u_n(t,x) \rightrightarrows u(t,x), \quad c_n = \inf u_{0n}(x) \to c. \quad (38)$$

As was already proved, for each $n \in \mathbb{N}$, $u_n(t,\cdot) \rightrightarrows c_n$ as $t \to +\infty$.

In view of (38) we can pass to the limit as $n \to \infty$ in the above relation and derive the desired result (31). □

Remark 4.8. In the case of concave hamiltonian $H(v)$

$$u(t,\cdot) \rightrightarrows c = \sup u_0(x) \quad \text{as } t \to +\infty. \quad (39)$$

Indeed, as is easy to verify, the function $w = -u(t,-x)$ is a v.s. of the problem

$$w_t - H(w_x) = 0, \quad w(0,x) = -u_0(-x),$$

with the convex hamiltonian $-H(v)$. By Theorem 4.7

$$w(t,x) = -u(t,-x) \rightrightarrows \inf -u_0(-x) = -\sup u_0(x) \quad \text{as } t \to +\infty,$$

which reduces to (39).

Remark 4.9. In [28] the decay property (31) was extended to the general multidimensional case (1). Namely, (31) was established for a convex hamiltonian $H(v) \in C(\mathbb{R}^n)$ satisfying the following non-degeneracy condition at point $v = 0$ in "resonant" directions $\xi \in M_0$:

$$\forall \xi \in M_0,\ \xi \ne 0, \quad \text{the functions } s \to H(s\xi)$$
$$\text{are not linear in any interval } |s| < \delta,\ \delta > 0.$$

We recall that M_0 is the additive group generated by the spectrum of initial function $u_0 \in AP(\mathbb{R}^n)$. Notice that in periodic setting the similar condition was introduced in [2] under the name "non-resonance condition".

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