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Group Pursuit in Recurrent Differential Games in the Class of Positional Strategies with Guide

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The article deals with the linear pursuit problem with n pursuers and one evader with the same dynamic and inertial abilities of the players. Sufficient conditions both in the class of quasi-strategies, and in the class of positional strategies with a guide are obtained.

Keywords: Differential game, pursuer, evader, recurrent function, guide system.

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1. Introduction

To date, various formalizations of the differential game have been proposed, which differ, in particular, in the awareness of the players accepted in them [4, 5, 6, 7, 8, 9, 10, 11, 14, 15, 16, 17, 19]. In many works [2, 4, 6, 8, 12, 13, 17] devoted to the problems of group pursuit, it is assumed that all pursuers at every moment of time know the controls of the evader either at the moment or throughout the period of continuation of the game up to a given moment, that is, the pursuers know the background of the game. This formalization of conflict interaction is not always natural. The most real is the assumption about the knowledge of the phase coordinates of the participants and the use of positional strategies by the participants of the game. The use of positional strategies leads to significant difficulties in the study of the problems of group persecution.

In [11] a two-player game was considered, for the analysis of which an auxiliary differential game (guide-system) was introduced, by the movement of which the pursuer corrected his trajectory at certain fixed points in time for the positions of the original and auxiliary games. In the auxiliary game, the pursuer built his control, knowing the control of the evader.

In [3] a guide-system was used to study the problem of simple group pursuit, and in work [18] to study the problem of group pursuit in a linear stationary differential game.

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In this paper, a linear non-stationary group pursuit problem is considered, for which the solvability conditions for the pursuit problem are obtained both in the class of quasi-strategies, and in the class of positional strategies with a guide.

The obtained results are planned to be applied to the construction of optimal strategies for the players using the Hamilton-Jacobi equations in the group pursuit problem with the payoff function "capture time".

2. Statement of the problem

In the space $\mathbb{R}^k$ $(k \geq 2)$, a differential game Γ of $n+1$ objects with n pursuers $P_1, \ldots, P_n$ and the evader E is considered.

The law of motion of each pursuer P_i has the form

$$\dot{x}_i = A(t)x_i + u_i, \quad u_i \in V, \quad x_i(t_0) = x_i^0. \tag{1}$$

The law of motion of evader E has the form

$$\dot{y} = A(t)y + v, \quad v \in V, \quad y(t_0) = y^0, \tag{2}$$

where $x_i, y, u_i, v \in \mathbb{R}^k$, $i \in I = \{1, \ldots, n\}$, V is a strictly convex compact set in $\mathbb{R}^k$, $A(t)$ is a continuous matrix function of dimension $k \times k$ on semi-axis $[t_0, \infty)$.

Instead of (1)–(2), we consider the system

$$\dot{z}_i = A(t)z_i + u_i - v, \quad z_i(t_0) = z_i^0 = x_i^0 - y^0. \tag{3}$$

We assume that $z_i^0 \neq 0$ for all $i \in I$.

In addition, we consider an auxiliary differential game $\tilde{\Gamma}$ with pursuers $\tilde{P}_1, \ldots, \tilde{P}_n$ and an evader $\tilde{E}$, which is described by a system of the form

$$\dot{w}_i = A(t)w_i + \tilde{u}_i - \tilde{v}, \quad \tilde{u}_i, \tilde{v} \in V, \quad w_i(t_0) = w_i^0. \tag{4}$$

In system (4), the matrix $A(t)$ and the set V are the same as in system (3).

A measurable function $v\colon [t_0, \infty[\to \mathbb{R}^k$ is called to be an *admissible control* if $v(t) \in V$ for all $t \geq t_0$. We call the prehistory $\tilde{v}_t(\cdot)$ of the evader $\tilde{E}$ at an instant t, $t \in [t_0, +\infty)$, the restriction of the function $\tilde{v}$ to the interval $[t_0, t]$.

The set of all measurable functions $v\colon [t_0, \infty) \to V$ is denoted by $\mathcal{V}$.

Definition 2.1. [19, p. 24] A *quasi-strategy* $\tilde{U}_i$ of a pursuer $\tilde{P}_i$ is map $\alpha_i\colon \mathcal{V} \to \mathcal{V}$ satisfying the following condition: let $t^0 \in [t_0, \infty)$, $v^1, v^2 \in \mathcal{V}, v^1(t) = v^2(t)$ be almost everywhere on $[t_0, t^0)$, then $u_i^1(t) = u_i^2(t)$ almost everywhere on $[t_0, t^0)$, where $u_i^j(\cdot) = \alpha_i(v^j(\cdot))$, $j = 1, 2$. □

We define *positional strategies with a guide* on the segment $[t_0, T]$ of the pursuers $P_1, \ldots, P_n$ in the game Γ.

Let $\Delta = \{t_0 = \tau_0 < \tau_1 < \ldots < \tau_r = T\}$ be a partition of the segment $[t_0, T]$. On the interval $[\tau_0, \tau_1)$ we choose the permanent control of the pursuers P_i, $i \in I$,

$$u_i^0(t) = U_i(z_i^0, w_i^0), \quad i \in I, \quad t \in [\tau_0, \tau_1).$$

The selected control $u_i^0(t)$ paired with some control $v(t),\ t \in [\tau_0, \tau_1)$, evader E generates motion $z_i(t),\ t \in [\tau_0, \tau_1)$, system (3) on the segment $[\tau_0, \tau_1)$.

On the segment $[\tau_0, \tau_1)$ in system (4), we set the control $\tilde{v}(t)$ of the evader $\tilde{E}$ and quasi-strategy $\tilde{U}_1, \ldots, \tilde{U}_n$ of pursuers $\tilde{P}_1, \ldots, \tilde{P}_n$. Thus, the functions $w_i(t),\ i \in I$, are defined, which are solutions of system (4) on the segment $[\tau_0, \tau_1)$.

At the instant $t = \tau_1$, based on the values $z_1(\tau_1), \ldots, z_n(\tau_1)$, $w_1(\tau_1), \ldots,\ w_n(\tau_1)$, select the permanent controls

$$u_i^1(t) = U_i(z_i(\tau_1), w_i(\tau_1)), \quad i \in I, \quad t \in [\tau_1, \tau_2),$$

of the pursuers $P_1, \ldots, P_n$ in the game Γ. The selected controls $u_i^1(t),\ i \in I$, of the pursuers $P_i,\ i \in I$, paired with some control $v(t),\ t \in [\tau_1, \tau_2)$, of the evader E, generates motion $z_i(t)$ system (3) on the segment $[\tau_1, \tau_2)$.

On the segment $[\tau_1, \tau_2)$ in system (4), we set the control $\tilde{v}(t)$ of the evader $\tilde{E}$ and quasi-strategy $\tilde{U}_1, \ldots, \tilde{U}_n$ of pursuers $\tilde{P}_1, \ldots, \tilde{P}_n$. Thus, the functions $w_i(t),\ i \in I$, are defined, which are solutions of system (4) on the segment $[\tau_1, \tau_2)$.

Let $z_i(\tau_s), w_i(\tau_s),\ i \in I$, be the positions in which the system (3) and the guide system (4) came at the instant $t = \tau_s$. At the instant $t = \tau_s$, based on the values $z_i(\tau_s), w_i(\tau_s),\ i \in I$, select the permanent controls

$$u_i^s(t) = U_i(z_i(\tau_s), w_i(\tau_s)), \quad i \in I, \quad t \in [\tau_s, \tau_{s+1}),$$

of the pursuers $P_1, \ldots, P_n$ in the game Γ.

In system (4), we set the control $\tilde{v}(t)$ of the evader $\tilde{E}$ and quasi-strategy $\tilde{U}_1, \ldots, \tilde{U}_n$ of pursuers $\tilde{P}_1, \ldots, \tilde{P}_n$ on the segment $[\tau_s, \tau_{s+1})$.

We continue this process until T.

3. Capture of the evader in the class of quasi-strategies

Definition 3.1. There is a *capture in the class of quasi-strategies* from the initial positions $z^0 = (z_1^0, \ldots, z_n^0)$ in the game Γ if there are an instant $T > t_0$, a quasi-strategy $U_1, \ldots, U_n$ of the pursuers $P_1, \ldots, P_n$ such that, for any measurable function $v(\cdot)$, $v(t) \in V$, $t \in [t_0, T]$, there are an instant $\tau \in [t_0, T]$ and a number $p \in I$ such that $z_p(\tau) = 0$. □

Definition 3.2. [20] A matrix function $A\colon \mathbb{R}^1 \to M_{kk}$ (M_{kk} is the set of all square matrices of dimension $k \times k$) is called *recurrent* (by V. I. Zubov) iff for any $\varepsilon > 0$ there exists $T(\varepsilon) > 0$ such that for any $a, t \in \mathbb{R}^1$, there exists $\tau(t) \in [a, a + T(\varepsilon)]$, for which it is true that $\|A(t + \tau(t)) - A(t)\| < \varepsilon$.

A matrix function $A\colon [t_0, \infty) \to M_{kk}$ is called *recurrent* on $[t_0, \infty)$ iff there exists a recurrent matrix function $F\colon \mathbb{R}^1 \to M_{kk}$ such that $A(t) = F(t)$ for all $t \in [t_0, \infty)$.

If for each $\varepsilon > 0$, the number $\tau(t)$ is independent of t, then the function f is called *almost periodic*. □

Denote by $\Phi(t)$ the fundamental matrix of the system

$$\dot{\omega} = A(t)\omega, \quad \Phi(t_0) = E,$$

where E is the identity matrix.

Assumption 3.3. *The fundamental matrix Φ is a recurrent function, and its derivative is uniformly bounded on $[t_0, \infty)$.*

We denote by Int$X$, co X, $\overline{X}$ respectively, the interior, the convex hull and closure of the set X.

$$\lambda(h, v) = \sup\{\lambda \geq 0 \mid -\lambda h \in V - v\}, \ D_\varepsilon(a) = \{z \in \mathbb{R}^k \mid \|z - a\| < \varepsilon\}.$$

Lemma 3.4. *Suppose that for some $b_1, \ldots, b_n \in \mathbb{R}^k$ with $b_i \neq 0$ for all $i \in I$ we have*

$$\min_{v \in V} \max_{i \in I} \lambda(b_i, v) > 0\,.$$

Then there exists $\varepsilon > 0$ such that for any $z_1, \ldots, z_n \in \mathbb{R}^k$ and $z_i \in D_\varepsilon(b_i)$, $i \in I$, the following inequality holds:

$$\min_{v \in V} \max_{i \in I} \lambda(z_i, v) > 0.$$

Proof. By Lemma 1.3.13 [12, p. 30], the function $\lambda(b, v)$ is continuous on the set $B \times V$, where B is an compact set such that $0 \notin B$. Therefore, the functions

$$g(z_1, \ldots, z_n, v) = \max_{i \in I} \lambda(z_i, v) \ \text{ and } \ G(z_1, \ldots, z_n) = \min_{v \in V} g(z_1, \ldots, z_n, v).$$

are continuous. The assertions of the lemma follow from the continuity of the function G. The lemma is proved. □

Further, let $F_i(t) = \int\limits_0^t \lambda(\Phi(s) z_i^0, v(s)) ds$, $i \in I$, $D(\varepsilon) = D_\varepsilon(z_1^0) \times \ldots \times D_\varepsilon(z_n^0)$.

Lemma 3.5. *Suppose that Assumption* 3.3 *holds and*

$$\min_{v \in V} \max_{i \in I} \lambda(z_i^0, v) > 0;$$

then there exists an instant $T > t_0$ such that for any admissible function $v(\cdot)$ there exists $p \in I$ such that $F_p(T) \geq 1$.

Proof. By Lemma 3.4, there exists $\varepsilon > 0$ such that for all $z_i \in D_\varepsilon(z_i^0)$, $i \in I$, the inequality $\min\limits_{v \in V} \max\limits_{i \in I} \lambda(z_i, v) > 0$ holds. Let $0 < \varepsilon_0 < \varepsilon$ such that

$$\delta = \min_{z \in \overline{D}(\varepsilon_0)} \min_{v \in V} \max_{i \in I} \lambda(z_i, v) > 0.$$

Denote by $\mu(A)$ the Lebesgue's measure of the set A,

$$\Delta = \{t \geq t_0 \big| \ \Phi(t)z_i^0 \in D_{\varepsilon_0}(z_i^0) \text{ for all } i \in I\}, \quad d = \max_i \|z_i^0\|, \quad M = \sup_{t \geq t_0} \|\dot{\Phi}(t)\|.$$

Since Φ is a recurrent function, there exists $T_0 > 0$ such that for every $s = 1, 2, \ldots,$ there is an instant $\tau_s \in [(s-1)T_0, sT_0)$, for which

$$\|\Phi(t_0 + \tau_s) - \Phi(t_0)\| < \frac{\varepsilon_0}{2d}.$$

Let $t_s = t_0 + \tau_s$. Then $\|\Phi(t_s)z_i^0 - z_i^0\| = \|\Phi(t_s)z_i^0 - \Phi(t_0)z_i^0\| < \frac{\varepsilon_0}{2}$.
Consequently, $\Phi(t_s)z_i^0 \in D_{\varepsilon_0/2}(z_i^0)$ for all $i \in I$. From the mean value theorem [1, p. 135], it follows that for any T_1, T_2, i the inequality

$$\|\Phi(T_1)z_i^0 - \Phi(T_2)z_i^0\| \leq Md|T_1 - T_2|$$

holds. Therefore, if $\|\Phi(T_1)z_i^0 - \Phi(T_2)z_i^0\| \geq \frac{\varepsilon_0}{2}$, then $|T_1 - T_2| \geq \tau_0 = \dfrac{\varepsilon_0}{2Md}$. So

$$[t_s, t_s + \tau_0] \subset \Delta_s = \{t \in [t_s, t_{s+1}) \big| \ \Phi(t)z_i^0 \in D_{\varepsilon_0}(z_i^0) \text{ for all } i \in I\}.$$

Hence $\bigcup\limits_s \Delta_s \subset \Delta$ and therefore $\mu(\Delta) = +\infty$. Then, we have

$$\begin{aligned}\max_{i \in I} F_i(t) &= \max_{i \in I} \int_{t_0}^{t} \lambda(\Phi(s)z_i^0, v(s))ds \geq \max_{i \in I} \int_{[t_0,t]\cap\Delta} \lambda(\Phi(s)z_i^0, v(s))ds \\ &\geq \frac{1}{n} \int_{[t_0,t]\cap\Delta} \sum_{i \in I} \lambda(\Phi(s)z_i^0, v(s))ds \geq \frac{1}{n} \int_{[t_0,t]\cap\Delta} \max_{i \in I} \lambda(\Phi(s)z_i^0, v(s))ds \\ &\geq \frac{\delta}{n}\mu([t_0, t] \cap \Delta).\end{aligned}$$

Since $\mu(\Delta) = +\infty$, then $\lim\limits_{t \to \infty} \mu([t_0, t] \cap \Delta) = +\infty$. Therefore, for the instant T from the condition $\frac{\delta}{n}\mu([t_0, T] \cap \Delta) \geq 1$ we obtain the inequality $\max\limits_{i \in I} F_i(T) \geq 1$. The lemma is proved. □

Remark 3.6. From the method of construction u_i it follows that the constructed strategies satisfy the definition of quasi-strategies. □

Let
$$\hat{T} = \min\{t \geq t_0 \big| \ \inf_{v(\cdot)} \max_{i \in I} F_i(t) \geq 1\}. \tag{5}$$

By Lemma 3.5, $\hat{T} < +\infty$.

Theorem 3.7. *Suppose that Assumption* 3.3 *holds and that*

$$\min_{v \in V} \max_{i \in I} \lambda(z_i^0, v) > 0; \tag{6}$$

then in the game Γ *capture occurs in the class of quasi-strategies.*

Proof. By Cauchy's formula, the solution of problem (3) for any admissible control has the form

$$z_i(t) = \Phi(t)\big(z_i^0 + \int_{t_0}^{t} \Phi^{-1}(s)(u_i(s) - v(s))ds\big). \tag{7}$$

Let $v(\cdot)$ be an arbitrary admissible control of the evader E. It follows from the definition of the instant $\hat{T}$ that there exists an instant $\tau \in [t_0, \hat{T}]$, being the root of the function

$$F(t) = 1 - \max_{i\in I} \int_{t_0}^{t} \lambda(\Phi(s)z_i^0, v(s))ds$$

and a number $p \in I$ such that

$$1 - \int_{t_0}^{\tau} \lambda(\Phi(s)z_p^0, v(s))ds \leq 0.$$

Let $t_\alpha \in [t_0, \infty)$ be the least root of the function $1 - F_\alpha(t), \alpha \in I$, if t_α exists.

Note that $t_\alpha \in [t_0, \tau]$ for all $\alpha \in I$.

We define the controls of the pursuers P_i, $i \in I$, setting

$$u_i(t) = \begin{cases} v(t) - \lambda(\Phi(t)z_i^0, v(t))\Phi(t)z_i^0, t \in [t_0, \min\{t_i, \tau\}], \\ v(t), t \in [\min\{t_i, \tau\}, \hat{T}]. \end{cases}$$

Then from (7), we obtain

$$\begin{aligned} z_p(T) &= \Phi(T)\big(z_p^0 - \int_{t_0}^{T} \lambda(\Phi(s)z_p^0, v(s))ds z_p^0\big) \\ &= \Phi(T)z_p^0\big(1 - \int_{t_0}^{\tau} \lambda(\Phi(s)z_p^0, v(s))ds\big) = 0. \end{aligned}$$

The proof of the theorem is complete. □

Corollary 3.8. *Suppose that Assumption 3.3 holds, V is a strictly convex compact set with smooth boundary, and*

$$0 \in \operatorname{Int} \operatorname{co}\{z_1^0, \ldots, z_n^0\}. \tag{8}$$

Then in the game Γ capture occurs in the class of quasi-strategies.

Proof. From condition (8) it follows that $\delta = \min_{v\in V} \max_{i\in I} \lambda(z_i^0, v) > 0$.

By Theorem 3.7, in the game Γ capture occurs in the class of quasi-strategies. □

Corollary 3.9. *Suppose that $A(t) = A$ for all $t \geq t_0$, that all the eigenvalues of the matrix A are purely imaginary and pairwise distinct, and that the inequality* (6) *holds. Then in the game Γ capture occurs in the class of quasi-strategies.*

Proof. It follows from the assumption that the fundamental matrix Φ is almost periodic, and therefore recurrent. □

Corollary 3.10 *Suppose that V is a strictly convex compact set with a smooth boundary, Assumption 3.3 holds, and there exists an instant $\tau \geq t_0$ such that*

$$0 \in \operatorname{Int} \operatorname{co}\{\Phi(\tau) z_i^0,\ i \in I\}.$$

Then in the game Γ capture occurs in the class of quasi-strategies.

Proof. Setting $u_i(t) = v(t)$ for all $t \in [t_0, \tau]$, we obtain $z_i(\tau) = \Phi(\tau) z_i^0$. Taking the instant τ for the initial, we obtain the validity of the assertion. □

4. Capture of the evader in the class of positional strategies with guide

Definition 4.1 There is a capture in the class of positional strategies with guide from the initial positions $z^0 = (z_1^0, \ldots, z_n^0)$ in the game Γ if for any $\varepsilon > 0$ there is an instant $T(\varepsilon) > t_0$, positional control strategies with a guide $U_1, \ldots, U_n$ of the pursuers $P_1, \ldots, P_n$ such that, for any measurable function $v(\cdot)$, $v(t) \in V$, $t \in [t_0, T(\varepsilon)]$, there are an instant $\tau \in [t_0, T(\varepsilon)]$ and number $p \in I$ such that $\|z_p(\tau)\| < \varepsilon$.

Theorem 4.2. *Suppose that Assumption* 3.3 *holds and*

$$\min_{v \in V} \max_{i \in I} \lambda(z_i^0, v) > 0.$$

Then in the game Γ capture occurs in the class of positional strategies with guide.

Proof. As $T(z^0)$, we take the moment $\hat{T}$, defined by formula (5).

Let $\Delta = \{t_0 = \tau_0 < \tau_1 < \ldots < \tau_r = \hat{T}\}$ is the partition of the segment $[t_0, \hat{T}]$. Consider the game $\tilde{\Gamma}$, described by system (4). We select $w_i^0 = z_i^0$ for all $i \in I$.

Step 1. Consider the segment $[\tau_0, \tau_1]$. We assign the constant control $\tilde{v}$ of the evader $\tilde{E}$ in the game $\tilde{\Gamma}$ to be arbitrary, and the control of the pursuers $\tilde{P}_i$ in the game $\tilde{\Gamma}$ is assumed

$$\tilde{u}_i(t) = \begin{cases} \tilde{v}(t) - \lambda(\Phi(t) z_i^0, \tilde{v}(t)) \Phi(t) z_i^0, & t \in [\tau_0, \tau_0^i], \\ \tilde{v}(t), \quad t \in [\tau_0^i, \tau_1], \end{cases}$$

where $\tau_0^i \leq \tau_1$ is the root of the function $F_i(t) = 1 - \int_{\tau_0}^t \lambda(\Phi(s) z_i^0, \tilde{v}(s)) ds$, if it exists. The control u_i pursuers $P_i, i \in I$, in the game Γ on $[\tau_0, \tau_1)$ is chosen arbitrary and constant.

Step 2. Consider the segment $[\tau_1, \tau_2]$. Let $s_i(\tau_1) = z_i(\tau_1) - w_i(\tau_1)$. Select the constant control $\tilde{v}$ of the evader $\tilde{E}$ in the game $\tilde{\Gamma}$ so that the equality

$$\Big(\sum_{i \in I} s_i(\tau_1), \tilde{v}\Big) = \min_{v \in V} \Big(\sum_{i \in I} s_i(\tau_1), v\Big)$$

holds. The control $\tilde{u}_i$ of the pursuers $\tilde{P}_i$ in the game $\tilde{\Gamma}$ on $[\tau_1, \tau_2)$ assume

$$\tilde{u}_i(t) = \begin{cases} \tilde{v}(t) - \lambda(\Phi(t)z_i^0, \tilde{v}(t))\Phi(t)z_i^0, & t \in [\tau_1, \tau_1^i], \\ \tilde{v}(t), \quad t \in [\tau_1^i, \tau_2], \end{cases}$$

where $\tau_1^i \le \tau_2$ is the root of the function $F_i(t) = 1 - \int_{\tau_0}^t \lambda(\Phi(s)z_i^0, \tilde{v}(s))ds$, if it exists.

The control u_i^1 pursuers P_i, $i \in I$ in the game Γ choose constant on $[\tau_1, \tau_2]$ so that the following equality holds:

$$(s_i(\tau_1), u_i^1) = \min_{u_i \in V}(s_i(\tau_1), u_i).$$

Step 3. Consider further the segment $[\tau_l, \tau_{l+1}]$. Let $s_i(\tau_l) = z_i(\tau_l) - w_i(\tau_l)$. Select the constant control $\tilde{v}$ of the evader $\tilde{E}$ in the game $\tilde{\Gamma}$ so that the equality

$$\Big(\sum_{i\in I} s_i(\tau_l), \tilde{v}\Big) = \min_{v\in V}\Big(\sum_{i\in I} s_i(\tau_l), v\Big). \tag{9}$$

holds. The control $\tilde{u}_i$ of the pursuers $\tilde{P}_i$ in the game $\tilde{\Gamma}$ on $[\tau_l, \tau_{l+1})$ assume

$$\tilde{u}_i(t) = \begin{cases} \tilde{v}(t) - \lambda(\Phi(t)z_i^0, \tilde{v}(t))\Phi(t)z_i^0, & t \in [\tau_l, \tau_l^i], \\ \tilde{v}(t), \quad t \in [\tau_l^i, \tau_{l+1}], \end{cases}$$

where $\tau_l^i \le \tau_{l+1}$ is the root of the function $F_i(t) = 1 - \int_{\tau_0}^t \lambda(\Phi(s)z_i^0, \tilde{v}(s))ds$, if it exists.

The control u_i^l pursuers P_i, $i \in I$ in the game Γ choose constant on $[\tau_l, \tau_{l+1}]$ so that the following equality holds:

$$(s_i(\tau_l), u_i^l) = \min_{u_i \in V}(s_i(\tau_l), u_i), \; i \in I. \tag{10}$$

Thus, the pursuers P_i, $i \in I$, in system (3) build their controls step by step, knowing at each step the real position of the game and the position in the auxiliary guide system (4). Definew

$$M = \max_{t\in[t_0, \hat{T}]} \|A(t)\|, \; d = \max_{v\in V} \|v\|, \quad \delta = \max_l(\tau_{l+1} - \tau_l), \; s_i(t) = z_i(t) - w_i(t).$$

Note that for any measurable functions $u_i, v\colon [t_0, \hat{T}] \to V$ for solving Cauchy's problem of system (3) the inequality

$$\begin{aligned} \|z_i(t)\| &\le \|z_i^0\| + \int_0^t \|A(s)z_i(s) + u_i(s) - v(s)\|ds \\ &\le \|z_i^0\| + 2d(t - t_0) + M\int_{t_0}^t \|z_i(s)\|ds, \end{aligned}$$

holds for all $t \in [t_0, \hat{T}]$.

Therefore, by the Gronwall-Bellman inequality for all $t \in [t_0, \hat{T}]$ the inequality

$$\|z_i(t)\| \le B, \text{ where } B = \|z_i^0\| + 2d(\hat{T} - t_0) + e^{M(\hat{T}-t_0)}.$$

holds. This means for all $i \in I$, $t \in [t_0, \hat{T}]$, the inequality

$$\|\dot{z}_i(t)\| \le K, \text{ where } K = MB + 2d,$$

holds. Consequently, for all $t_1, t_2 \in [t_0, \hat{T}]$, we have the inequalities

$$\|z_i(t_1) - z_i(t_2)\| \le K|t_1 - t_2|, \quad \|w_i(t_1) - w_i(t_2)\| \le K|t_1 - t_2|,$$
$$\|s_i(t_1) - s_i(t_2)\| \le 2K|t_1 - t_2|.$$

Let $t \in [\tau_l, \tau_{l+1}]$. Then for an arbitrary function $v(\cdot)$

$$\begin{aligned} \frac{d}{dt}\|s_i(t)\|^2 &= 2(s_i(t), \dot{s}_i(t)) = 2(s_i(t), A(t)s_i(t) + u_i^l - \tilde{u}_i(t) + \tilde{v}(t) - v(t)) \\ &= 2(s_i(t), A(t)s_i(t)) + 2(s_i(t), u_i^l - \tilde{u}_i(t)) + 2(s_i(t), \tilde{v}(t) - v(t)). \end{aligned} \tag{11}$$

From inequality (10) it follows that for all $t \in [\tau_l, \tau_{l+1}]$ the inequality

$$(s_i(\tau_l), u_i^l - \tilde{u}_i(t)) \le 0,$$

holds. Therefore,

$$(s_i(t), A(t)s_i(t)) \le M\|s_i(t)\|^2, \tag{12}$$

$$\begin{aligned} &(s_i(t), u_i^l - \tilde{u}_i(t)) = (s_i(\tau_l), u_i^l - \tilde{u}_i(t)) + (s_i(t) - s_i(\tau_l), u_i^l - \tilde{u}_i(t)) \\ &\le (s_i(t) - s_i(\tau_l), u_i^l - \tilde{u}_i(t)) \le 2d\|s_i(t) - s_i(\tau_l)\| \le 4dK|t - \tau_l| \le 4dK\delta. \end{aligned} \tag{13}$$

Consequently, from equality (11) and inequalities (12), (13) we obtain

$$\frac{d}{dt}\Big(\sum_{i\in I}\|s_i(t)\|^2\Big) \le 2M\sum_{i\in I}\|s_i(t)\|^2 + 4dKn\delta + 2\Big(\sum_{i\in I} s_i(t), \tilde{v} - v(t)\Big). \tag{14}$$

As for all $t \in [\tau_l, \tau_{l+1}]$ we have $\Big(\sum\limits_{i\in I} s_i(\tau_l), \tilde{v} - v(t)\Big) \le 0$, then

$$\Big(\sum_{i\in I} s_i(t), \tilde{v}-v(t)\Big) = \Big(\sum_{i\in I} s_i(\tau_l), \tilde{v}-v(t)\Big) + \Big(\sum_{i\in I}(s_i(t)-s_i(\tau_l)), \tilde{v}-v(t)\Big) \le 4Kdn\delta\,.$$

Therefore, from inequality (14) it follows that for all $t \in [t_0, \hat{T}]$ the inequality

$$\frac{d}{dt}\Big(\sum_{i\in I}\|s_i(t)\|^2\Big) \le 2M\sum_{i\in I}\|s_i(t)\|^2 + C\delta.$$

holds. From the condition of the problem $\sum\limits_{i\in I}\|s_i(t_0)\| = 0$ is true.

Consequently, from the last inequality we get

$$\sum_{i\in I}\|s_i(t)\|^2 \leq C\delta(t-t_0) + 2M\int\limits_{t_0}^{t}\Big(\sum_{i\in I}\|s_i(t)\|^2\Big)dt.$$

By the Gronwall-Bellman inequality, we have

$$\sum_{i\in I}\|s_i(t)\|^2 \leq C\delta(t-t_0)e^{2M(t-t_0)} \leqslant C\delta(\hat{T}-t_0)e^{2M(\hat{T}-t_0)}.$$

Taking $\delta > 0$ so that the inequality $C\delta(\hat{T}-t_0)e^{2M(\hat{T}-t_0)} < \varepsilon^2$ holds, we finally get $\sum_{i\in I}\|s_i(\hat{T})\|^2 < \varepsilon^2$. Therefore, $\|s_i(\hat{T})\| < \varepsilon$ for all i. Since $w_p(\hat{T}) = 0$ with some p, then $\|z_p(\hat{T})\| < \varepsilon$. The theorem is proved. □

Corollary 4.3. *Suppose that Assumption* 3.3 *holds, V is a strictly convex compact set with a smooth boundary and inequality* (6) *is satisfied. Then in the game* Γ *capture occurs in the class of positional strategies with guide.*

Example 4.4. Let in the system (3) $A(t) = 0$ for all t and inequality (6) is satisfied. Then in the game Γ, capture occurs in class of positional strategies with guide.

Example 4.5. Let $t_0 = 0$ in system (3), and the matrix $A(t) = a(t)E$, where

$$a(t) = \begin{cases} 0, & \text{if } t \in [0, 2\pi], \\ \sin t, & \text{if } t > 2\pi. \end{cases}$$

Then the matrix $\Phi(t)$ has the form

$$\Phi(t) = \begin{cases} E, & \text{if } t \in [0, 2\pi], \\ e^{1-\cos t}E, & \text{if } t > 2\pi. \end{cases}$$

The function $\Phi(t)$ is recurrent and, therefore, Assumption 1 is fulfilled. We note that the function $\Phi(t)$ is not almost periodic. Let inequality (6) is satisfied. Then in the game Γ, capture occurs in class of positional strategies with guide.

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