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Hamiltonian Systems for Control Reconstruction Problems

Nina N. Subbotina, Evgenii A. Krupennikov
Krasovskii Institute of Mathematics and Mechanics, Ural Branch of the Russian Academy of Sciences, Yekaterinburg 620108, Russia;
Ural Federal University, Yekaterinburg 620002, Russia
subb@uran.ru, krupennikov@imm.uran.ru

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We present and justify a new algorithm for solving dynamic reconstruction problems. The algorithm is based on solutions of Hamiltonian systems that arise in auxiliary variational problems with concave-convex discrepancy functionals. Modifications of Hamiltonian systems, improving the effectiveness of the new algorithm, are suggested and justified. Results of numerical solution of a reconstruction problem in medicine are reported.

Keywords: Dynamic reconstruction problems, convex-concave discrepancy, variational problems, Hamiltonian systems.

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1. Introduction

This paper is devoted to the problem of dynamic reconstruction of a control, generating the realized trajectory of a dynamic controlled system. The reconstruction bases on inaccurate measurements of the realized trajectory, which are discrete and arrive step-by-step in real-time.

Dynamic reconstruction problems are actual and have been studied by many authors. See, for example, [1, 9, 11, 12, 13], where many methods were suggested to solve these inverse problems. These methods are based on ideas and results of algebra, geometry, functional analysis, the theory of perturbations and so on.

A well-known approach has been suggested by A. V. Kryazhimskii and Yu. S. Osipov [7, 8]. This approach uses a regularized (Tikhonov regularization [18]) procedure of extremal aiming on an auxiliary stable model, a guide. This idea has roots in the works of N. N. Krasovskii's school on the theory of optimal feedback control [4].

Many new reconstruction problems arise in modern applications and require new effective methods.

A new approach to dynamic reconstruction problems was suggested by N. N. Subbotina, E. A. Krupennikov and T. B. Tokmantcev in [5, 6, 14, 15, 16, 17]. It also uses auxiliary extremal problems. Variational problems on extremum of a regularized integral discrepancy functional are introduced. Necessary optimality

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conditions in these variational problems have the form of Hamiltonian ODE's. These conditions are used as a base to solve reconstruction problems. This approach allows to reduce the reconstruction problem to integration of systems of ODE's. Similarly to the procedure, based on extremal aiming on an auxiliary dynamics of a stable model-guide, the new approach bases on Hamiltonian system of ODE's, constructed by expanding the dynamics with so-called adjoint system of ODE's.

The key feature of the approach is application of a discrepancy functional, which is convex in controls and concave in state variables. This allows to construct a Hamiltonian system in the variational problem, whose solutions have stable oscillating character and can be considered as the reconstruction problem solution.

We consider the dynamics to be affine in controls. The number of the control parameters is assumed to be greater or equal to the number of the state variables of the controlled system.

The novel result presented in this paper is the description and justification of a new algorithm for solving dynamic reconstruction problems. This algorithm is a modification and improvement of the algorithms suggested in the papers [5, 6, 14, 15, 16, 17].

In comparison with the previously developed versions, the new algorithm simplifies the structure of Hamiltonian systems that are to be numerically integrated.

The numerical simulation of the algorithm's application is performed on an example from the area of medicine [20].

2. Statement

The following control reconstruction problem is under consideration.

2.1. Model

We consider controlled systems with dynamics of the form

$$\frac{dx(t)}{dt} = G(t,x(t))u(t) + f(t,x(t)), \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ t \in [0,T], \tag{1}$$

$$m \geq n, \quad T < \infty, \tag{2}$$

where x is the vector of the state coordinates and u is the vector of the controls.

We understand admissible controls as measurable functions satisfying the restriction

$$u(t) \in \mathbf{U} \subset \mathbb{R}^m, \quad t \in [0,T], \tag{3}$$

where $\mathbf{U}$ is a compact convex set.

2.2. Input Data

We assume that one can observe some real motion of the system (1), namely, a trajectory $x^*(\cdot)\colon [0,T] \to \mathbb{R}^n$, which is generated by an admissible control. Discrete inaccurate measurements of this 'basic' trajectory arrive step-by-step in real time:

$$\{y_k^\delta :\ \|y_k^\delta - x^*(t_k)\| \leq \delta, \quad t_k = k\Delta t, \quad k = \overline{0,N}, \quad T = N\Delta t\}. \tag{4}$$

where $\delta \in (0, \delta^*]$ is the measurement error and $\Delta t \in (0, \Delta^*]$ is the measuring step. The symbol $\|\cdot\|$ denotes the Euclidean norm.

An admissible control that generates the basic trajectory is unknown and is to be reconstructed.

2.3. Assumptions

We consider the model (1)–(4) under the following assumptions. There exist positive constants $\delta_0 < \delta^*$, $\Delta_0 < \Delta^*$ and a compact set $\Psi \subset \mathbb{R}^n$ with

(A1) For any $\delta \in (0, \delta_0]$ and any $\Delta t \in (0, \Delta_0]$

$$\bigcup_{k=\overline{0,N}} B_{2\delta_0}[y_k^\delta] \subset \Psi.$$

The notation $B_r[x]$ means closed ball of the radius r with the center in x.

(A2) The elements of the matrix function $G(\cdot) : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and the vector function $f(\cdot) : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ from (1) are twice continuously differentiable with respect to all variables for each $(t,x) \in D_0 \overset{\triangle}{=} [0,T] \times \Psi$. These elements are not oscillating.

(A3) The rank of the matrix $G(t,x)$ equals n for each $(t,x) \in D_0$.

2.4. Control reconstruction problem

Let $\mathbb{U}^*$ be the set of admissible controls generating $x^*(\cdot)$. It is non-empty since $x^*(\cdot)$ is already generated by some admissible control. But, in general case, it may consist of more than one element. So, the control reconstruction problem for the basic trajectory is incorrect. To state the correct reconstruction problem we will consider the so-called 'normal' control, namely, an admissible control which belongs to the set $\mathbb{U}^*$ of all admissible controls that generate the basic trajectory and has the minimal norm in the space L^2.

To prove that the normal control is unique, consider the following auxiliary problem.

Problem 2.1. *For $t \in [0,T]$ find an element $u \in \mathbf{U}$, which minimises the expression $0.5u^2$ under the following additional restriction*

$$\frac{dx^*(t)}{dt} = f(t, x^*(t)) + G(t, x^*(t))u. \tag{5}$$

Following Lagrange, one can reduce this problem to the problem on conditional extremum [2].

Problem 2.2. *Find an element $u \in \mathbf{U}$, which minimises the expression*

$$0.5u^2 + \langle \psi, \frac{dx^*(t)}{dt} - f(t, x^*(t)) - G(t, x^*(t))u \rangle, \quad \psi \in R^n, \ \|\psi\| \neq 0. \tag{6}$$

In agreement with (A3) and [10], the last problem has the unique solution

$$u_0 = G^\top(t, x^*(t))\psi, \quad \psi = \Big(G(t, x^*(t))G^\top(t, x^*(t))\Big)^{-1}\left[\frac{dx^*(t)}{dt} - f(t, x^*(t))\right]. \quad (7)$$

The symbol $G^\top$ denotes the transpose. Therefore, the normal control $u^*(\cdot)$ is uniquely defined for each $t \in [0, T]$, where $dx^*(t)/dt$ exists.

So, we can consider the following dynamic reconstruction problem (DRP):

Problem 2.3. *For any $\delta \in (0, \delta_0]$, $\Delta t \in (0, \Delta_0]$ and the set of measurements $\{y_k^\delta\}$ (4), arriving in real-time with time step Δt, find an admissible control $u^\delta(\cdot) \in U[0, T]$ such that at the end instant T of the reconstruction process, the following relations are satisfied:*

(C1) *It generates a trajectory $x^\delta(\cdot) : [0, T] \to \mathbb{R}^n$ with the boundary condition $x^\delta(0) = y_0^\delta$ such that*

$$\lim_{\delta \to 0} \|x^\delta(\cdot) - x^*(\cdot)\|_{C[0,T]} = 0; \quad (8)$$

(C2)

$$u^\delta(\cdot) \overset{w^*}{\to} u^*(\cdot) \text{ as } \delta \to 0, \quad (9)$$

where the notion $\|x(\cdot)\|_{C[0,T]}$ means the norm in the space $C([0, T], R^n)$ and $\overset{w^*}{\to}$ denotes weak-convergence in the space $(C([0, T], R^n))^*$.*

3. DRP Solution

We suggest the following algorithm for constructing a solution to Problem 2.3 under the assumptions (A1)–(A3).

The algorithm is performed step-wise with the time step Δt. Each k-th step ($k = \overline{2, N+1}$) is performed when a new measurement point $y_k^\delta \in \mathbb{R}^n$ arrives. On each step the desired control u^δ is extended on the corresponding time interval $[t_{k-2}, t_{k-1}]$.

3.1. Measurements interpolation

On each step (at the instants t_k, $k = \overline{2, N+1}$, when a new measurement node arrives) we consequentially construct continuous interpolation $y^\delta(t) : [0, T] \to R^n$ of the measurements $\{y_k^\delta\}$ (4). The function $y^\delta(\cdot)$ must satisfy the following conditions:

(I1) It is twice continuously differentiable on each interval $[t_{k-1}, t_k]$, $k = \overline{1, N}$.

(I2) It satisfies the estimates

$$\|y^\delta(t)\| \le K, \quad \left\|\frac{dy^\delta(t)}{dt}\right\| \le K, \quad t \in [0, T]; \quad (10)$$

$$\left\|\frac{d^2 y^\delta(t)}{dt^2}\right\| \le K, \quad t \in [0, T] \setminus \Theta^\delta, \quad (11)$$

where the constant $K > 0$ is chosen based on assumptions (A1), (A2), and the sets Θ^δ have measures $\beta(\Theta^\delta) \le \beta^\delta(\Delta t) \overset{\Delta t \to 0}{\longrightarrow} 0$.

(I3) It approximates the basic trajectory:

$$\|y^\delta(t) - x^*(t)\| \le 2\delta, \quad t \in [0,T].$$

Remark 3.1. These conditions can be met by constructing cubic splines on intervals $[t_{k-2}, t_k]$ with nodes at the points $y^\delta_{k-2}, y^\delta_{k-1}, y^\delta_k$.

3.2. Auxiliary variational problem

Now, we introduce the following auxiliary variational problem (AVP) on the k-th step.

Problem 3.2. *Find a pair of functions*

$$(x_k(\cdot), u_k(\cdot)) \in C_1([t_{k-2}, t_k], R^n) \times C_1([t_{k-2}, t_k], R^m)$$

such that:

(1) *They satisfy equation* (1) *and the following structure holds:*

$$u_k(t) = -\frac{1}{\alpha^2} G^\top(t_{k-2}, y^\delta(t_{k-2}))s_k(t),$$
$$t \in [t_{k-2}, t_k], \quad s_k(\cdot) \in C_1([t_{k-2}, t_k], R^n), \quad k = \overline{2, N+1}. \tag{12}$$

(2) *They satisfy the boundary conditions*

$$x_2(0) = y^\delta(0), \quad u_2(0) = G^+(0, y^\delta(0))\Big(\frac{dy^\delta(0)}{dt} - f(0, y^\delta(0))\Big),$$
$$x_k(t_{k-2}) = y^\delta(t_{k-2}), \quad s_k(t_{k-2}) = s_{k-1}(t_{k-2}), \quad k = \overline{3, N+1}, \tag{13}$$

where $G^+ \overset{def}{=} G^T(GG^T)^{-1}$ *is the generalized inverse of the matrix* G [10].

(3) *They minimize the functional*

$$I(x(\cdot), u(\cdot)) = \int\limits_{t_{k-2}}^{t_k} \Big[-\frac{\|x(t) - y^\delta(t)\|^2}{2} + \frac{\alpha^2\|u(t)\|^2}{2}\Big]\, dt, \tag{14}$$

where α *is a small regularising parameter in the sense of* [18].

Note that since Assumption (A3) is true, $G^+(0, y^\delta(0))$ exists.

3.3. Necessary optimality conditions for AVP

One can get necessary optimality conditions for AVP (1),(12)–(14) in the Lagrangian form [2]. The Lagrangian for our AVP is

$$L(x, u, \dot{x}, \lambda, t) = -\frac{\|x - y^\delta(t)\|^2}{2} + \frac{\alpha^2\|u\|^2}{2} + \big\langle \lambda^T, \dot{x} - G(t,x)u - f(t,x)\big\rangle, \tag{15}$$

where λ is the Lagrange multipliers vector and $\langle \cdot, \cdot \rangle$ is the scalar product.

The corresponding Euler equations are, for $i = \overline{1,n}$,

$$\frac{d\lambda_i(t)}{dt}+(x_i(t)-y_i^\delta(t))+\sum_{j=1}^{n}\left[\lambda_j(t)\left(\sum_{k=1}^{m}\frac{\partial}{\partial x_i}g_{jk}(t,x(t))u_k(t)+\frac{\partial}{\partial x_i}f_j(t,x(t))\right)\right]=0, \quad (16)$$

$$\alpha^2 u(t) - G^T(t,x(t))\lambda(t) = 0. \quad (17)$$

Equation (17) defines the relation between the controls $u(t)$ and the Lagrange multipliers $\lambda(t)$. We can substitute it into (16) and (1) to rewrite them in the form of a Hamiltonian system, where $s(t) = -\lambda(t)$ is the adjoint variables vector:

$$\frac{dx_k(t)}{dt} = -\alpha^{-2}G(t,x_k(t))G^T(t,x_k(t))s_k(t) + f(t,x_k(t)),$$

$$\frac{s_{k,i}(t)}{dt} = x_{k,i}(t) - y_i^\delta(t) + \alpha^{-2}\left\langle s_k(t), \frac{\partial}{\partial x_{k,i}}\, G(t,x_k(t))\, G^T(t,x_k(t))s_k(t)\right\rangle +$$

$$+ \left\langle s_k(t), \frac{\partial}{\partial x_{k,i}} f(t,x_k(t))\right\rangle,$$

$$t \in [t_{k-2}, t_k], \quad i \in \overline{1,n}, \quad k = \overline{2, N{+}1} \quad (18)$$

with the boundary conditions

$$x_2(0) = y^\delta(0),\ s_2(0) = -\alpha^2\Big(G(0,y^\delta(0))\, G^T(0,y^\delta(0))\Big)^{-1}\left(\frac{dy^\delta(0)}{dt} - f(0,y^\delta(0))\right),$$

$$x_k(t_{k-1}) = y^\delta(t_{k-1}), \quad s_k(t_{k-1}) = s_{k-1}(t_{k-1}), \quad k = \overline{3, N{+}1}. \quad (19)$$

Remark 3.3. The suggested below algorithm uses equations based on the necessary optimality conditions (18),(19). However, it is not verified if the extremum is actually reached in the AVP, since it is only important that the solution of (18) provides just a stationary point of the functional (14), but not necessary an optimal point.

3.4. Auxiliary linearized dynamics

We construct a continuous interpolation $y^\delta(\cdot) : [0,T] \to R^n$ of measurements $\{y_k^\delta\}$ (4) of the states $x^*(t_k)$ step by step. The interpolation is based on each step (interval $[t_{k-2}, t_k]$, $k = \overline{2, N{+}1}$) on the points $y_{k-2}^\delta, y_{k-1}^\delta, y_k^\delta$ $(y_{N+1}^\delta = y_N^\delta)$ and the relations

$$y^\delta(0) = y^\delta(\Delta t) = y_0^\delta, \quad \frac{dy^\delta(0)}{dt} = f(0, y^\delta(0)), \quad (20)$$

and for $k = \overline{3, N+1}$,

$$y^\delta(t_{k-2}) = y_{k-2}^\delta,\ y^\delta(t_{k-1}) = y_{k-1}^\delta,\ y^\delta(t_k) = y_k^\delta,\ \frac{dy^\delta(t_{k-2})}{dt} = f(t_{k-2}, y^\delta(t_{k-2})). \quad (21)$$

On each step we consider the following boundary problem for a linearized version of the Hamiltonian system (18):

$$\begin{cases} \bar{z}_k(t) \triangleq \bar{x}_k(t) - y^\delta(t), \quad \dfrac{d\bar{z}_k(t)}{dt} = -\alpha^{-2}Q_k\bar{s}_k(t) + f_k - \dfrac{dy^\delta(t)}{dt}, \\ \dfrac{d\bar{s}_k(t)}{dt} = \bar{z}_k(t), \quad t \in [t_{k-2}, t_k], \quad k = \overline{2, N+1}, \\ \bar{s}_1(0) = -\alpha^2\Big(G(0, y^\delta(0))G^T(0, y^\delta(0))\Big)^{-1}\left(\dfrac{dy^\delta(0)}{dt} - f(0, y^\delta(0))\right), \\ \bar{x}_1(0) = y^\delta(0), \quad \bar{z}_k(t_{k-2}) = 0, \quad \bar{s}_k(t_{k-2}) = \bar{s}_{k-1}(t_{k-2}), \quad k = \overline{3, N+1}, \end{cases} \tag{22}$$

where

$$Q_k \triangleq G(y^\delta(t_{k-2}), t_{k-2})G^T(y^\delta(t_{k-2}), t_{k-2}), \quad f_k \triangleq f(y^\delta(t_{k-2}), t_{k-2}). \tag{23}$$

The system (22) has the unique solution $(\bar{x}_k(\cdot),\ \bar{s}_k(\cdot)) : [t_{k-2}, t_k] \to \mathbb{R}^n \times \mathbb{R}^n$. This solution is used as a base for construction of the solution of DRP (Problem 2.3). Namely, we consider the cut-off function

$$\hat{u}^{\alpha,\delta}(t) = \begin{cases} u^{\alpha,\delta}(t), & u^{\alpha,\delta}(t) \in \mathbf{U}, \\ \operatorname*{argmin}_{w \in \mathbf{U}} \|u^{\alpha,\delta}(t) - w\|, & u^{\alpha,\delta}(t) \notin \mathbf{U}. \end{cases}, \tag{24}$$

$$u^{\alpha,\delta}(t) = -\alpha^{-2}G^T(t_{k-2}, y^\delta(t_{k-2}))\bar{s}_k(t),\ t \in [t_{k-2}, t_{k-1}].$$

3.5. Main result

Theorem 3.4. *Let the Assumptions* (A1)–(A3) *hold. Then the controls* $\hat{u}^{\alpha,\delta}(t)$ *of the form* (24) *solve* DRP (C1), (C2).

4. Proof of Theorem 3.4

4.1. Properties of auxiliary linearized dynamics

Consider the properties of the $n \times n$-matrices Q_k from (23).

The matrices $G(t_{k-2}, y^\delta(t_{k-2}))$, $k = \overline{2, N+1}$ are non-degenerate because of Assumption (A3). Therefore, see [10], the matrices Q_k are positive definite. According to the matrix theory in [10], the Schur decomposition

$$Q_k = H_k^\top \Lambda_k H_k \tag{25}$$

holds, where Λ_k is the diagonal $n \times n$-matrix with the diagonal elements $\lambda_{k,i}$, $i \in \overline{1, n}$, that are the positive eigenvalues of the matrix Q_k and the $n \times n$-matrix H_k is orthogonal (that is $H_k^\top = H_k^{-1}$) and it's i-th row $h_{k,i}$ is the eigenvector of Q_k corresponding to it's eigenvalue $\lambda_{k,i}$.

Consider now the system of nonhomogeneous linear ODEs (22).

We fix the interval $[t_{k-2}, t_k]$, $k \in \overline{2, N+1}$ and introduce the new variables

$$\breve{z}_k = H_k\bar{z}_k, \qquad \breve{s}_k = H_k\bar{s}_k. \tag{26}$$

The boundary problem (22) has the following form in these variables:

$$\frac{d\breve{z}_k(t)}{dt} = -\frac{1}{\alpha^2}\Lambda_k\breve{s}_k + H_k\Big(f_k - \frac{dy^\delta(t)}{dt}\Big), \quad \frac{d\breve{s}_k(t)}{dt} = \breve{z}_k \tag{27}$$

with the boundary conditions

$$\breve{z}_k(t_{k-2}) = 0, \quad \breve{s}_k(t_{k-2}) = \breve{s}_{k-1}(t_{k-2}) = H_{k-1}\bar{s}_{k-1}(t_{k-2}). \tag{28}$$

Since Λ_k in (27) is diagonal, we can reduce the $2n$-dimensional problem (27),(28) to n problems for each pair of coordinates $(\breve{z}_{k,i}(\cdot), \breve{s}_{k,i}(\cdot))$, $i \in \overline{1,n}$:

$$\frac{d\breve{z}_{k,i}(t)}{dt} = -\frac{1}{\alpha^2}\lambda_{k,i}\breve{s}_{k,i} + \Big\langle h_{k,i}, f_k - \frac{dy^\delta(t)}{dt}\Big\rangle, \quad \frac{d\breve{s}_{k,i}(t)}{dt} = \breve{z}_{k,i}(t) \tag{29}$$

with the boundary conditions

$$\breve{z}_{k,i}(t_{k-2}) = 0, \quad \breve{s}_{k,i}(t_{k-2}) = \breve{s}_{k-1,i}(t_{k-2}) = \langle h_{k-1,i}, \bar{s}_{k-1}(t_{k-2})\rangle, \tag{30}$$

where the vector $h_{k,i}$ is the eigenvector of Q_k corresponding to it's eigenvalue $\lambda_{k,i}$ and $\|h_{k,i}\| = 1$.

The solutions $(\breve{z}_{k,i}(\cdot), \breve{s}_{k,i}(\cdot))$, $i \in \overline{1,n}$, of the systems of nonhomogeneous linear ODEs (29)–(30), $k = \overline{2, N+1}$, can be obtained by the Cauchy formula:

$$\begin{aligned}|\breve{z}_{k,i}(t)| = &-\frac{\sqrt{\lambda_{k,i}}}{\alpha}\sin\Big(\frac{\sqrt{\lambda_{k,i}}}{\alpha}(t - t_{k-2})\Big)\breve{s}_{k,i}(t_{k-2}) + \\ &+ \int_{t_{k-2}}^{t}\cos\Big(\frac{\sqrt{\lambda_{k,i}}}{\alpha}(t-\tau)\Big)\Big\langle h_{k,i}, f_k - \frac{dy^\delta(\tau)}{d\tau}\Big\rangle d\tau,\end{aligned} \tag{31}$$

$$\begin{aligned}|\breve{s}_{k,i}(t)| = &\cos\Big(\frac{\sqrt{\lambda_{k,i}}}{\alpha}(t - t_{k-2})\Big)\breve{s}_{k,i}(t_{k-2}) + \\ &+ \int_{t_{k-2}}^{t}\frac{\alpha}{\sqrt{\lambda_{k,i}}}\sin\Big(\frac{\sqrt{\lambda_{k,i}}}{\alpha}(t-\tau)\Big)\Big\langle h_{k,i}, f_k - \frac{dy^\delta(\tau)}{d\tau}\Big\rangle d\tau,\end{aligned} \tag{32}$$

where $t \in [t_{k-2}, t_k]$, $i = \overline{1,n}$. From (20),(21),(31),(32) and the Cauchy–Bunyakovsky–Schwarz inequality one can get the following estimates at $t = t_1$:

$$|\breve{z}_{k,i}(t_1)| \le \Big[\frac{(\Delta t)^3}{2}(\Delta t + \frac{\alpha}{2\sqrt{\lambda_*}})\Big]^{1/2}\Big[2K(1 + \frac{\beta^\delta}{(\Delta t)^3})\Big]^{1/2}\sqrt{2K}\|h_{k,i}\| \overset{\triangle}{=} r(\Delta t, \alpha), \tag{33}$$

$$|\breve{s}_{k,i}(t_1)| \le \frac{\alpha}{\sqrt{\lambda_*}}r(\Delta t, \alpha). \tag{34}$$

We define $\lambda_* = \min\limits_{i\in\overline{1,n},(t,x)\in D_0}\lambda_i(t,x)$, and $\lambda^* = \max\limits_{i\in\overline{1,n},(t,x)\in D_0}\lambda_i(t,x)$, where $\lambda_i(t,x)$ are positive eigenvalues of the matrix $Q(t,x) = G(t,x)G^\top(t,x)$. Since H_k is orthogonal, $\|h_{k,i}\| = 1$, $i = \overline{1,n}$ [10].

We continue the recurrent estimates and get that at $t = t_k,\ k = \overline{2, N+1}$

$$|\breve{z}_{k,i}(t_k)| \le kK\frac{\sqrt{\lambda^*}}{\sqrt{\lambda_*}}r(\Delta t, \alpha), \tag{35}$$

$$|\breve{s}_{k,i}(t_k)| \le kK\frac{\alpha}{\sqrt{\lambda_*}}r(\Delta t, \alpha). \tag{36}$$

We get from (26),(35) and (36) the following estimates for the solutions $(\bar{z}(\cdot), \bar{s}(\cdot))$ of system (22) for $t \in [0, T]$:

$$\|\bar{z}(t)\| \le 2nN\frac{\sqrt{\lambda^*}}{\sqrt{\lambda_*}}r(\Delta t, \alpha), \tag{37}$$

$$\|\bar{s}(t)\| \le 2nN\frac{\alpha}{\sqrt{\lambda_*}}r(\Delta t, \alpha). \tag{38}$$

Consider the control $u^{\alpha,\delta}(t)$ from (24):

$$u^{\alpha,\delta}(t) = -\alpha^{-2}G^T(t_{k-2}, y^\delta(t_{k-2}))\bar{s}_k(t), \quad t \in [t_{k-2}, t_k], \quad k = \overline{2, N+1}. \tag{39}$$

We estimate it by application of (38):

$$\|u^{\alpha,\delta}(t)\| \le \frac{2nN}{\alpha\sqrt{\lambda_*}}\|G\| \cdot r(\Delta t, \alpha), \qquad t \in [0, T], \tag{40}$$

where
$$\|G\| = \max_{i\in\overline{1,n}, j\in\overline{1,m}, (t,x)\in\Psi} |g_{i,j}(t.x)|.$$

It follows from (33),(40) that

$$\|u^{\alpha,\delta}(t)\| \le \frac{2n^2T\sqrt{2K}}{\sqrt{\lambda_*}}\|G\| \left[\frac{\Delta t}{\alpha}\right]^{1/2} \left[\frac{1}{2}\left(\frac{\Delta t}{\alpha} + \frac{\alpha}{2\sqrt{\lambda_*}}\right)\right]^{1/2} \left[2K(1 + \frac{\beta^\delta}{(\Delta t)^3}\right]^{1/2}. \tag{41}$$

We assume
$$\lim_{\alpha\to 0, \Delta t\to 0} \frac{\Delta t}{\alpha} = K_0 < \infty, \qquad \lim_{\alpha\to 0, \Delta t\to 0} \frac{\beta^\delta(\Delta t)}{(\Delta t)^3} = \beta^0 < \infty. \tag{42}$$

It follows from (43) that the controls $u^{\alpha,\delta}(\cdot)$ are restricted, and for sufficiently small $\alpha, \Delta t$ the following relation holds

$$\|u^{\alpha,\delta}(t)\| \le K_1 < \infty. \tag{43}$$

4.2. Estimates for discrepancy

We substitute the controls $u^{\alpha,\delta}(\cdot)$ into the system (1) and get the dynamics

$$\frac{dx(t)}{dt} = G(t, x(t))u^{\alpha,\delta}(t) + f(t, x(t)), \quad t \in [0, T]. \tag{44}$$

Let us estimate for $t \in [t_{k-2}, t_{k-1}],\ k = \overline{2, N+1}$ the discrepancy

$$\|x(t) - \bar{x}_k(t)\| = \|(x(t) - y^\delta(t)) - (\bar{x}_k(t) - y^\delta(t))\| \stackrel{\triangle}{=} \|z(t_{k-1}) - \bar{z}_k(t_{k-1})\|, \tag{45}$$

where $x(\cdot) : [0, T] \to \mathbb{R}^n$ is the solution of (44) with the boundary condition $x(0) = y^\delta(0)$ and each $\bar{x}_k(\cdot)$ is the solution of (22).

Note that due to (22),(24)

$$\frac{d\bar{x}_k(t)}{dt} = G_k u^{\alpha,\delta}(t) + f_k, \quad t \in [t_{k-2}, t_{k-1}].$$

Also note that $x(0) = \bar{x}_0(0) = y^\delta(0)$, $\bar{x}_k(t_{k-2}) = y^\delta(t_{k-2})$.
We get from (45) for any $k = \overline{2, N+1}$, the inequality

$$\|z(t_{k-1}) - \bar{z}_k(t_{k-1})\| \le \|x(t_{k-2}) - y^\delta(t_{k-2})\| + \\ + \int_{t_{k-2}}^{t_{k-1}} \|f(\tau, x(\tau)) - f_k\| + \Big\|\Big(G(\tau, x(\tau)) - G_k\Big)u^{\alpha,\delta}(\tau)\Big\| d\tau. \tag{46}$$

It follows from the Assumptions (A1), (A2) that

$$\|f(\tau, x(\tau)) - f_k\| = \|f(\tau, x(\tau)) - f(t_{k-2}, y^\delta(t_{k-2})\|$$
$$\le \Big[\|f(\tau, x(\tau)) - f(\tau, x(t_{k-2}))\| + \|f(\tau, y^\delta(t_{k-2})) - f(t_{k-2}, y^\delta(t_{k-2}))\|\Big] + \\ + \Big[\|f(\tau, x(t_{k-2}))) - f(\tau, y^\delta(t_{k-2})))\Big]$$
$$\le \Big[K_2\Big\|\frac{\partial f}{\partial x}\Big\| + \Big\|\frac{\partial f}{\partial t}\Big\|\Big]\Delta t + \Big\|\frac{\partial f}{\partial x}\Big\|\|x(t_{k-2}) - y^\delta(t_{k-2})\|, \quad K_2 \triangleq \|G\|K_1 + \|f\|,$$
$$\Big\|\Big(G(\tau, x(\tau)) - G_k\Big)u^{\alpha,\delta}(\tau)\Big\| = \Big\|\Big(G(\tau, x(\tau)) - G(t_{k-2}, y^\delta(t_{k-2})\Big))u^{\alpha,\delta}(\tau)\Big\|$$
$$\le \Big[\|G(\tau, x(\tau)) - G(\tau, x(t_{k-2}))\| + \|G(\tau, y^\delta(t_{k-2})) - G(t_{k-2}, y^\delta(t_{k-2}))\|\Big]\|u^{\alpha,\delta}(\tau)\| + \\ + \Big[\|G(\tau, x(t_{k-2}))) - G(\tau, y^\delta(t_{k-2})))\Big]\|u^{\alpha,\delta}(\tau)\|$$
$$\le \Big[K_2\Big\|\frac{\partial G}{\partial x}\Big\| + \Big\|\frac{\partial G}{\partial t}\Big\|\Big]nK_1\Delta t + nK_1\Big\|\frac{\partial G}{\partial x}\Big\|\|x(t_{k-2}) - y^\delta(t_{k-2})\|,$$

where
$$\Big\|\frac{\partial f}{\partial x}\Big\| = n \max_{\substack{i,j\in\overline{1,n} \\ (t,x)\in D_0}} \Big|\frac{\partial f_i(t,x)}{\partial x_j}\Big|, \quad \Big\|\frac{\partial f}{\partial t}\Big\| = n \max_{\substack{i\in\overline{1,n} \\ (t,x)\in D_0}} \Big|\frac{\partial f_i(t,x)}{\partial t}\Big|,$$

$$\Big\|\frac{\partial G}{\partial x}\Big\| = nm \max_{\substack{i,j\in\overline{1,n} \\ l\in\overline{1,m} \\ (t,x)\in D_0}} \Big|\frac{\partial g_{i,l}(t,x)}{\partial x_j}\Big|, \quad \Big\|\frac{\partial G}{\partial t}\Big\| = nm \max_{\substack{i\in\overline{1,n} \\ l\in\overline{1,m} \\ (t,x)\in D_0}} \Big|\frac{\partial g_{i,l}(t,x)}{\partial t}\Big|,$$

and $\|f\| = n \max_{i\in\overline{1,n},(t,x)\in D_0} |f_i(t,x)|$. These estimates and (46) imply the relations

$$\|z(t_{k-1})\| = \|x(t_{k-1}) - y^\delta(t_{k-1})\| \\ \le R_1(\Delta t)^2 + (1 + R_2\Delta t)\|x(t_{k-2}) - y^\delta(t_{k-2})\| + \|\bar{z}_{k-1}(t_{k-2})\|, \tag{47}$$

where
$$R_1 = \Big[K_2\Big\|\frac{\partial G}{\partial x}\Big\| + \Big\|\frac{\partial G}{\partial t}\Big\|\Big]nK_1, \quad R_2 = nK_1\Big\|\frac{\partial G}{\partial x}\Big\|. \tag{48}$$

The relations (33),(37),(42) yield

$$\|\bar{z}_{k-1}(t_{k-2})\| \leq 2n^2K^2 \left[2K+\beta^0\right]^{1/2} \left[\frac{1}{2}\left(1+\frac{1}{2K_0\sqrt{\lambda_*}}\right)\right]^{1/2} [\Delta t]^2 \triangleq R_3(\Delta t)^2. \quad (49)$$

We consequently apply the estimates (47),(49) for each $k = \overline{1, N+1}$, beginning from $\|z(0)\| = \|x(0) - y^\delta(0)\| = 0$ and obtain

$$\|z(t_{k-1})\| \leq (1 + R_2\Delta t)\|z(t_{k-2}\| + (\Delta t)^2(R_1 + R_3) \quad (50)$$

$$\leq (R_1 + R_3)(k-1)(\Delta t)^2 e^{R_2(k-1)\Delta t}. \quad (51)$$

The relations (51) imply that the following estimates hold:

$$\|z(t)\| \leq (R_1 + R_3)\, T e^{R_2 T} \Delta t, \quad t \in [0, T]. \quad (52)$$

Assuming (42), it follows from (52) that

$$\|x(\cdot) - y^\delta(\cdot)\|_{C[0,T]} = \|z(\cdot)\|_{C[0,T]} \to 0 \quad \text{as } \delta \to 0,\ \Delta t \to 0,\ \alpha \to 0. \quad (53)$$

The relations (53) imply that (C1) is fulfilled.

4.3. Properties of reconstructed controls

The restrictions (43) on the controls $u^{\alpha,\delta}(\cdot)$ and the weak compactness of the ball in the space L_1 guarantee that, see [19], there exist sequences $\alpha_l \to 0$, $\delta_l \to 0$, $(\Delta t)_l \to 0$, $l \to \infty$, such that $u_l(\cdot) = u^{\alpha_l,\delta_l}(\cdot)$ weak*-converges to a generalized control $u_0(\cdot) \in (C[0,T], \mathbb{R}^n)^*$ as $l \to \infty$:

$$u_l(\cdot) \xrightarrow{w^*} u_0(\cdot), \quad l \to \infty.$$

This implies that we can obtain from

$$x_l(t) = \int_{t_0}^{t} f(\tau, x_l(\tau)) + G(\tau, x_l(\tau))u_l(\tau)d\tau \quad (54)$$

by tending l to infinity that

$$x^*(t) = \int_{0}^{t} f(\tau, x^*(\tau)) + G(\tau, x^*(\tau))u_0(\tau)d\tau, \quad t \in [0, T]. \quad (55)$$

This means that the equality

$$\frac{dx^*(t)}{dt} = f(t, x^*(t)) + G(t, x^*(t))u_0(t) \quad (56)$$

holds almost everywhere on $[0, T]$, where $u_0(t) = \int\limits_{\mathbf{U}_1} u\, d\mu_0(t)$, $\mathbf{U}_1 = B_{K_1}[\mathbf{0}]$, $\mu_0(t)$ is a regular probability measure on the set $\mathbf{U}_1$, and the function $t \to \mu_0(t)$ is measurable [19]. We obtain from the representation (24) for $u_l(t)$ that

$$u_0(t) = G^\top(t, x^*(t))\tilde{s_0}(t), \quad (57)$$

where $\tilde{s_0}(t)$ is a measurable function with the values

$$\tilde{s_0}(t) \in S(t) \subset \mathbb{R}^n, \quad S(t) = \mathrm{Co}\Big\{s_0 \in \mathbb{R}^n : s_0 = \lim_{r\to\infty} -\frac{\bar{s}_{l_r}(t)}{\alpha^2_{l_r}},\ l_r \overset{r\to\infty}{\to} \infty\Big\},$$

where $t \in [0,T]$ and the symbol $\mathrm{Co}(F)$ means the convex hull of the set F.

We substitute (57) into (56) and obtain the unique representation for $\tilde{s_0}(t)$:

$$\tilde{s_0}(t) = \Big(G(t,x^*(t))G^\top(t,x^*(t))\Big)^{-1}\Big[\frac{dx^*(t)}{dt} - f(t,x^*(t))\Big]. \tag{58}$$

This means that for $u_0(\cdot)$ (57)

$$u_0(t) \in \mathbf{U}, \quad \text{a.e.w. } t \in [0,T], \quad u_0(\cdot) \in \mathbb{U}^*. \tag{59}$$

One can see that $u_0(\cdot)$ (57) has the minimal L^2-norm among all elements of $\mathbb{U}^*$. Indeed, we have expressions (7) for the normal control. They coincide with the expressions (58), (59). Therefore, $u_0(\cdot)$ is the normal control.

The values of the cut-off controls $\hat{u}_l(\cdot) = \hat{u}^{\alpha_l,\delta_l}(\cdot)$ (24) are close to $u_l(\cdot) = u^{\alpha_l,\delta_l}(\cdot)$. Hence, the controls $\hat{u}_l(\cdot)$ tend in L^2-norm to the normal control $u_0(\cdot)$ as $l \to \infty$. It implies that the corresponding trajectories $x_l(\cdot)$ of (1) generated by the cut-off controls tend in C-norm to the basic trajectory $x^*(\cdot)$. It means that the controls $\hat{u}_l(\cdot)$ (24) satisfy (C1) and (C2) and thus solve DRP.

5. Example

We consider an example from the area of medicine to illustrate numerical simulation of the proposed above method.

Inverse problems for dynamic models are an important part of modern medicine. Some parameters involved into the modeled treatment processes can be unknown. Reconstruction methods may provide means to obtain them. The normalizing nature of the suggested method also guarantees that the reconstructed controls are optimal (in the sense of the L^2 norm).

The model in the example represents the fed-batch penicillin fermentation process. It is described in details in [20]. The dynamics is

$$\begin{pmatrix}\dot{X}(t)\\ \dot{P}(t)\\ \dot{S}(t)\\ \dot{V}(t)\end{pmatrix} = \begin{pmatrix} X(t) & -\dfrac{X(t)}{S_F V(t)} & 0 & 0\\ 0 & -\dfrac{X(t)}{S_F V(t)} & X(t) & 0\\ -\dfrac{X(t)}{Y_{X/S}} & \dfrac{S_F - S(t)}{S_F} & -\dfrac{X(t)}{Y_{P/S}} & -\dfrac{S(t)X(t)}{K_m + S(t)}\\ 0 & \dfrac{1}{S_F} & 0 & 0\end{pmatrix}\begin{pmatrix}\mu(t)\\ U(t)\\ \rho(T)\\ M_s(t)\end{pmatrix} + \begin{pmatrix}0\\ -K_{deg}P(t)\\ 0\\ 0\end{pmatrix},$$

$$\begin{gathered} Y_{X/S} = 0.47, \quad Y_{P/S} = 1.2, \quad S_F = 500, \quad K_m = 0.0001, \quad K_{deg} = 0.01,\\ X(0) = 1.5, \quad P(0) = 0, \quad S(0) = 0.01, \quad V(0) = 7, \\ \mu(t) \in [0,1], \quad U(t) \in [0,50], \quad \rho(t) \in [0,1], \quad M_s(t) \in [0,1], \quad t \in [0,25]. \end{gathered} \tag{60}$$

The state variables are the concentrations of biomass X, product P and substrate S, and the volume of biomass V. The unknown parameters are the specific biomass growth rate μ, the substrate feed rate U, the specific penicillin production rate ρ and the maintenance consumption rate M_s. The feed rate U is the control parameter. The parameters μ,ρ and M_s describe some a-priori unknown properties of the patient's organism. We will formally consider them as the unknown controls as well.

Assumption 1 holds in the example due to the boundary conditions in (60) and because all state variables are positive on $(0, T]$ since their biological meaning.

First, we numerically construct the base trajectory $(X^*(\cdot), P^*(\cdot), S^*(\cdot), V^*(\cdot))$ generated by the controls

$$\mu(t) = 0.11, \quad U(t) = 15 + \frac{5t}{T}, \quad \rho(t) = 0.0055, \quad M_s(t) = 0.029. \tag{61}$$

Then we generate the sets of inaccurate measurements

$$\{y_X^\delta\},\ \{y_P^\delta\},\ \{y_S^\delta\},\ \{y_V^\delta\} \tag{62}$$

by random perturbation of the constructed base trajectory in discrete points.

Problem 1 in this example is to find approximations of the normal controls $\mu^*(\cdot)$, $U^*(\cdot)$, $\rho^*(\cdot)$, $M_s^*(\cdot)$, generating the considered basic trajectory. Remark that in the dynamics (60) the last equation can be integrated separately. Therefore, generally speaking, the number of controls is greater than the number of independent state variables. So, the normal controls may not coincide with the initial controls (61).

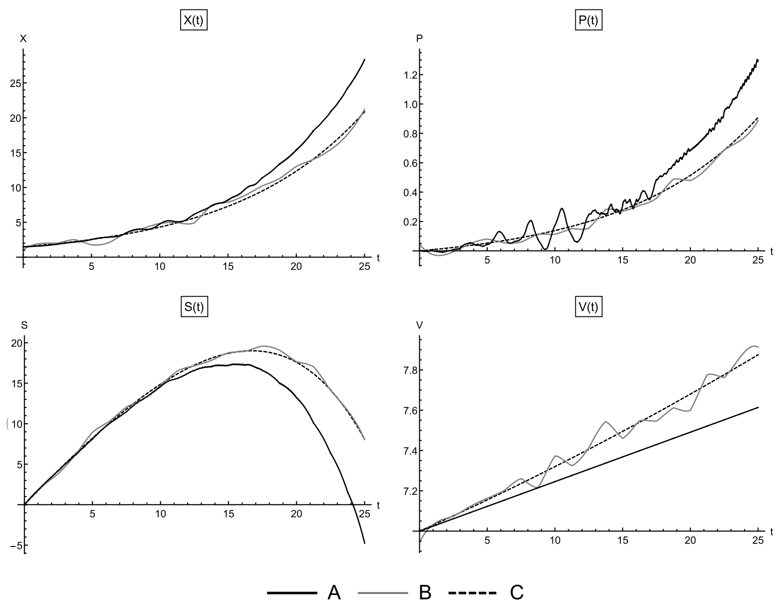

Figure 5.1: The trajectories (rough approximation). **A** is the reconstructed trajectory, **B** is the interpolation of the measurements, **C** is the base trajectory.

The suggested algorithm has been applied for the dynamics (60) and the measurements (62) generated for various approximation parameters δ, h_δ, α. Figures 5.1 and 5.2 represent the results obtained for rough approximation parameters $\delta_X = 1$, $\delta_P = 0.05$, $\delta_S = 1$, $\delta_V = 0.1$, $h_\delta = T/20$, $\alpha = 0.5$.

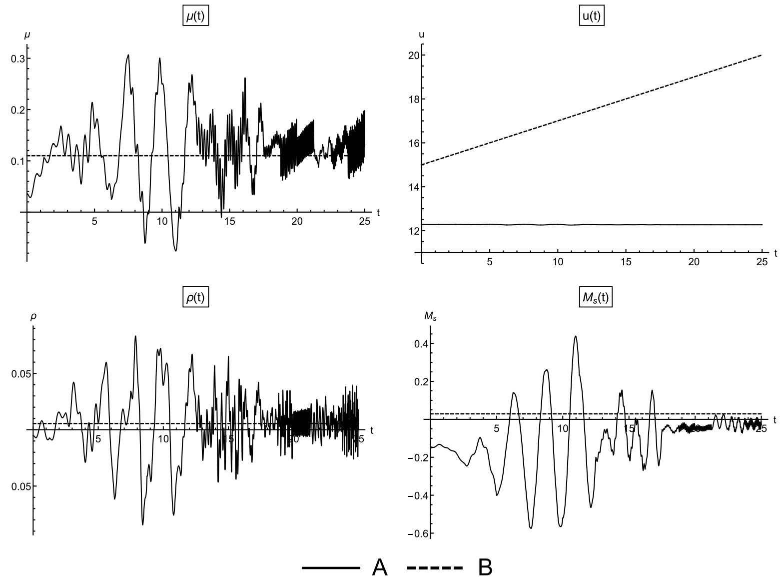

Figure 5.2: The controls (rough approximation). **A** is the reconstructed approximation of the normal control, **B** is the initial control.

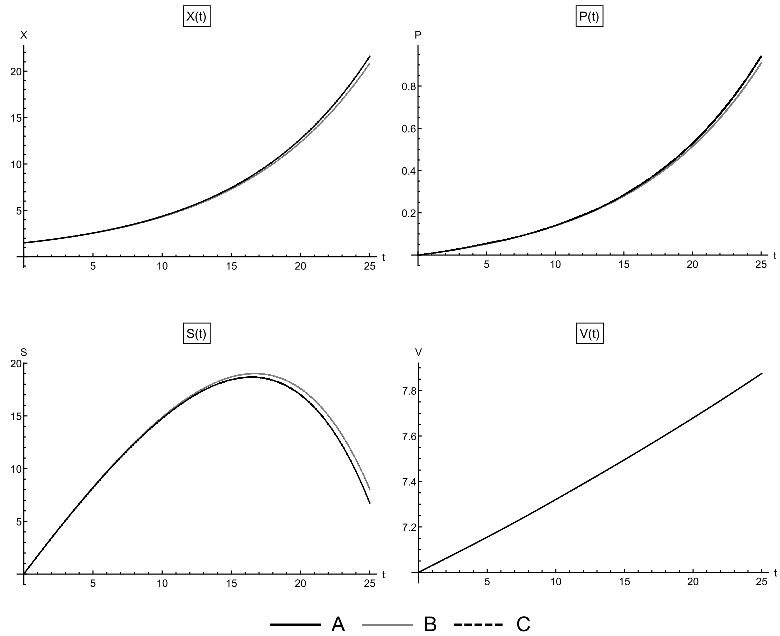

Figure 5.3: The trajectories (more accurate approximation). **A** is the reconstructed trajectory, **B** is the interpolation of the measurements, **C** is the base trajectory.

Figures 5.3 and 5.4 represent more accurate approximations for $\delta_X = 0.1$, $\delta_P = 0.005$, $\delta_S = 0.1$, $\delta_V = 0.01$, $h_\delta = T/200$, $\alpha = 0.05$. Figures 5.2 and 5.4 demonstrate that the reconstructed normal controls $U^*(\cdot)$, $M_s^*(\cdot)$ differ form controls (61). That means that the initial controls are not optimal and the same result (e.g. the same trajectory) can be achieved through a smaller substrate feed U.

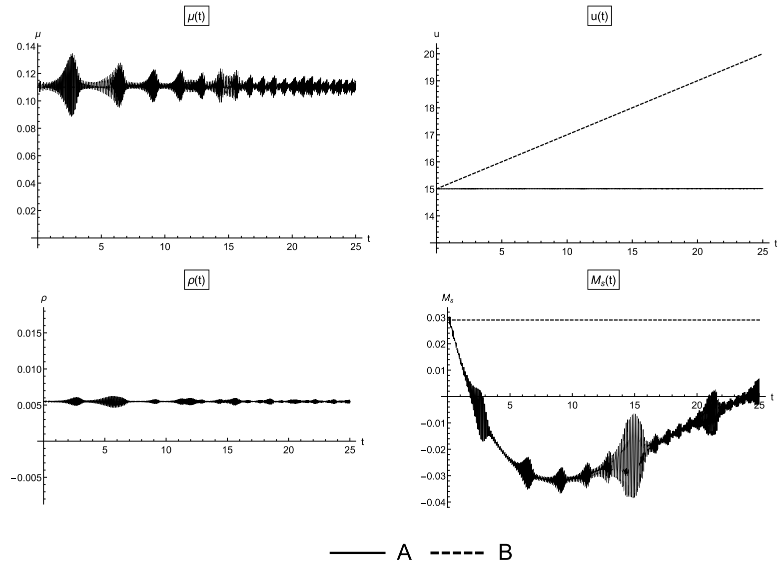

Figure 5.4: The controls (more accurate approximation). **A** is the reconstructed approximation of the normal control, **B** is the initial control.

6. Conclusion

The dynamic control reconstruction problem (DRP) has been considered. It is the problem of reconstruction of the normal control, generating a trajectory of a dynamic system in real-time. The inaccurate measurements of the realized trajectory are used. A new method for solving this problem is suggested and justified. It is a modification of the previously proposed approach. The novelty of the modification is simplification of the algorithm's procedures. Namely, the DRP problem is reduced to solving systems of linear ODEs with constant parameters.

Moreover, we need to perform matrix inversion only once at the beginning of the reconstruction process. This provides better computational effectiveness of the method in comparison with our previous suggestions and algebraic methods for solving DRP.

The results of the simulation of the method's application to an example from the area of medicine are reported.

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