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Approximate Optimal Control in the Infinite Time Horizon Problem with Phase Constraints

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The paper investigates a control problem of optimal distribution of investments directed towards improvement of the resource productivity and/or purchasing the natural resources. The problem is based on the growth model of the effective resource consumption under the condition on resources' exhaustion. The paper provides an interpretation of phase variables in terms of the reliability theory. The optimal solution of the problem is approximated by piecewise constant controls that guarantee convergence of the trajectories to the unique equilibrium of the Hamiltonian system. The derived approximation satisfies the phase constraints and matches with the stabilized solutions constructed on the basis of the nonlinear regulator approach.

Keywords: Optimal control problem, phase constraints, reliability theory, piecewise constant controls, Hamiltonian system, growth models.

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1. Introduction

The research is devoted to the problem on optimization of the resource use efficiency in resource-dependent economies. The problem design is conducted within the endogenous economic growth theory [2, 5]. The penalty for the wasteful use of resources is expressed by the *price-formation mechanism* that is inverse proportional to the current stock of natural resources. The rapidly raising costs of natural resources has a negative impact on the consumption index that estimates the quality of the control process [7].

Nowadays, the rate of the resource consumption is rapidly increasing (see, for example [8]), therefore, the optimization problem for the resource productivity comes to the fore.

In this paper, we present the optimization model of the resource use and give an interpretation of the production factors serving as phase variables in terms

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of the reliability theory. The control parameter in the model is the share of the output invested into the resource productivity. The quality of a control strategy is estimated by the integral logarithmic consumption index discounted on the infinite time interval. Using the Pontryagin maximum principle [4, 1], we investigate the control problem and provide the qualitative analysis of the Hamiltonian system, specifically, we derive the conditions on model parameters that guarantee the existence of the steady state. As it is proved in [9], once a steady state exists, its is possible to construct a nonlinear regulator that stabilizes solutions of the Hamiltonian system.

The paper develops a scheme for constructing an approximate solution of the problem using a piecewise control regime. The control parameters and the switching time are determined by means of two conditions that are the given phase constraints and the existence of a saturation level at the steady state of the Hamiltonian system.

The paper has the following structure. We start from the model description and the statement of the optimal control problem. The second section provides an interpretation of the phase variables in terms of the reliability theory. Next, we move towards the problem analysis and provide the qualitative characterization of the Hamiltonian systems. Then, an algorithm is provided for constructing approximate solutions of the optimal control problem with infinite horizon. Finally, we compare the stabilized trajectories of the Hamiltonian system and the approximate solutions in a vicinity of the steady state.

2. Optimization model and control problem

In [7] the authors have developed the model and discussed its features and properties. Here, we give a brief overview of the basic model parameters and dependencies that are required for the problem statement.

2.1. Optimization model of resource consumption

In the basic concept, the main phase variables of the model are presented by the current production $y = y(t)$, the resource use $m = m(t)$, and the cumulative resource consumption $M = M(t)$, where symbol t stands for the time variable $t \in [0, +\infty)$. The cumulative resource consumption $M(t)$ is introduced as the integrated material use $m(t)$

$$M(t) = \int\limits_0^t m(s)ds. \tag{1}$$

The condition on the limited natural resources is given by the integral equality

$$M(t) \leqslant \overline{M} = \int\limits_0^{+\infty} m(s)ds. \tag{2}$$

The initial consumption level $m(0) = m_0$ is assumed to be known, and $M(0) = 0$. The resource productivity $z(t)$ at time t is determined by the formula

$$z(t) = \frac{y(t)}{m(t)}. \tag{3}$$

In the definition of the price formation mechanism we put into the basis the concept of raising prices $p(t)$ on the natural resources in the case of their limitation or exhaustion

$$p(t) = \frac{p_0\overline{M}}{\overline{M} - M(t)}, \quad p(0) = p_0. \tag{4}$$

It is assumed that prices are growing according to the inversely proportional rule of resources exhaustion. Formula (4) envisages that prices $p(t)$ can grow rapidly to infinity according to the hyperbolic law when the integrated material use $M(t)$ reaches its limitation $\overline{M}$ (2) .

Under the assumption of closeness of the economy the balance equation takes the form

$$y(t) = c(t) + p(t)m(t) + u(t)y(t), \tag{5}$$

i.e. the production $y(t)$ in period t is shared between consumption $c(t)$ and the growing cost of the natural resources $p(t)m(t)$ plus investments $u(t)$ in improving the resource productivity $z(t)$. The balance relation (5) allows to derive the current consumption level $c(t)$ in the following form

$$c(t) = y(t)\left(1 - \frac{p(t)m(t)}{y(t)} - u(t)\right), \tag{6}$$

and also to indicate the constraints for the control parameter $u(t)$,

$$0 \leq u(t) \leqslant \overline{u}. \tag{7}$$

The output $y(t)$ depends on the resource use $m(t)$ in accordance to the *production function* of the Cobb-Douglas type

$$y(t) = ae^{bt}m^{\alpha}(t), \quad a > 0, b \geqslant 0, 0 \leqslant \alpha < 1, \tag{8}$$

where the parameter a is a scale factor, the symbol b stands for the growth rate of the output $y(t)$ due to the development of the production factors (*i.e.* the resource use $m(t)$), and the constant α is an elasticity coefficient.

Remark 2.1. It is assumed that the model parameters satisfy the inequality $m_0/\overline{M} < b/\alpha$ that is verified statistically for many countries [7].

2.2. Optimal control problem

Let assume that the relative raise in the resource productivity $z(t)$ is proportional to the portion of the assigned investment $u(t)$ (see [2])

$$\dot{z}(t) = \beta u(t)z(t), \quad \beta \geqslant 0, \tag{9}$$

where parameter β indicates the effectiveness of the investment process $u(t)$. The equality (9), together with the formulas (3) and (8), provide the dynamics of the material use

$$\dot{m}(t) = \frac{b - \beta u(t)}{1 - \alpha} m(t). \tag{10}$$

For the model analysis, we introduce new phase variables in the following form

$$x_1(t) = \frac{m(t)}{\overline{M} - M(t)}, \quad x_2(t) = \frac{p(t)m(t)}{y(t)}. \tag{11}$$

The first variable $x_1(t)$ denotes the share of the current stock of the natural resources that is used in the current period of time t, and the second variable $x_2(t)$ is the share of the output that is spent on the purchase of the natural resources. The dynamic relations for the new variables $x(t) = (x_1(t), x_2(t))$ can be derived using equalities (1), (4), (8)–(10) as follows

$$\begin{cases} \dot{x}_1(t) = x_1(t)\left(x_1(t) + \dfrac{b - \beta u(t)}{1 - \alpha}\right) =: f_1(x(t), u(t)), & x_1^0 = \dfrac{m_0}{\overline{M}} \\ \dot{x}_2(t) = x_2(t)\left(x_1(t) - \beta u(t)\right) =: f_2(x(t), u(t)), & x_2^0 = \dfrac{p_0 m_0^{1-\alpha}}{a} \end{cases} \tag{12}$$

In the variables $x(t) = (x_1(t), x_2(t))$ (11), the logarithmic consumption index can be expressed by the formula

$$\ln c(t) = \ln \frac{x_1(t)}{x_2(t)} + \ln\left(1 - u(t) - x_2(t)\right) + \ln\left(p_0 \overline{M}\right) = w(x, u) + \ln\left(p_0 \overline{M}\right). \tag{13}$$

The quality of the control process is determined by the integral consumption index discounted on the infinite time interval

$$J(x(\cdot), u(\cdot)) = \int\limits_0^{+\infty} e^{-\rho t} \ln w(x(t), u(t))dt. \tag{14}$$

The *optimal control problem* is to maximize the utility function (14) along the trajectories $(x(t), u(t))$ of the dynamic system (12) that satisfy the initial conditions $x^0 = (x_1^0, x_2^0)$ in the phase variables $x(t) = (x_1(t), x_2(t))$ and the constraints (7) in the control variable $u(t)$.

3. Model analysis within the reliability theory

The first model variable $x_1(t)$ can be interpreted as the risk of running out of the natural resources. Let us consider the share of the current use of materials $m(t)$ to the initial resource stock $\overline{M}$ as the density, and the value of $M(t)/\overline{M}$ as the distribution of a random variable indicating that by the time moment t the stock of the natural resources is going to exhaustion

$$f_m(t) = \frac{m(t)}{\overline{M}}, \quad F_m(t) = \frac{M(t)}{\overline{M}}. \tag{15}$$

Using the introduced definitions of the model variables (see (1) and (15)), one can consider the ratio

$$\lambda(t) = \frac{f_m(t)}{1 - F_m(t)} = \frac{1}{1 - F_m(t)} \frac{dF_m(t)}{dt}, \tag{16}$$

which coincides with the phase variable x_1 and has a specific meaning in the reliability theory. The function $(1 - F_m(t))$ indicates the probability that the natural resources do not run out by the time moment t. The Formula (16) determines the value of $\lambda(t)$ as a local characteristic of the stock reliability that is the reliability of the resource stock at any given instant of time. In other words, parameter $\lambda(t)$ is the probability that the non empty resource stock up to time t will be exhausted in the next span of time $[t, t + \Delta t]$, where Δt is supposed to be small enough. In terms of probability theory, parameter $\lambda(t)$ is the density of the conditional failure probability at time t provided that up to this moment the resource stock has not been run out. The function $\lambda(t)$ determined from formula (16) is called the *danger of failure.*

The distribution $F_m(t)$ can be explicitly found from the equation (16). Indeed, integrating (16) under the initial condition $F_m(0) = 0$, one gets

$$F_m(t) = 1 - e^{-\int\limits_0^t \lambda(s)ds}.$$

3.1. An exponential case

For the constant *danger of failure* $\lambda(t)$, the random variable indicating the exhaustion time of the natural resource stock would have the exponential distribution. Consequently, using the definition of $\lambda(t)$ (16) and density $f_m(t)$ (15), we derive

$$f_m(0) = \frac{m(0)}{\overline{M}} = \frac{m_0}{\overline{M}} = \lambda(0) \quad \Rightarrow \quad F_m(t) = 1 - e^{-(m_0/\overline{M})t}.$$

It results in the following law of the material use

$$m_{\exp}(t) = m_0 e^{-(m_0/\overline{M})t}, \tag{17}$$

ensuring the constant investments directed towards increasing the resource productivity. The investment level can be evaluated by equation (10) in the form

$$-\frac{m_0^2}{\overline{M}} e^{-(m_0/\overline{M})t} = \frac{b-\beta u}{1-\alpha} m_0 e^{-(m_0/\overline{M})t} \quad \Rightarrow \quad u_{\exp}(m_0) = \frac{b}{\beta} + \frac{1-\alpha}{\beta} \cdot \frac{m_0}{\overline{M}}. \tag{18}$$

Remark 3.1. The obtained investment level has to satisfy the control restrictions (7), *i.e.* the material use reduction rate $m_0/\overline{M}$ should not exceed the value $(\beta\overline{u} - b)/(1 - \alpha)$.

In the exponential case, other model factors such as output $y(t)$ (8), prices $p(t)$ (4) and the resource productivity $z(t)$ (3) are determined according to their definitions using relations (17) and (18).

$$y_{\exp}(t) = ae^{bt}m^{\alpha}_{\exp}(t) = am_0^{\alpha} \exp\left\{\left(b - \alpha\frac{m_0}{\overline{M}}\right)t\right\} \tag{19}$$

$$p_{\exp}(t) = \frac{p_0}{1 - (M(t)/\overline{M})} = \frac{p_0}{1 - F_m(t)} = p_0 e^{(m_0/\overline{M})t} \tag{20}$$

$$z_{\exp}(t) = \frac{y(t)}{m(t)} = am_0^{\alpha-1} \exp\left\{\left(b + (1-\alpha)\frac{m_0}{\overline{M}}\right)t\right\} \tag{21}$$

The obtained relations (19)–(21) ensure the exponential growth of the output $y(t)$ and the resource productivity $z(t)$, while the prices $p(t)$ on the natural resources increase exponentially as well. It is worth to note that the total expenditure on the materials remains the same at any moment of time, since

$$p_{\exp}(t) \cdot m_{\exp}(t) := p_0 e^{(m_0/\overline{M})t} \cdot m_0 e^{-(m_0/\overline{M})t} = p_0 \cdot m_0.$$

3.2. Utility function along the exponential trajectories

In parallel with the derived solution corresponding the exponential distribution $F_m(t)$ of the relative resource use, one can estimate the utility function $V_{\exp}(m_0)$ by substituting formulas (17)–(20) into relation (14)

$$J_{\exp}(m_0) = \int\limits_0^{+\infty} e^{\rho t}\left(\ln y_{\exp}(t) + \ln\left(1 - u_{\exp}(m_0) - \frac{p_{\exp}(t)m_{\exp}(t)}{y_{\exp}(t)}\right)\right)dt =$$

$$= \frac{\ln\left(am_0^{\alpha}\right)}{\rho} + \frac{\gamma(m_0)}{\rho^2} + \int\limits_0^{+\infty} e^{-\rho t} \ln\left(1 - u_{\exp}(m_0) - \frac{p_0 m_0^{1-\alpha}}{a}e^{-\gamma t}\right)dt,$$

where the parameter $\gamma = \gamma(m_0) = \alpha\left(\dfrac{b}{\alpha} - \dfrac{m_0}{\overline{M}}\right)$ is positive due to the assumption mentioned in Remark 2.1. The last integral is evaluated using the following change of variables

$$\xi(t; m_0) = \xi_0(m_0)e^{-\gamma(m_0)t}, \quad \xi_0(m_0) := \xi(0; m_0) = \frac{p_0 m_0^{1-\alpha}}{a(1 - u_{\exp}(m_0))}$$

and partial integration. The result is expressed through the Lerch-transcendent function, $\Phi(\xi, \omega_0) = \sum\limits_{n=0}^{\infty} \dfrac{\xi^n}{n + \omega_0}$, in the following way

$$\int\limits_0^{+\infty} e^{-\rho t} \ln\left(1 - u_{\exp}(m_0) - \frac{p_0 m_0^{1-\alpha}}{a}e^{-\gamma(m_0)t}\right)dt =$$

$$= \frac{1}{\rho}\ln\left(1 - u_{\exp}(m_0) - \frac{p_0 m_0^{1-\alpha}}{a}\right) + \frac{1}{\rho}\left(\xi_0(m_0)\right)^{-\frac{\rho}{\gamma(m_0)}} \int\limits_0^{\xi_0(m_0)} \frac{\xi^{\frac{\rho}{\gamma(m_0)}}}{1 - \xi}d\xi.$$

Calculating the last integral, we derive the following relation

$$\frac{1}{\rho}\big(\xi_0(m_0)\big)^{-\frac{\rho}{\gamma(m_0)}}\int\limits_0^{\xi_0(m_0)}\frac{\xi^{\frac{\rho}{\gamma(m_0)}}}{1-\xi}d\xi=\frac{1}{\rho}\big(\xi_0(m_0)\big)^{-\frac{\rho}{\gamma(m_0)}}\int\limits_0^{\xi_0(m_0)}\xi^{\frac{\rho}{\gamma(m_0)}}\sum_{n=0}^{\infty}\xi^n d\xi=$$

$$=\frac{1}{\rho}\big(\xi_0(m_0)\big)^{-\frac{\rho}{\gamma(m_0)}}\sum_{n=0}^{\infty}\int\limits_0^{\xi_0(m_0)}\xi^{\frac{\rho}{\gamma(m_0)}+n}d\xi=\frac{1}{\rho}\sum_{n=0}^{\infty}\frac{\xi_0^n}{n+\frac{\rho}{\gamma(m_0)}}-\frac{\gamma(m_0)}{\rho^2}.$$

Substituting the obtained relations into the functional $J_{\exp}(m_0)$, we get the utility function in the form

$$J_{\exp}(m_0)=\frac{\ln\left(am_0^{\alpha}\right)}{\rho}+\frac{1}{\rho}\ln\left(1-u_{\exp}(m_0)-\frac{p_0m_0^{1-\alpha}}{a}\right)+\frac{1}{\rho}\sum_{n=0}^{\infty}\frac{\xi_0^n}{n+\frac{\rho}{\gamma(m_0)}}. \quad (22)$$

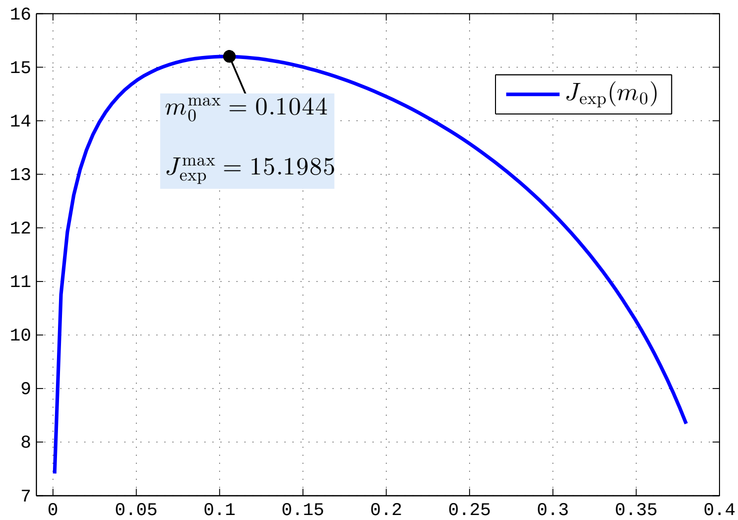

Figure 3.1: The utility functional $J_{\exp}$ (22) as a function of the initial material use m_0.

In conclusion, the obtained results can be considered as a possible approximate solution of the control problem if the model parameters satisfy the inequalities $0<\xi_0<1$ and $\gamma>0$, which in the original model parameters, look like as follows

$$\frac{p_0m_0^{1-\alpha}}{a}+\frac{b}{\beta}+\frac{1-\alpha}{\beta}\frac{m_0}{\overline{M}}<1,\qquad \frac{b}{\alpha}-\frac{m_0}{\overline{M}}>0.$$

These inequalities together with the constraints on the control parameter u (see Remark 3.1) impose restrictions on the initial material use m_0

$$m_0<\min\left\{\left(\frac{a(1-\overline{u})}{p_0}\right)^{\frac{1}{1-\alpha}},\ \frac{\overline{M}(\beta\overline{u}-b)}{1-\alpha},\ \frac{b}{\alpha}\overline{M}\right\}.$$

Figure 3.1 demonstrates the utility functional as a function of m_0 for the constant *danger of failure* $\lambda(t)$, which determines the relative reduction rate of the material use at the level $m_0/\overline{M}$. For the calculations, we use the numerical data indicated in [7, 8]. More specifically, the following estimates of the model parameters are used for the experiments: $\overline{M} = 1.81{\cdot}10^6$, $\overline{u} = 0.12$, $\rho = 0.18$, $a = 64.34$, $b = 0.069$, $\alpha = 0.4091$, $\beta = 1.523$, and $p_0 = 100$. The utility function is concave and achieves its maximum $J^{\max}_{\exp} = 15.1985$ at the point $m_0^{\max} = 0.1044$. The investments $u_{\exp}(m_0)$ under this level of the initial material use equals $u_{\exp}\left(m_0^{\max}\right) = 0.0453$.

4. Problem analysis within the maximum principle

The problem analysis is conducted within the Pontryagin maximum principle [4] generalized for the optimal control problems in the infinite time interval [1].

4.1. Hamiltonian Function and Hamiltonian systems

The stationary Hamiltonian function has the form

$$\widehat{H}(x,\psi,u) = \ln w(x,u) + \psi^\top f(x,u), \quad f(x,u) = \big(f_1(x,u), f_2(x,u)\big)^\top, \tag{23}$$

where $\psi = (\psi_1,\psi_2)^\top$ is a vector of conjugate variables, and $f(x,u)$ is a vector function determined the right-hand parts of the dynamics (12).

It is convenient to introduce the adjoint variables of the form $x_3 = \psi_1 x_1$, $x_4 = \psi_2 x_2$, and denote by symbol X the vector of phase and adjoint variables, *i.e.* $(x_2, x_2, x_3, x_4)^\top =: X$. The Hamiltonian function (23) $\widehat{H}(\cdot,u)$ in variable u takes the maximum value at $u = u^0$

$$u_0 = \begin{cases} 0, & X \in D_1 := \{X : v(X) \geq 1 - x_2\} \\ 1 - x_2 - v(X), & X \in D_2 := \{X : 1 - \overline{u} - x_2 \leq v(X) \leq 1 - x_2\} \\ \overline{u}, & X \in D_3 := \{X : v(X) \leq 1 - \overline{u} - x_2\}, \end{cases} \tag{24}$$

where the function $v(X)$ is determined by $v(X) = -\dfrac{1-\alpha}{\beta}\dfrac{1}{x_3 + (1-\alpha)x_4}$.

In the domains D_1 and D_3 of constant control regimes, the Hamiltonian system has the form

$$\begin{cases} \dot{x}_1(t) = x_1(t)\left(x_1(t) + \dfrac{b - \beta\widetilde{u}}{1-\alpha}\right) \\ \dot{x}_2(t) = x_2(t)(x_1(t) - \beta\widetilde{u}) \\ \dot{x}_3(t) = \rho x_3(t) - x_1(t)(x_3(t) + x_4(t)) - 1 \\ \dot{x}_4(t) = \rho x_4(t) + \dfrac{1-\widetilde{u}}{1 - \widetilde{u} - x_2(t)} \end{cases} \quad \text{where} \quad \widetilde{u} = \begin{cases} 0, & X(t) \in D_1, \\ \overline{u}, & X(t) \in D_3. \end{cases} \quad X(t) := \big(x_1(t), x_2(t), x_3(t), x_4(t)\big)^\top$$

In the domain D_2 of the transient control regime $u(t) = 1 - x_2(t) - v(X(t))$, the Hamiltonian system looks as follows

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\left(x_1(t) + \frac{\beta}{1-\alpha}(x_2(t) + v(X(t)) - 1) + \frac{b}{1-\alpha}\right), \\ \dot{x}_2(t) &= x_2(t)\left(x_1(t) + \beta(x_2(t) + v(X(t)) - 1)\right), \\ \dot{x}_3(t) &= \rho x_3(t) - x_1(t)(x_3(t) + x_4(t)) - 1, \\ \dot{x}_4(t) &= \rho x_4(t) - \frac{\beta}{1-\alpha}(x_3(t) + (1-\alpha)x_4(t))x_2(t) + 1. \end{aligned} \tag{25}$$

The conjugate variable x_4 in the domain D_2 is negative.

4.2. Steady state of the Hamiltonian system

The steady state of the Hamiltonian system does exist in the domain of the transient control regime if the model parameters belong to the set Ω depicted at the Fig. 4.1. The steady state coordinates can be found analytically by the formulas

$$\begin{aligned} x_1^* &= \frac{b}{\alpha}, & x_2^* &= \frac{(\alpha\rho - b)((1-\alpha)\rho + \alpha\beta - b)}{\alpha\beta\rho}, \\ x_3^* &= (1-\alpha)\left(1 - \frac{\alpha\rho}{\alpha\beta - b}\right)x_4^*, & x_4^* &= -\frac{\alpha\beta - b}{\alpha\rho(1-\alpha)(\beta-\rho) + b(\alpha\beta - b)}. \end{aligned} \tag{26}$$

The control u^* corresponding to the steady state has the value

$$u^* = \frac{b}{\alpha\beta} \leq \overline{u} < 1. \tag{27}$$

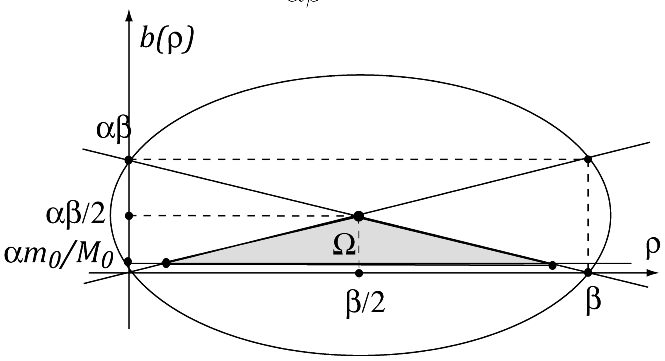

Figure 4.1: The steady state domain, Ω

According to the results [9], the existence of the steady state guarantees the local stabilizability of the Hamiltonian system. The paper [6] describes the algorithm for the system stabilization.

Remark 4.1. The exponential solution obtained within the reliability theory has another stationary levels. Indeed, the first phase variable $x_1(t) := \lambda(t) = m_0/\overline{M}$ is constant, and the second phase variable

$$x_2(t) = \frac{p(t)m(t)}{y(t)} = \frac{p_0 m_0^{1-\alpha}}{a} \exp\left\{-\alpha\left(\frac{b}{\alpha} - \frac{m_0}{\overline{\overline{M}}}\right)t\right\}$$

tends to zero over the time goes to infinity. However, the approach based on the Pontryagin maximum principle provides the saturation level for the phase variables at the point (x_1^*, x_2^*).

The next section is aimed to improve the first method for reaching the correct stationary level derived within the Pontryagin maximum principle using the piecewise constant controls.

5. Piecewise constant controls

Based on the constraints (2) and the probability $F_m(t)$, see (15), we have the following relation

$$\lim_{t \to +\infty} F_m(t) = 1. \tag{28}$$

For reaching the stationary level (26), one can change the control variable u in such a way, that before the time moment T it remains on the constant level of u_1, and after switches to the steady state value u^* (27), i.e.

$$\widehat{u}(t) = \begin{cases} u_1, & t < T \\ u^*, & t \geq T, \end{cases} \tag{29}$$

where parameter u_1 is the constant control satisfying the constraints (7), and parameter T is the moment of time that can be found by formula (28). For estimating the time T, we substitute relation (29) in the density of distribution (15)

$$f_m(t, u_1) = \frac{m_0}{\overline{M}} \cdot \begin{cases} \exp\left\{\dfrac{b - \beta u_1}{1 - \alpha} t\right\}, & t < T \\ \exp\left\{\dfrac{b - \beta u_1}{1 - \alpha} T\right\} \exp\left\{-\dfrac{b}{\alpha}(t - T)\right\}, & t \geq T. \end{cases} \tag{30}$$

The obtained density $f_m(t)$ (30) under the piecewise control $\widehat{u}(t)$ together with the relation (28) provide the following relation

$$1 = \int_0^{+\infty} f_m(t, u_1) dt = \frac{m_0(1 - \alpha)}{\overline{M}(b - \beta u_1)} \left(e^{\frac{b - \beta u_1}{1 - \alpha} T} - 1\right) + \frac{m_0 \alpha}{\overline{M} b} e^{\frac{b - \beta u_1}{1 - \alpha} T}. \tag{31}$$

The expression (31) can be considered as an equation with respect to the time parameter T. Solving this equation, we obtain that the switching time instant T satisfies the formula

$$T(u_1) = \begin{cases} \dfrac{1 - \alpha}{b - \beta u_1} \ln\left(\dfrac{b\overline{M}}{\alpha m_0} \dfrac{u_{\exp} - u_1}{u^* - u_1}\right), & \beta u_1 \neq b, \\ \dfrac{\overline{M}}{m_0} - \dfrac{\alpha}{b}, & \beta u_1 = b, \end{cases} \tag{32}$$

where control $u_{\exp}$ is introduced in formula (18) (see section 3). Noteworthy, the control level $u_{\exp}$ guarantees inexhaustibility of the natural resources for any finite period of time, i.e. $\lim_{u_1 \to u_{\exp}-0} T(u_1) = +\infty$. Moreover, for all u_1 larger than $u_{\exp}$, the switching time moment T does not exist (see Fig. 5.1). Hence, the condition (2) of the natural resources' exhaustibility is fulfilled if the constant control value u_1 does not exceed the level of $u_{\exp}$. According to the qualitative analysis of the Hamiltonian system (25), there exists the steady state satisfying relation (26). Using this fact, we estimate the constant control u_1 by assuming that at the switching time moment $T(u_1)$ the approximate solution $x_2(t, u_1)$ achieves the stationary level x_2^*, i.e.

$$\begin{aligned} x_2\big(T(u_1), u_1\big) \quad &:= \quad \frac{p_0 \overline{M}}{\overline{M} - M\big(T(u_1)\big)} \frac{m\big(T(u_1), u_1\big)}{y\big(T(u_1)\big)} = \\ &= \quad \frac{p_0 \overline{M} b}{a \alpha m_0^{\alpha}} \left(\frac{\alpha m_0}{b \overline{M}} \frac{u^* - u_1}{u_{\exp} - u_1} \right)^{\alpha\beta \frac{u^* - u_1}{b - \beta u_1}} = x_2^*. \end{aligned} \tag{33}$$

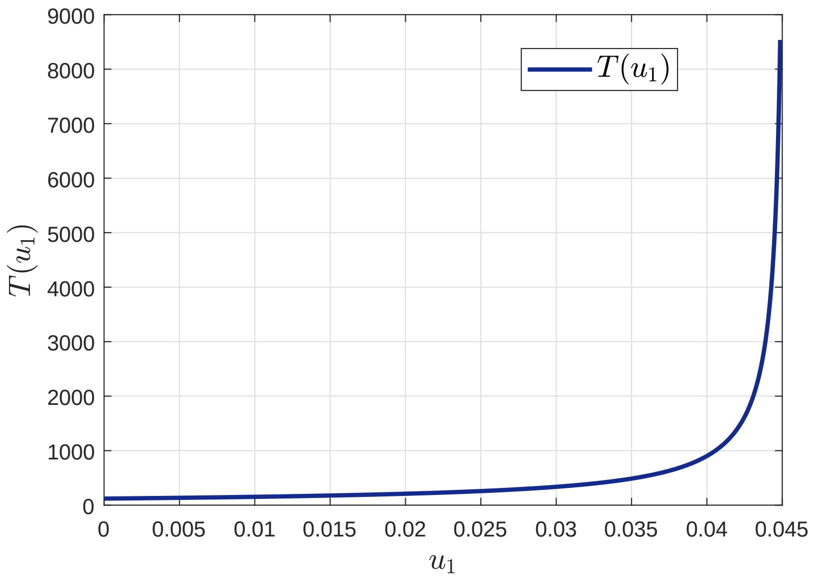

Figure 5.1: The switching moment T as a function of control u_1

Using formulas (1), (8), and (30), we find the control level u_1 numerically (see Fig. 5.2).

The utility functional $J(\cdot)$ (14) as a function of the initial material use m_0 calculated under the piecewise control regime (29) is depicted on Figure 5.3.

In this case, the utility can be found as a sum of two integral terms

$$J(m_0) = \int_0^T e^{-\rho t} \ln c(t) dt + \int_T^{+\infty} e^{-\rho t} \ln c(t) dt = J_T(m_0) + J_\infty(m_0), \tag{34}$$

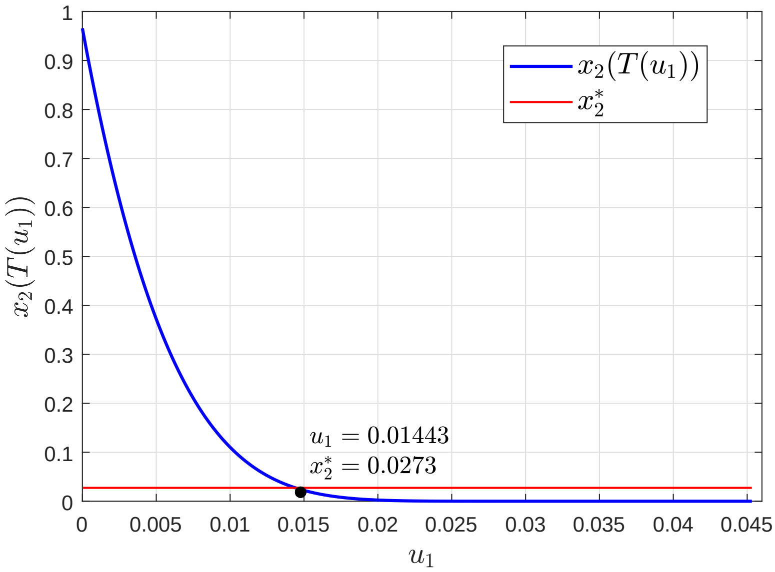

Figure 5.2: Numerical search for the initial control u_1

where

$$J_T(m_0) = \int_0^T e^{-\rho t} \ln\left(y_T(t)(1 - u_1(m_0)) - p_T(t)m_T(t)\right)dt, \tag{35}$$

and

$$\begin{aligned} J_\infty(m_0) &= \int_T^{+\infty} e^{-\rho t} \ln\left(y_\infty(t)(1 - u^*) - p_\infty(t)m_\infty(t)\right)dt \\ &= \frac{\ln c_\infty(m_0)}{\rho} e^{-\rho T(m_0)}. \end{aligned} \tag{36}$$

In the formulas (35) and (36) all terms can be calculated using relations between model parameters and equality (30). More precisely, elements of the integrand in $J_T(m_0)$ (35) are calculated according to the following relations

$$m_T(t) = m_0 \exp\left\{\left(\frac{b - \beta u_1(m_0)}{1 - \alpha}\right)t\right\}, \quad y_T(t) = ae^{bt} m_T^\alpha(t),$$

$$m_T(t)p_T(t) = p_0 \overline{M} \frac{f_m(t)}{1 - F_m(t)}.$$

Denoting the integrand in $J_T(m_0)$ (35) by $c_T(m_0; t)$, we obtain

$$c_T(m_0; t) = y_T(t)(1 - u_1(m_0)) - p_T(t)m_T(t).$$

For the second integral, the output $y_\infty(t)$ and the product $m_\infty(t)p_\infty(t)$ are constants with respect to time t. However, the value of y_∞ depends on the initial material use m_0, indeed

$$y_\infty = am_0^\alpha \exp\left\{\left(\frac{b-\alpha\beta u_1(m_0)}{1-\alpha}\right)T\right\},$$

$$m_\infty p_\infty = p_0\overline{M}\frac{f_m(t)}{1-F_m(t)} = p_0\overline{M}\frac{f_m(t)}{\int\limits_t^{+\infty} f_m(s)ds} = \frac{p_0\overline{M}b}{\alpha}$$

$$c_\infty(m_0) = y_\infty(1-u^*) - m_\infty p_\infty.$$

As a result, the utility function $J(m_0)$ (34) has the following form

$$J(m_0) = J_T(m_0) + J_\infty(m_0) = \int\limits_0^T e^{-\rho t}\ln c_T(m_0;t)dt + \frac{\ln c_\infty(m_0)}{\rho}e^{-\rho T(m_0)}.$$

Consequently, calculating the first integral numerically, we find the utility function $J(m_0)$ (34) for the piecewise constant control (see Figure 5.3).

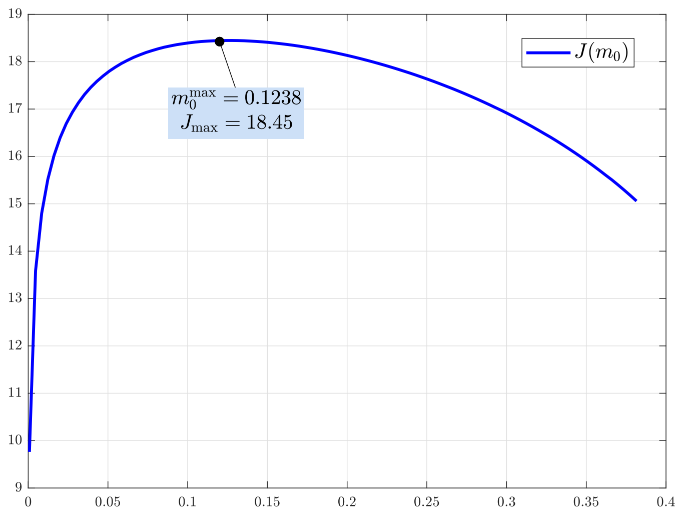

Figure 5.3: The utility functional $J(m_0)$ (14) for the piecewise control $\widehat{u}$ (29)

As can be seen, the utility functions $J_{\exp}(m_0)$ and $J_(m_0)$ (see Figure 3.1 and Figure 5.3, respectively) are strictly concave and have the identical behavior. Under the piecewise control regime the maximum value $J_{\max}(m_0) = 18.45$ of the functional (14) is reached at the point $m_0 = 0.1238$, while in the exponential case it takes smaller value $J_{\exp}^{\max} = 15.1985$ at $m_0^{\max} = 0.1044$. Consequently, from the optimality point of view, the piecewise control regime realizes approximate optimal strategy with better values of the optimized indices compared with the solutions obtained in the exponential case (see section 3).

6. Stabilized and approximate solutions

In accordance with the statistical data and the econometric analysis [7], the model parameters have the following estimates: $m_0 = 0.0985$, $\overline{M} = 1.81 \cdot 10^6$, $\overline{u} = 0.12$, $\rho = 0.18$, $a = 64.34$, $b = 0.069$, $\alpha = 0.4091$, $\beta = 1.523$, $p_0 = 100$. According to formulas (26) we find the steady state $X^* = (0.14, 0.03, 6.21, -2.22)^\top$, and the steady state control value $u^* = 0.11$. Eigenvalues of the Jacobian evaluated at the steady state of the Hamiltonian system (25) indicate the saddle character of the steady state, since eigenvalues are real numbers, and two of them are negative $\lambda_1 = -1.832$, $\lambda_2 = -0.011$, and two other eigenvalues are positive $\lambda_3 = 0.203$, $\lambda_4 = 2.000$.

Basing on the results in [3], it is shown that the stabilized solutions constructed by the algorithm provided in [9] approximate the optimal trajectories with the quadratic precision around the steady state. Therefore, it is worth to compare the approximate solution described in the present paper with the stabilized trajectories [9] in a vicinity of the steady state. The comparison results are presented in Figures 6.1 and 6.2.

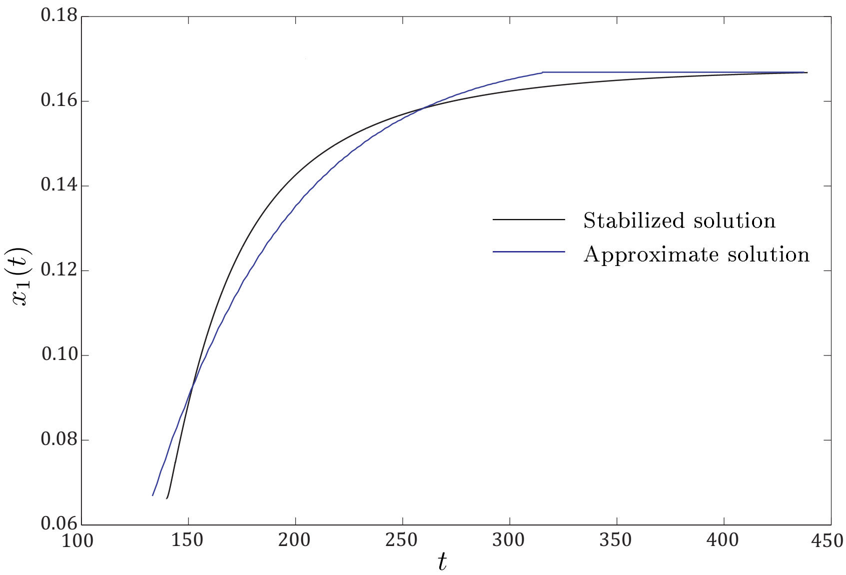

Figure 6.1: The stabilized trajectory and the approximate solution $x_1(t)$.

Figures 6.1 and 6.2 illustrate the phase trajectories $x_1(t)$ and $x_2(t)$ constructed by the algorithm developed in [6] and approximate solutions obtained in the present paper. The trajectories evolve in a similar way, however, the approximation accuracy is not good enough, especially, for the first phase component $x_1(t)$.

7. Conclusion

The paper investigates the control problem of optimization of the resource productivity, and proposes the novel interpretation of the phase variables in terms of the reliability theory. We compare the simulation results derived using the traditional techniques based on the Pontryagin maximum principle and the new

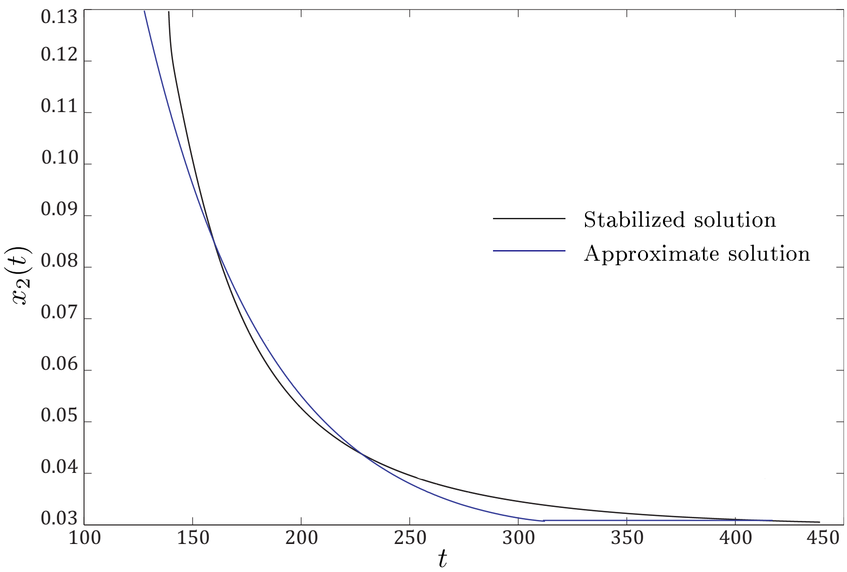

Figure 6.2: The stabilized trajectory and the approximate solution $x_2(t)$.

approach for approximation of optimal control by piecewise linear controls whose trajectories satisfy the phase constraints. Comparing the saturation property of optimal solutions with the exponential trajectories calculated under the piecewise constant controls, one can conclude that the proposed approximate solutions fit quite well to trends of optimal trajectories and demonstrate a similar behavior.

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